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By: Professor Dr. Adil H. Ahmad

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أ.د . عادل حميد احمد الشمري

Text Book : Engineering Electromagnetics (6^{th} ed.), **W. H. Hayt, Jr., J. A. Buck**

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CONTENTS

Chapter One

Vector Analysis and Coordinate systems

1.1 SCALARS AND VECTORS

The term *scalar* refers to a quantity whose value may be represented by a single (positive or negative) real number. The x, *y,* and z we used in basic algebra are scalars, and the quantities they represent are scalars. If we speak of a body falling a distance L in a time t, or the temperature T at any point in a bowl of soup whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars. Other scalar quantities are mass, density, pressure (but not force), volume, and volume resistivity. Voltage is also a scalar quantity, although the complex representation of a sinusoidal voltage, an artificial procedure, produces a *complex scalar,or phasor,* which requires two real numbers for its representation, such as amplitude and phase angle, or real part and imaginary part.

A *vector* quantity has both a magnitude[1] and a direction in space. We shall be concerned with two- and three-dimensional spaces only, but vectors may be defined in ^-dimensional space in more advanced applications. Force, velocity, acceleration, and a straight line from the positive to the negative terminal of a storage battery are examples of vectors. Each quantity is characterized by both a magnitude and a direction.

1.2 VECTOR ALGEBRA

With the definitions of vectors and vector fields now accomplished, we may proceed to define the rules of vector arithmetic, vector algebra, and (later) of vector calculus. Some of the rules will be similar to those of scalar algebra, some will differ slightly, and some will be entirely new and strange. This is to be expected, for a vector represents more information than does a scalar, and the multiplication of two vectors, for example, will be more involved than the multiplication of two scalars.

Vectorial addition follows the parallelogram law, and this is easily, if inaccurately, accomplished graphically. Fig. 1.1 shows the sum of two vectors, A and B. It is easily seen that A + B = B + A, or that vector addition obeys the commutative law. Vector addition also obeys the associative law,

A + (B + C) = (A + B) + C

Note that when a vector is drawn as an arrow of finite length, its location is defined to be at the tail end of the arrow.

Coplanar vectors, or vectors lying in a common plane, such as those shown in Fig. 1.1, which both lie in the plane of the paper, may also be added by expressing each vector in

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terms of "horizontal" and "vertical" components and adding the corresponding components.
The rule for the subtraction of vectors follows easily from that for addition, for we may always express A — B as A + (—B); the sign, or direction, of the second vector is

FIGURE 1.1
Two vectors may be added graphically either by drawing both vectors from a common origin and completing the parallelogram or by beginning the second vector from the head of the first and completing the triangle; either method is easily extended to three or more vectors.

reversed, and this vector is then added to the first by the rule for vector addition.
Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direc
tion when multiplied by a negative scalar. Multiplication of a vector by a scalar also obeys the associative and distributive laws of algebra, leading to

$$(r + s)(A + B) = r(A + B) + s(A + B) = rA + rB + sA + sB$$

Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar.
The multiplication of a vector by a vector is discussed in Secs. 1.6 and 1.7. Two vectors are said to be equal if their difference is zero, or A = B if A - B = 0.
In our use of vector fields we shall always add and subtract vectors which are defined at the same point. For example, the *total* magnetic field about a small horseshoe magnet will be shown to be the sum of the fields produced by the earth and the permanent magnet; the total field at any point is the sum of the individual fields at that point.

1.3 THE CARTESIAN COORDINATE SYSTEM

In order to describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. There are three simple methods of doing this, and about eight or ten other methods which are useful in very special cases. We are going to use only the three simple methods, and the simplest of these is the *cartesian, or rectangular, coordinate system.*

In the cartesian coordinate system we set up three coordinate axes mutually at right angles to each other, and call them the x, *y,* and z axes. It is customary to choose a *right-handed* coordinate system, in which a rotation (through the smaller angle) of the *x* axis into the *y* axis would cause a right-handed screw to progress in the direction of the *z* axis. If the right hand is used, then the thumb, forefinger, and middle finger may then be identified, respectively, as the x, y, and z axes. Fig. 1.2a shows a right-handed cartesian coordinate system.

