An Improved Algorithm For The Solution of Kepler's Equation For An Elliptical Orbit

Mohammed S.Rasheed*

Received on:17/2/2009
Accepted on:7/1/2010

Abstract

In this paper, a root finding method due to iterative method is used first to the solution of Kepler's equation for an elliptical orbit. Then the extrapolation technique in the form of Aitken $\Delta^2 -$acceleration is applied to improve the convergence of the iterative method.

In addition, by making use a new improvement to Aitken's method enables one to obtain efficiently the numerical solution of the Kepler's equation. The speed of the proposed algorithms is compared using different values of eccentricity (e) in the range $e \in (0,1)$ and for given mean anomaly (M).

Keywords: Kepler's equation; Iterative method; Aitken's method, extrapolation technique.

خوارزمية مُحسنة لحل معادلة كبلر للمدار الاهليجي

الخلاصة

في هذا البحث, تم استخدام اولاً طريقة لأيجاد الجذر بالنسبة الى الطريقة التكرارية لحل معادلة كبلر في المدار الاهليجي. تم تطبيق أسلوب الاستكمال بشكل (تعجيل - Δ^2) لأتكن لتحسين الاقتراب في الطريقة التكرارية.

أضافة الى ذلك, تم أستخدام تحسين جديد لطريقة أتكن للحصول على حل عددي كفوء لمعادلة كبلر. تم مقارنة سرعة الخوارزميات المُقترحه بأستخدام قيم مختلفة للانحراف الشاذ (e) في المجال $e \in (0,1)$, مع قيمة مقترحة للانحراف المتوسط M .

1. Introduction

The iterative on the solution of the Kepler problem is extensive. There is a large number of researchers were devoted to its solution in numerical form [1,2,45,7,10].

Nowadays, papers on Kepler's equation and its solution are still of scientific interest (see e.g. [3,6,8]) and it shall be in the future.

In this paper, we will discuss the solution of Kepler's equation in the conventional form,

$$f(E) = M - E + e\sin E = 0 \quad \ldots(1)$$

and for elliptic orbits.

Since the elliptical orbit is periodic, then solution of eq.(1) is considered with $0 \le M \le 2p$ for which the solution will be similarly bounded, $0 \le E \le p$.

Note that, there is no restriction on the values of the eccentricity (e) is considered. Indeed, from the point of view of celestial mechanics, Kepler's equation has no sence for

* Applied Sciences Department, University of Technology /Baghdad

$e \notin [0,1)$, therefore the ranges for (e) is taken to be $(0,1)$.

2. Solutions of Kepler's Equation

It is well known that Kepler's equation can be solved by means of an iterative method defined, in a natural way, from the equation itself. This method yields to the unique solution if the eccentricity is in the range of the elliptic orbits. In [11], Newton method was used to solve Kepler's equation while Thomas [9] used Gauss method to solve Kepler's equation which depend on Picard type iteration. A root –finding method due to Laguerre [4] was applied to the solution of the kepler problem.In addition, Danby [13] and Serafin [11] described numerical solution for solving Kepler's equation.

In this paper, Aitken's acceleration technique with an improvement is applied to find the solution of kepler equation.

2.1 The Iterative Method

The solution proceeds with the choice of a successive approximation algorithm and an appropriate starting value for the iteration, which we will call E_o.

The iterative method uses the following scheme: writing Kepler's equation eq.(1) in the form [2]:

$$E = M + e \sin E \quad \ldots(2)$$

and obtaining a first approximation E_o.

In numerical experiments, the following typical starting values tested are considered [2],

$$E_o = 0$$
$$E_o = M$$
$$E_o = M + e$$
$$E_o = M + 0.85\, sign(\sin M)\, e$$
$$E_o = M + \frac{0.85\, sign(\sin M)\, e}{1 + \sin M - \sin(M + e)}$$

In experimenting, found that simply replacing $E_o = 0$ by other starting values reduced the iterations.

The eccentric anomaly initial guess E_o that will be used in this paper is:

$$E_o = M + e\frac{\sin M}{1 - \sin(M+e) + \sin M} \quad \ldots(3)$$

Then precede further the procedure as indicated below:

$$\left\{\begin{array}{l} E_1 = M + e\sin E_o \\ E_2 = M + e\sin E_1 \\ E_3 = M + e\sin E_2 \\ \vdots \\ E_{n+1} = M + e\sin E_n \end{array}\right\} \quad \ldots(4)$$

Remark:

Aitken $\Delta^2 -$Acceleration [12] will be applied in this paper to find an approximate solution for Kepler's equation as well as an improvement to Aitken $\Delta^2 -$Acceleration is considered here which allowed some improvement at each step.

