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Study the Integro-Differential Equation in a Lorentzian Energy Spread Case

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Abstract

The fundamentals of free electron laser (FEL) theory are now well–established and can provide a sophisticated description of experiments over a wide range of parameters. While new technology is being developed for systems working from 1mm to 10nm wavelengths, the theory remains the same. In this work, the final consequences of Lorentzian energy distribution in FEL amplifier will be found by solving of Integro-differential equation using symbolic toolbox services in MATLAB software v. 6.5 and the relation between several parameters will be delimited and plotted in order to extract and present the complex relations between the FEL parameters.

Keyword: Evolutionary programming and Symbolic Analysis, Mathematical Modeling, FEL Amplifier Process

الخلاصة

ان أساسيات نظريةِ ليزرِ الألكترون الحُرِّ أصبحت الان معرفة ويُمْكِنُهـــا أَنْ تصـــف التجـــارب المعقّده لـــتشكيلة واسعة من العواملِ. بينما التقنيات الحديثة قد بدات تتُطوّر للأنظمةِ التي تَعْملُ مِنْ مليمترِ واحد إلى عشرة نانومتر من الطولِ موجة ، ومع ذلك يَبْقت النظريةُ بدون تغييـــر. في هذا العملِ، النتائج النهائية لتوزيعِ طاقةِ لورنزية في مضخّمِ ليزرِ الألكترون الحُرِّ سَـــتَوجدُ بحَلّ المعادلة التكاملية - التفاضليةِ حيـــث اَســـتعملت خصـــائص وميـــزات للغـــة البرمجـــة (MATLAB software v. 6) في هذا البحث. علاوة على ذلك تم وصف العلاقـــةِ بـــين عِـــدّة عوامل لأستخلاص وتمثيل العلاقاتَ المركّبةَ بين عدد من عواملِ ليزرِ الألكترون الحُرِّ.

1 Introduction

The problem of electromagnetic wave amplification in the undulator refers to a class of self-consistent problems [1]. It can be separated into two parts: *Solution of the dynamical problem* and *Solution of the electrodynamics' problem*.

To close the problem, the field equations and equations of motion should be solved simultaneously. In principle, modern super-computers allow one to perform direct simulation of the FEL process. The results of such simulations depend on a large number of problem parameters. They provide the possibility of obtaining a

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numerical answer for a specific set of input data.

A deeper insight into FEL physics can be obtained only by introducing some simplifying assumptions about the properties of the electron beam and of the electromagnetic field.

Theoretical investigation of the free electron laser should be performed in two stages. In the beginning one should study the general properties of the FEL, namely the ideal mechanism of amplification. At the next stage different complications can be introduced into the FEL model allowing it to extend the number of additional effects influencing the operation of the FEL amplifier. These factors can be divided into two groups. The space charge effects and diffraction effects are fundamental. On the other hand, there are a lot of other factors such as energy spread effects, non-ideality of the undulator field, etc. the principal difference of the fundamental effects and all the others is that the fundamental effects depend on the same physical parameters as the ideal FEL amplification mechanism itself.

The one-dimensional model is an important one from the methodological point of view. The following assumptions are used in the one-dimensional, steady-state model of the FEL amplifier for linear region:

- The electron beam has Gaussian density distribution in the direction perpendicular to the undulator axis;
- The electron moves along identical trajectories parallel to the undulator axis;
- The amplified wave is a monochromatic plane wave;
- The electron beam is infinitely long.

2 Theory

The description of the Free Electron Laser given here is based on Kim [1] and Reiche [2]. A more detailed description can be found in [3] and [4] and the references therein.

The treatment in this search will be considered the case of an electron beam with Lorentzian energy distribution as [2]:

$$F(e-e_o)=\frac{1}{p}\frac{\Lambda_T}{(e-e_o)^2+\Lambda_T^2} \quad (1)$$

where:

$F(e-e_o)$ is the distribution function of the canonical momentum.

Λ_T is the normalized energy spread parameter.

e is the energy, and

e_o is the nominal energy of an electron.

The corresponding expression for the reduced distribution function $\hat{F}$ has the form (see Figure (1));

$$\hat{F}(\hat{P})=\frac{1}{p}\frac{\hat{\Lambda}_T}{\hat{P}^2+\hat{\Lambda}_T^2} \quad (2)$$

where

$$\hat{P}=(e-e_o)/(r\,e_o),$$

is the normalized energy deviation.

with a normalized energy spread $\hat{\Lambda}_T=\Lambda_T/(e_o\,r)$

and

$$\Lambda_T=\left\langle(\Delta e)^2\right\rangle,$$

which is at least an approximation for the more realistic Gaussian distribution.