A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of a perpendicular dropped from the point to the x, y, and z axes. An alternative method of interpreting coordinate values, and a method corresponding to that which *must* be used in all other coordinate systems, is to consider the point as being at thecommon intersection of three surfaces, the planes x =

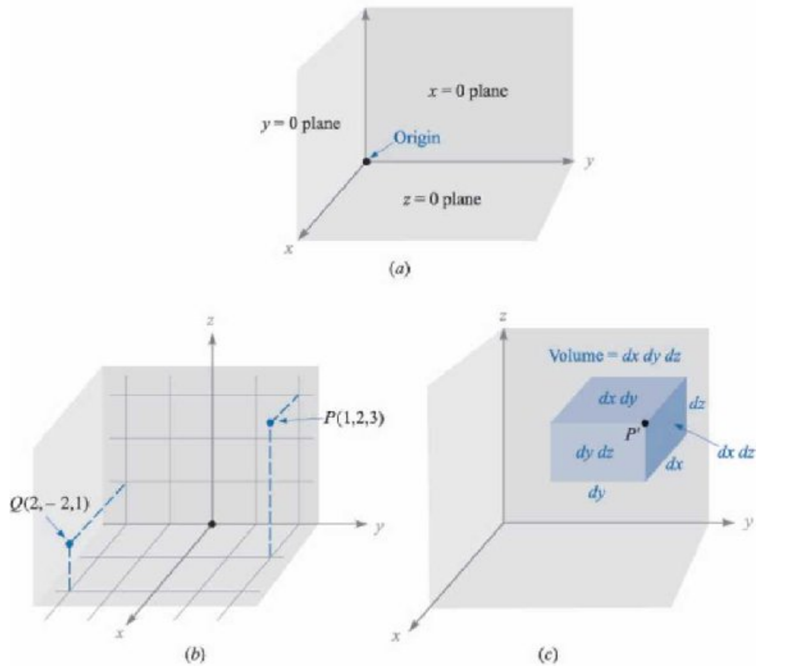

FIGURE 1.2
(a) A right-handed cartesian coordinate system. If the curved fingers of the right hand indicate the direction through which the *x* axis is turned into coincidence with the *y* axis, the thumb shows the direction of the *z* axis. (b) The location of points *P(1,* 2, 3) and *Q(2,* —2, 1).(c) The differential volume element in cartesian coordinates; dx, dy, and *dz* are, in general, independent differentials.

constant, y = constant, and z = constant, the constants being the coordinate values of the point.

Fig. *1.2b* shows the points P and Q whose coordinates are (1, 2, 3) and (2, —2, 1), respectively. Point P is therefore located at the common point of intersection of the planes x = 1, y = 2, and z = 3, while point Q is located at the intersection of the planes x = 2, y = — 2, z = 1.

As we encounter other coordinate systems in Secs. 1.8 and 1.9, we should expect points to be located at the common intersection of three surfaces, not necessarily planes, but still mutually perpendicular at the point of intersection.

If we visualize three planes intersecting at the general point P, whose coordinates are x, y, and z, we may increase each coordinate value by a differential amount and obtain three slightly displaced planes intersecting at point Pwhose coordinates are x + *dx, y + dy,* and z + *dz.* The six planes define a rectangular parallelepiped whose volume is *dv = dxdydz;* the surfaces have differential areas *dS* of dxdy, *dydz,* and dzdx. Finally, the distance dL from P to P' is the diagonal of the parallelepiped and has a length of y $(dx)^2 + (dy)^2 + (dz)^2$. The volume element is shown in Fig. 1.2c; point P' is indicated, but point P is located at the only invisible corner.