2.2 Aitken $\Delta^2 -$Acceleration

Aitken's method is a way of solving KE quickly and accurately enough that astronomers might make repeated calculations for determining orbits. Given (e) an M, taking E_o from eq.(3), then calculating

$$E_1 = M + e \sin E_o$$
$$E_2 = M + e \sin E_1$$

and using

$$E = E_2 - \frac{(E_2 - E_1)^2}{E_2 - 2E_1 + E_o} = \frac{E_2\, E_o - E_1^2}{E_2 - 2E_1 + E_o}$$

as the solution of KE. In general, when an iteration

$$E_{n+1} = q\,(E_n)$$

where

$$q\,(E_n) = M + e \sin E_n \quad , \; n = 0,1,....$$

leads to a linearly convergent sequence $\{E_n\}$, one method of improving the convergence is to use extrapolation in the form of Aitken $\Delta^2 -$acceleration.

Let the sequence $\{\overline{E}_n\}$be defined by

$$\overline{E}_n = E_{n+2} - \frac{(E_{n+2} - E_{n+1})^2}{E_{n+2} - 2E_{n+1} + E_n} \quad , \; n = 0,1,... \qquad \ldots(5)$$

Equivalently eq.(5) for acceleration the convergence can be written as

$$\overline{E}_n = E_n - \frac{(E_{n+1} - E_n)^2}{E_{n+2} - 2E_{n+1} + E_n} \quad , \; n = 0,1,... \qquad \ldots(6)$$

or in terms of the forward difference operator Δ, as

$$\overline{E}_n = E_n - \frac{(\Delta E_n)^2}{\Delta^2 E_n} \quad , \quad n = 0,1,...$$

and hence the name $\Delta^2 -$acceleration.

2.3 The Improved Aitken's Method

In this section, the improved Aitken's method is proposed. It will show the difference in the number of iterations required to converge to a given accuracy.

The improved Aitken's method uses the following scheme:

Given e and M , taking E_o from eq.(3), then calculating

$$\overline{E}_o = E_2 - \frac{(E_2 - E_1)^2}{E_2 - 2E_1 + E_o}$$

$$\overline{E}_1 = E_3 - \frac{(E_3 - E_2)^2}{E_3 - 2E_2 + E_1}$$

$$\overline{E}_2 = E_4 - \frac{(E_4 - E_3)^2}{E_4 - 2E_3 + E_2}$$

where E_o is determining using eq.(3) while E_1, E_2 are determining from eq.(4).

Then define the improve value $\overline{\overline{E}}$ using

$$\overline{\overline{E}} \approx \overline{E}_2 - \frac{(\overline{E}_2 - \overline{E}_1)^2}{\overline{E}_2 - 2\overline{E}_1 + \overline{E}_o} \qquad \ldots(7)$$

In general, the improved sequence $\{\overline{\overline{E}}\}$ can be defined by:

$$\overline{\overline{E}}_n = \overline{E}_{n+2} - \frac{(\overline{E}_{n+2} - \overline{E}_{n+1})^2}{\overline{E}_{n+2} - 2\,\overline{E}_{n+1} + \overline{E}_n} \quad , n = 0,1 \qquad \ldots(8)$$

3. Results and Conclusions

Newton's method for iterative solution of equations is a standard technique. The procedure of Newton's method provides correct values of the true anomaly, but can take a large number of iterations, especially for large values of eccentricity. The method becomes unstable for certain values of mean anomaly and eccentricity. For certain kinds of function, the method either does not converge, or generates very large values before converging. The efficient the algorithms described in section two in solving Kepler's equation. As shown in table (1) giving the number of iterations required for the algorithms to converge to $|\Delta E| < 10^{-}$

9. In every algorithm the starting value is

$$E_o = M + e\frac{\sin M}{1 - \sin(M+e) + \sin M}$$

and the number of iterations required is noted for each (e-M) pair of starting values, where e is varied from 0.1 to 0.9 in steps of 0.1 and M is 151.7425 deg.

Clearly, the numbers of iterations increase as e approaches to 0.9.

Note the remarkable improvement in the number of iterations required when the improved Aitken's method was used. Finally, table (1) shows the value of mean anomaly lies in the range of 151.7425 $\leq$ E $\leq$ 151.7425+e.