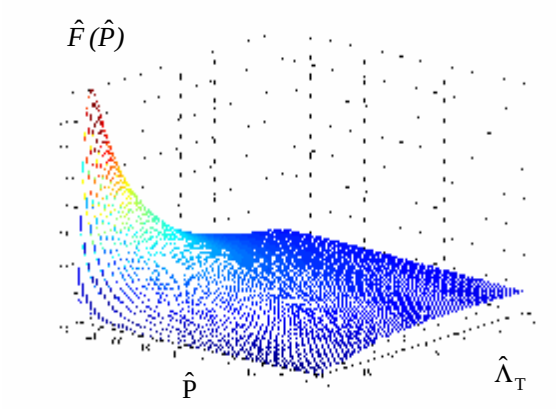


Figure (1)
Distribution function of the reduced canonical momentum $\hat{F}(\hat{P})$ versus reduced canonical momentum $\hat{P}$ and energy spread parameter $\hat{\Lambda}_T$.

Substituting the Lorentzian distribution function into the final form of integro-differential equation solution [3], we find that $\hat{D}$ is given by the expression:

$$\hat{D} = i\,(\,l + \hat{\Lambda}_T + i\hat{C}\,)^{-2} \qquad (3)$$

And a lengthily calculation yields the cubic dispersion equation [2]:

$$((l - i\,\hat{\Lambda}_T + \hat{C})^2 - \hat{\Lambda}_p^2)\,l + 1 = 0 \qquad (4)$$

This equation is similar to equation for a mono-energetic beam but with complex coefficients [5].

3 Results and Discussion

For the realization, it can be found the general solution of integro-differential equation for a Lorentzian Energy Spread case using the Laplace transform technique [6] and MATLAB symbolic toolbox [7,8]. The same result can be extract using another technique, which the resultant cubic equation (in normalized form) can be written as equation (5):

$$((l - i\,\hat{\Lambda}_T + \hat{C})^2 - \hat{\Lambda}_p^2)\,l + 1 = 0\,, \qquad (5)$$

The general solution of above equation can be written as follow:

$$\lambda_1 = \frac{1}{6}\sqrt[3]{\alpha} - 6J + \mathrm{i}\frac{2}{3}\hat{\Lambda}_T - \frac{2}{3}\hat{\mathrm{C}} \qquad (6a)$$

$$\lambda_{2,3} = -\frac{1}{12}\sqrt[3]{\alpha} + 3J + \mathrm{i}\,\frac{2}{3}\hat{\Lambda}_T - \frac{2}{3}\hat{\mathrm{C}}$$

$$\pm\,\mathrm{i}\,\frac{\sqrt{3}}{2}\left[\frac{1}{6}\sqrt[3]{\alpha} + 6J\right] \qquad (6b)$$

where;

$$J = \frac{\hat{\Lambda}_T^2 + \mathrm{i}\,2\,\hat{\mathrm{C}}\,\hat{\Lambda}_T - \hat{\mathrm{C}}^2 - 3\,\hat{\Lambda}_\mathrm{p}^2}{9\,\sqrt[3]{\alpha}}$$

$$\begin{aligned}\alpha = {} & \mathrm{i}8\hat{\Lambda}_T^3 - 24\hat{\Lambda}_T^2\,\hat{\mathrm{C}} - \mathrm{i}24\hat{\Lambda}_T\,\hat{\mathrm{C}}^2 + 8\hat{\mathrm{C}}^3 + \mathrm{i}72\hat{\Lambda}_T\,\hat{\Lambda}^2 \\ & - 72\hat{\Lambda}_\mathrm{p}^2\,\hat{\mathrm{C}} - 108 + 12\big[81 - \mathrm{i}48\hat{\Lambda}_T^3\,\hat{\mathrm{C}}\,\hat{\Lambda}_\mathrm{p}^2 \\ & + \mathrm{i}48\hat{\Lambda}_T\,\hat{\mathrm{C}}^3\,\hat{\Lambda}_\mathrm{p}^2 - \mathrm{i}48\hat{\Lambda}_T\,\hat{\mathrm{C}}\,\hat{\Lambda}_\mathrm{p}^4 - 12\hat{\mathrm{C}}^3 \\ & - 12\hat{\Lambda}_T^4\,\hat{\Lambda}_\mathrm{p}^2 + 24\hat{\Lambda}_T^2\,\hat{\Lambda}_\mathrm{p}^4 + 12\hat{\mathrm{C}}^4\hat{\Lambda}_\mathrm{p}^2 - 24\hat{\mathrm{C}}^2\hat{\Lambda}_\mathrm{p}^4 \\ & - \mathrm{i}12\hat{\Lambda}_T^3 + 36\hat{\Lambda}_T^2\,\hat{\mathrm{C}} + 108\hat{\mathrm{C}}\,\hat{\Lambda}_\mathrm{p}^2 + 72\hat{\Lambda}_T^2\,\hat{\mathrm{C}}^2\,\hat{\Lambda}_\mathrm{p}^2 \\ & + \mathrm{i}36\hat{\Lambda}_T\,\hat{\mathrm{C}}^2 + \mathrm{i}108\hat{\Lambda}_T\,\hat{\Lambda}_\mathrm{p}^2 + 12\hat{\Lambda}_\mathrm{p}^6\big]^{1/2}\end{aligned}$$