All this is familiar from trigonometry or solid geometry and as yet involves only scalar quantities. We shall begin to describe vectors in terms of a coordinate system in the next section.

1.4 VECTOR COMPONENTS AND UNIT VECTORS

To describe a vector in the cartesian coordinate system, let us first consider a vector r extending outward from the origin. A logical way to identify this vector is by giving the three *component vectors,* lying along the three coordinate axes, whose vector sum must be the given vector. If the component vectors of the vector r are x, y, and z, then r = x + y + z. The component vectors are shown in Fig. 1.3a. Instead of one vector, we now have three, but this is a step forward, because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes.

In other words, the component vectors have magnitudes which depend on the given vector (such as r above), but they each have a known and constant direction. This suggests the use of *unit vectors* having unit magnitude, by definition, and directed along the coordinate axes in the direction of the increasing coordinate values. We shall reserve the symbol a for a unit vector and identify the direction of the unit vector by an appropriate subscript. Thus a_x, a_y, and a_z are the unit vectors in the cartesian coordinate system.[2] They are directed along the x, y, and z axes, respectively, as shown in Fig. 1.3b.

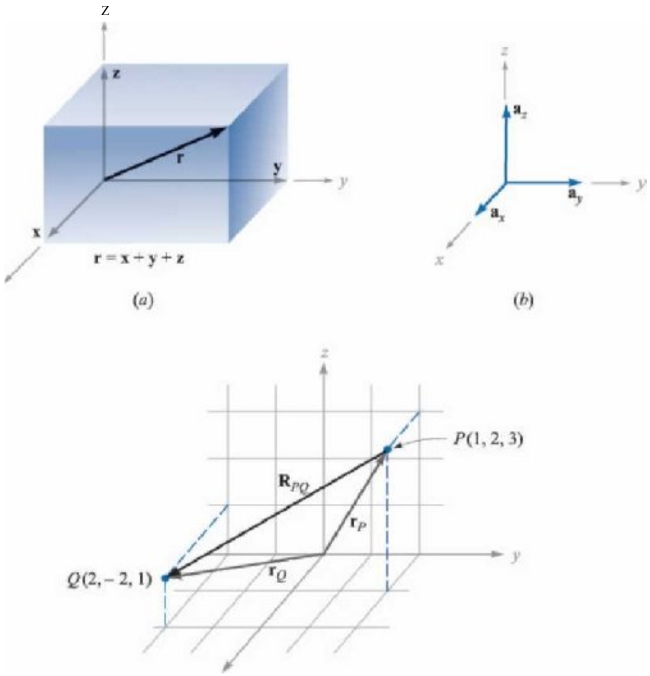

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FIGURE 1.3

(a) The component vectors x, y, and z of vector r.(b) The unit vectors of the cartesian coordinate system have unit magnitude and are directed toward increasing values of their respective variables. (c) The vector R_PQ is equal to the vector difference rQ — r_P.

If the component vector y happens to be two units in magnitude and directed toward increasing values of y, we should then write y = $2a_y$. A vector r_P pointing from the origin to point *P(1,* 2, 3) is written $r_P = a_x + 2a_y + 3a_z$. The vector from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from P(1, 2, 3) to Q(2, —2, 1) is therefore

RPQ = rQ — rp = (2 — 1)ax + (—2 — 2)ay + (1 — 3)a_z The vectors r_P, rQ, and R_PQ are

shown in Fig.1.3c.

This last vector does not extend outward from the origin, as did the vector r we initially considered. However, we have already learned that vectors having the same magnitude and pointing in the same direction are equal, so we see that to help our visualization processes we are at liberty to slide any vector over to the origin before determining its component vectors. Parallelism must, of course, be maintained during the sliding process.