References

[1]. Branham, R.L., "Double Star orbits with semi-definite programming and alternative", A&A 507,1107-1115, ESO, 2009.

[2]. Chobotov, V-A., "Orbital Mechanics, by the American Institute of Aeronautics and Astronautics", Inc., 1996.

[3].Colwell, P., "Solving Kepler's equation over three centuries", by Willmann- Bell, Inc., 1993.

[4].Conway, B.A., "An Improved Algorithm Due to Laguerne for the Solution of Kepler's equation", Celestial Mechanics 39(199-211), by D.Reidel Publishing Company, 1986.

[5].Lanchares, V. and Perez, I.L., "The Dynamics of Kepler's equation", Monografias De la Real Academia de Ciencias de Zaragoza, zz: 75-82, 2003.

[6].Murison, M., A., "A Practical Method for Solving the Kepler's equation", 2006. http://www.alpheratz.net/murison /dynamics/twobody/kepler Iterations-summary.pdf.

[7]Nijenhuis, A., "Solving Kepler's equation with high efficiency and accuracy", Celestial mechanics and Dynamic alastronomy SI: 319-330, 1991.

[8].Sharaf, M.A. & Selim, H.H., "Homotopy Continuation method for Solving hyperbolic form of Kepler's equation", Contrib. Astron. Obs. Shalnate Pleso, 36, 71-76, 2006.

[9]. Thomas, J.F., "Extension of Gauss method for the solution of solve Kepler's equation", Msc, thesis, Pennsylvania state university,1976.

[10].Toshio, F., "A method solving Kepler's equation for hyperbolic case", Celestial mechanics and Dynamical astronomy, 68:121-137, 1997.

[11].Serafin, R.A., "on solving Kepler's equation for nearly parabolic orbits", Celestial mechanics and Dynamical astronomy 65:389-398, 1997.

[12]. Wait, R., "The Numerical Solution of Algebric Equations", John Wiley & Sons, Ltd., 1979.

[13]"Celestial Computing with matlab by Sciences software", 2008.

Table (1) shows the number of iterations needed to converge by four algorithms

e	E	Number of iterations			
		Newton	Iteration	Aitken	Mod. Aitken
0.1	154.23320094	75	8	1	-
0.2	156.34097686	170	14	4	2
0.3	158.14199629	230	21	6	3
0.4	159.695403729	290	26	8	5
0.5	161.04707996	328	31	9	6
0.6	162.23279417	400	38	14	7
0.7	163.28065271	450	48	13	8
0.8	164.21294339	480	87	35	19
0.9	165.04750916	550	165	66	46

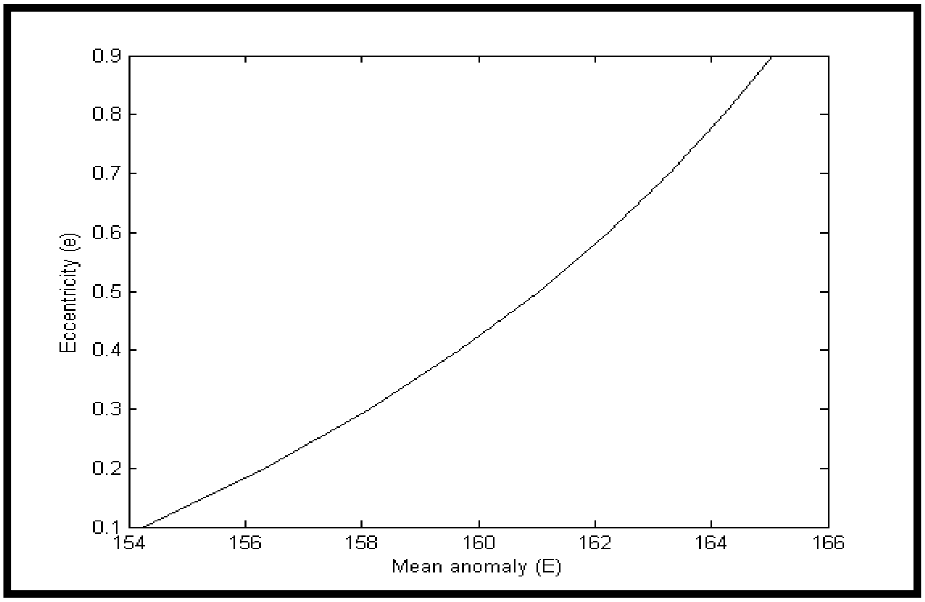

Figure (1) The relation between eccentricity (e) and mean anomaly (E)