The three roots are not symmetrically located in the complex plane and there might be more than one growing and decaying modes.

In Figure (2) the largest growth rate of all three solution is plotted depending on different settings of the energy spread $\hat{\Lambda}_T$. With increasing energy spread the growth rate for a fixed value of $\hat{C} < 0$ is reduced almost in a linear way. The steep edge of the gain curved at $\hat{C} \approx 2$ is smeared out

yielding a slightly increased growth rate for $\hat{C}$ >1.9 with increasing spread.

Beside the predicted reduction of the growth rate the gain curve becomes more and more antisymmetric for $\hat{\Lambda}_T$ > 0.5. For an injection below the resonant energy ($\hat{C}$ < 0), where all roots have positive imaginary parts, all modes are exponentially decaying.

The complete energy of the radiation field is transferred to the electron beam.

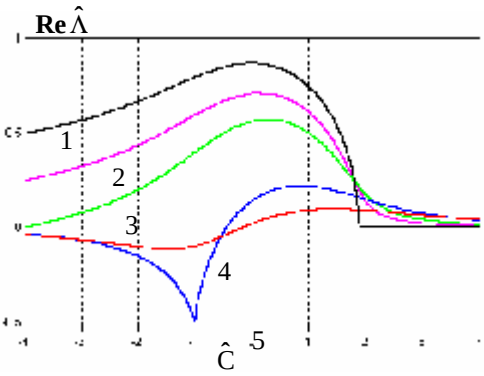


Figure (2)

Reduced field growth rate $\mathrm{Re}\,\hat{\Lambda}$ **versus the detuning parameter** $\hat{C}$**, Curve 1:** $\hat{\Lambda}_T \to 0$**, Curve 2:** $\hat{\Lambda}_T \to 0.25$**, Curve 3:** $\hat{\Lambda}_T \to 0.5$**, Curve 4:** $\hat{\Lambda}_T \to 1.5$**, Curve 5:** $\hat{\Lambda}_T \to 2.5$ **at** $\hat{\Lambda}_p^2 \to 0$.

This can be understood regarding the electron motion in the longitudinal phase space. The bucket of the ponderomotive wave is filled by the initial distribution in such a way that the distribution thins out towards higher energy. This happens if the mean energy of the distribution lies at the lower border of the separatrix and the tail towards higher energy covers the bucket.

Due to the FEL interaction more electrons in the center of the distribution gain energy than the electrons lose in the tail. The width of the bucket is reduced and electrons may be detrapped if the center of the electron distribution is close to the separatrix of the bucket. This principle is repeated till the bucket completely vanishes.

The final formula of field amplitude $\tilde{E}(\hat{z})$ in case of Lorentian energy distribution can be written as [15]:

$$\tilde{E}(\hat{z}) = E_{ext} \sum_j \frac{\exp(l_j \hat{z})}{1-2i(l_j + \hat{\Lambda}_T + i\hat{C})\, l_j^2} \quad (6)$$

where

l_j are the roots of the cubic equation (Eq. (5a,b)).