If we are discussing a force vector F, or indeed any vector other than a displacement-type vector such as r, the problem arises of providing suitable letters for the three component vectors. It would not do to call them x, y, and z, for these are displacements, or directed distances, and are measured in meters (abbreviated m) or some other unit of length. The

problem is most often avoided by using *component scalars,* simply called *components,* F_x, F_y, and F_z. The components are the signed magnitudes of the component vectors. We may then write F = $F_x a_x + F_y a_y + F_z a_z$. The component vectors are $F_x a_x$, $F_y a_y$, and $F_z a_z$.
Any vector B then may be described by B = $B_x\boldsymbol{a_x} + B_y\boldsymbol{a_y} + B_z\boldsymbol{a_z}$. The magnitude of B written |B| or simply B, is given by

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} \qquad (1)$$

Each of the three coordinate systems we discuss will have its three fundamental and mutually perpendicular unit vectors which are used to resolve any vector into its component vectors. However, unit vectors are not limited to this application. It is often helpful to be able to write a unit vector having a specified direction. This is simply done, for a unit vector in a given direction is merely a vector in that direction divided by its magnitude. A unit vector in the r direction is r/^$x^2 + y^2 + z^2$, and a unit vector in the direction of the vector B is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|} \qquad (2)$$

Example 1.1

Specify the unit vector extending from the origin toward the point *G(2,* —2, —1).

Solution. We first construct the vector extending from the origin to point G,
G = 2ax — 2ay — a_z We continue by finding the magnitude of G,

|G| = J$(2)^2 + (—2)^2 + (—1)^2$ = 3
and finally, expressing the desired unit vector as the quotient, G
a_G = — = 2 ax — f ay — 1 az = 0.667ax — 0.667ay — 0.333a_z 1G1
A special identifying symbol is desirable for a unit vector so that its character is immediately apparent. Symbols which have been used are u_B, a_B, 1_B, or even b. We shall consistently use the lowercase a with an appropriate subscript.

1.5 THE VECTOR FIELD

We have already defined a vector field as a vector function of a position vector. In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question. Since we have considered only the cartesian coordinate system, we should expect the vector to be a function of the variables x, y, and z.
If we again represent the position vector as r, then a vector field G can be expressed in functional notation as G(r); a scalar field T is written as T(r).
If we inspect the velocity of the water in the ocean in some region near the surface where tides and currents are important, we might decide to represent it by a velocity vector which is in any direction, even up or down. If the z axis is taken as upward, the x axis in a northerly direction, the y axis to the west, and the origin at the surface, we have a right-handed coordinate system and may write the velocity vector as v = i^ax + i>yay + i>zaz, or v(r) = i>x(r)ax + iy(r)ay + iz(r)az; each of the components i_x, i_y, and i_z may be a function of the

three variables x, y, and z. If the problem is simplified by assuming that we are in some portion of the Gulf Stream where the water is moving only to the north, then i_y, and i_z are zero. Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be v = $2e^{z/100}a_x$. We have a velocity of 2m/s (meters per second) at the surface and a velocity of 0.368 x 2, or 0.736 m/s, at a depth of 100 m (z = — 100), and the velocity continues to decrease with depth; in this example the vector velocity has a constant direction.

While the example given above is fairly simple and only a rough approximation to a physical situation, a more exact expression would be correspondingly more complex and difficult to interpret. We shall come across many fields in our study of electricity and magnetism which are simpler than the velocity example, an example in which only the component and one variable were involved (the x component and the variable z). We shall also study more complicated fields, and methods of interpreting these expressions physically will be discussed then.

1.6 THE DOT PRODUCT

We nowconsider the first of two types of vector multiplication. The second type will be discussed in the following section.

Given two vectors A and B, the *dot product,or scalar product,* is defined as the product of the magnitude of A, the magnitude of B, and the cosine of the smaller angle between them,

$$A \bullet B = |A||B| \cos 6_{AB}$$

The dot appears between the two vectors and should be made heavy for emphasis. The dot, or scalar, product is a scalar, as one of the names implies, and it obeys the commutative law,

$$A. B = B. A \tag{4}$$

for the sign of the angle does not affect the cosine term. The expression A • B is read "A dot B."