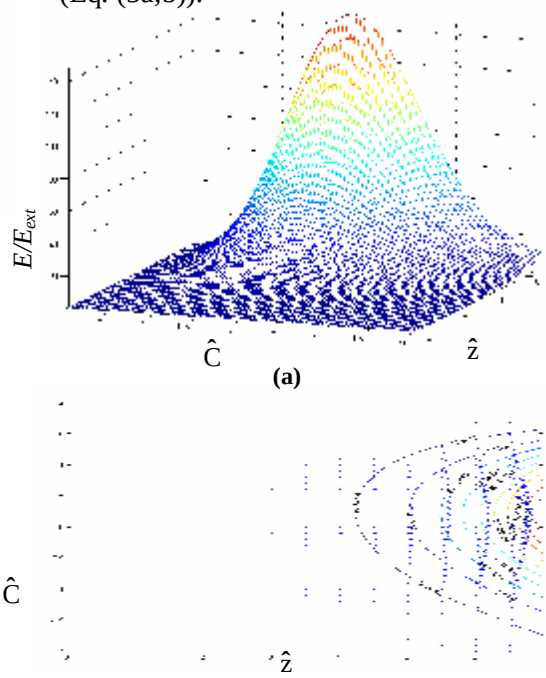
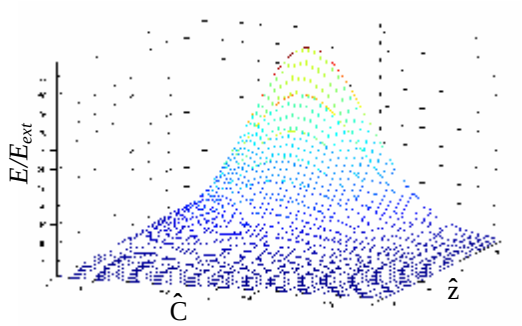


Figure (3)

Field gain E/E_{ext} **versus** $\hat{C}$ **and the reduced length of undulator** $\hat{z}$**. Here** $\hat{\Lambda}_p^2 \to 0$ **and** $\hat{\Lambda}_T^2 \to 0$.



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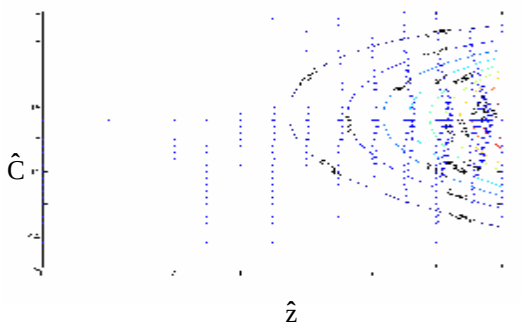
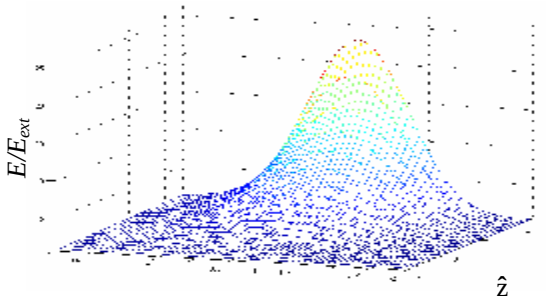


(b)

Figure (4)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \rightarrow$ 0.25 and $\hat{\Lambda}_T^2 \rightarrow 0$.



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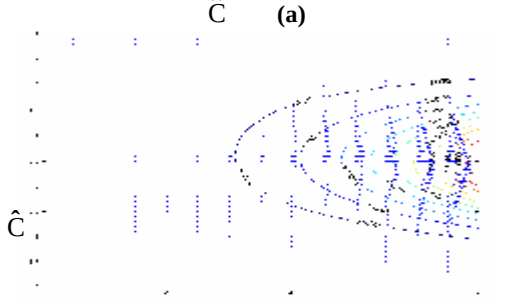
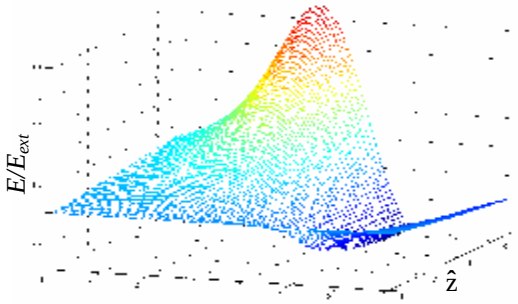


(b)

Figure (5)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \rightarrow$ 0.5 and $\hat{\Lambda}_T^2 \rightarrow 0$.



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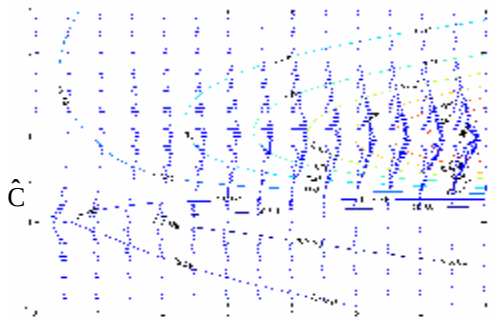

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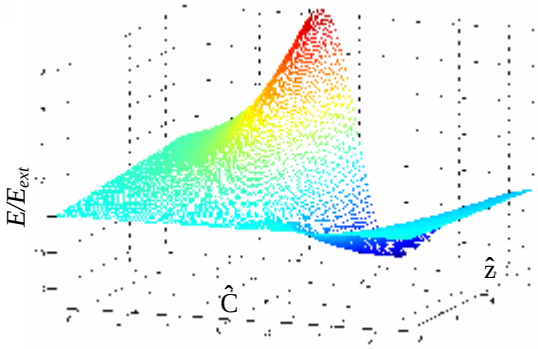
Figure (6)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \rightarrow$ 1.5 and $\hat{\Lambda}_T^2 \rightarrow 0$.



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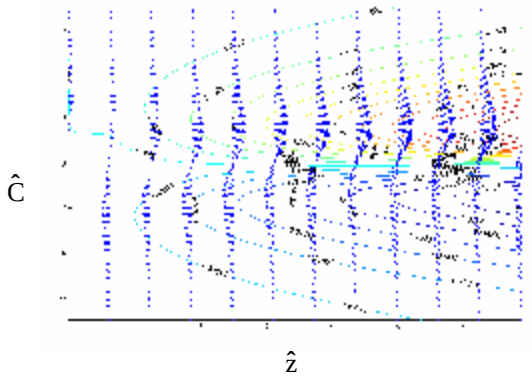
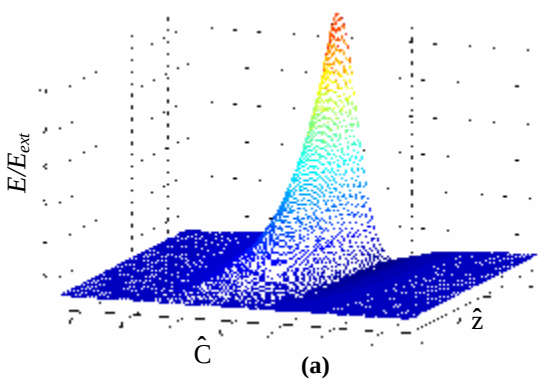


(b)

Figure (7)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \rightarrow$ 2.5 and $\hat{\Lambda}_T^2 \rightarrow 0$.



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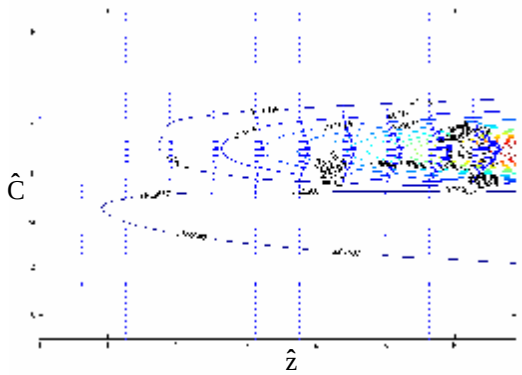
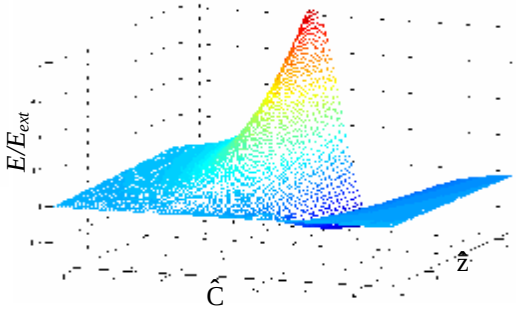


(b)

Figure (8)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \rightarrow$ 0.5 and $\hat{\Lambda}_T^2 \rightarrow 1$.



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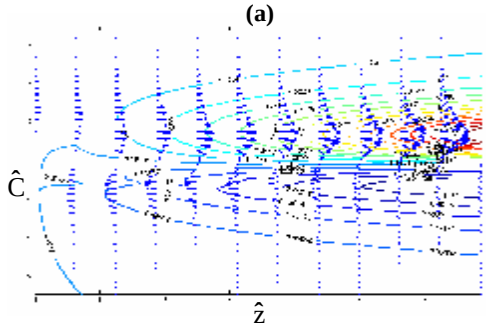
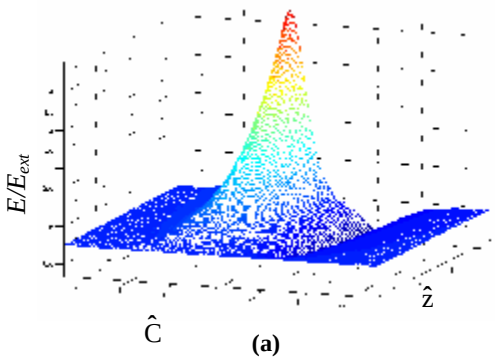


(b)

Figure (9)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \rightarrow$ 1.5 and $\hat{\Lambda}_T^2 \rightarrow 1$.



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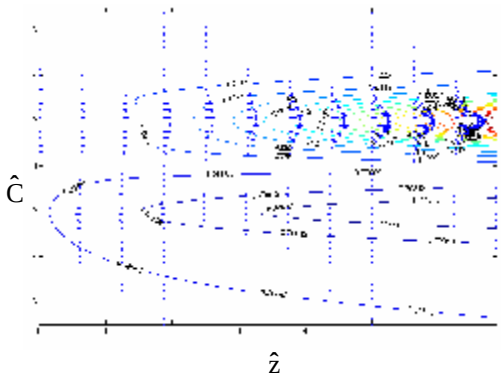
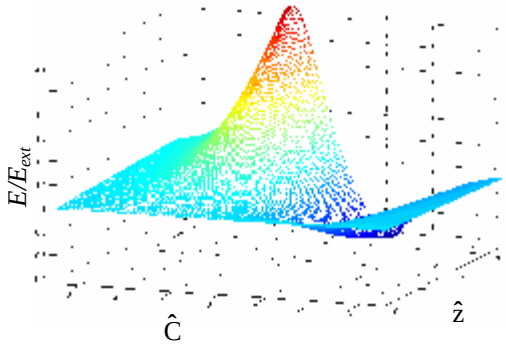


(b)

Figure (10)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \to 0.5$ and $\hat{\Lambda}_T^2 \to 4$.



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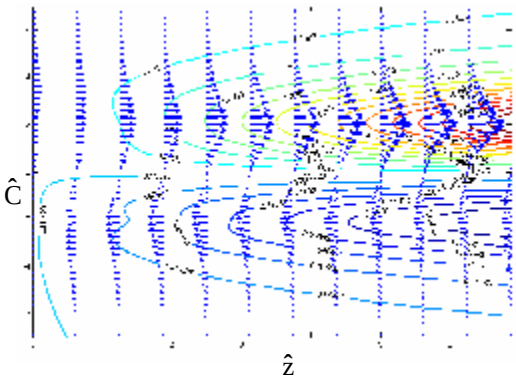

(b)

Figure (11)

Field gain E/E_{ext} versus $\hat{C}$ and the reduced length of undulator $\hat{z}$. Here $\hat{\Lambda}_p^2 \to 1.5$ and $\hat{\Lambda}_T^2 \to 4$.

4 Conclusion

Figures (3) to (11) illustrate the dependence of the field gain on the detuning parameter $\hat{C}$, space charge parameter $\hat{\Lambda}_p^2$ and spread energy parameter $\hat{\Lambda}_T^2$ along the undulator length.

According to our process in the previous equations, the vectors shown in Figures (3b) to (11b), and the directions of these vectors in Figures (3b) to (5b) will be from the undulator entrance in the direction of the end of undulator length where this phenomena occur only with neglecting the space charge parameter $\hat{\Lambda}_p^2$.

When we enter the effects of the space charge, the field gain equation (see Figures (6) to (8)), the negative peak in the field gain amplitude will be observed along the detuning parameter $\hat{C}$ axis.

The negative concavity values in the field gain amplitude will be increases when we increase the effective value of the space charge parameter $\hat{\Lambda}_p^2$ (see Figures (6) to (11)). Also, the universal distribution of the field gain amplitude will be shifted in the positive side of the detuning parameter values.

The rigorous results obtained in reduced form furnish universal plots for calculation the output characteristics of the FEL amplifier in the linear mode of operation. These solutions serve as a reliable basis for the development of numerical methods.

The analysis of nonlinear processes refers to problems solvable only numerically by a computer. On the other hand, testing of the numerical

simulation codes would be difficult without the use of rigorous results of FEL amplifier linear theory as a primary standard.

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