Perhaps the most common application of the dot product is in mechanics, where a constant force F applied over a straight displacement L does an amount of work *FL* cos 6, which is more easily written F • L.

Another example might be taken from magnetic fields, a subject about which we shall have a lot more to say later. The total flux <J> crossing a surface of area S is given by BS if the magnetic flux density B is perpendicular to the surface and uniform over it. We define a *vector surface* S as having the usual area for its magnitude and having a direction *normal* to the surface (avoiding for the moment the problem of which of the two possible normals to take). The flux crossing the surface is then B • S. This expression is valid for any direction of the uniform magnetic flux density.

Finding the angle between two vectors in three-dimensional space is often a job we would prefer to avoid, and for that reason the definition of the dot product is usually not used in its basic form. A more helpful result is obtained by considering two vectors whose cartesian components are given, such as A = $\hat{}4_xa_x + y4_ya_y + \hat{}4_za_z$ and B = $B_xa_x + B_ya_y + B_za_z$. The dot product also obeys the distributive law, and, therefore, A • B yields the sum of nine scalar terms, each involving the dot product of two unit vectors. Since the angle between two different unit vectors of the cartesian coordinate system is 90°, we then have

$$^ax \wedge y = {}^ay \wedge {}^ax = {}^ax \wedge {}^az = {}^az \wedge {}^ax = {}^ay \wedge {}^az = {}^az \wedge {}^ay = {}^0$$

The remaining three terms involve the dot product of a unit vector with itself, which is unity, giving finally

$$A \bullet B = ixBx + \hat{}yBy + \hat{}zBz \tag{5}$$

which is an expression involving no angles.
A vector dotted with itself yields the magnitude squared, or

$$A \cdot A = |A|^2 \qquad (6)$$

and any unit vector dotted with itself is unity,

$a \cdot a = 1$

One of the most important applications of the dot product is that of finding the component of a vector in a given direction. Referring to Fig. 1.4a, we can obtain the component (scalar) of B in the direction specified by the unit vector a as

$B \cdot a = |B||a| \cos\theta = |B| \cos\theta$

The sign of the component is positive if $0 < \theta_{Ba} < 90°$ and negative whenever $90° < \theta_{Ba} < 180°$.

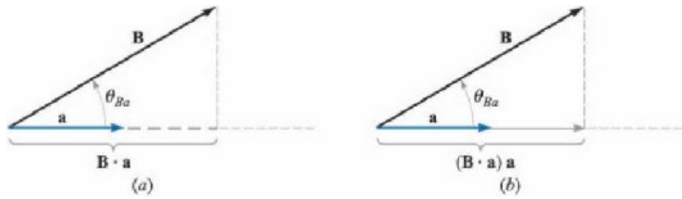


FIGURE 1.4
(a) The scalar component of B in the direction of the unit vector a is B ■ a . (f t) The vector component of B in the direction of the unit vector a is (B ■ a)a.

In order to obtain the component *vector* of B in the direction of a, we simply multiply the component (scalar) by a, as illustrated by Fig. 1.4ft. For example, the component of B in the direction of ax is B • ax = Bx, and the component vector is $5_x a_x$, or $(B \cdot a_x)a_x$. Hence, the problem of finding the component of a vector in any desired direction becomes the problem of finding a unit vector in that direction, and that we can do.
The geometrical term *projection* is also used with the dot product. Thus, B • a is the projection of B in the a direction.

Example 1.2

In order to illustrate these definitions and operations, let us consider the vector field $G = ya_x - 2.5xa_y + 3a_z$ and the point Q(4, 5, 2). We wish to find: G at Q; the scalar component of G at Q in the direction of $a_N = {}^1 (2a_x + a_y - 2a_z)$; the vector component of G at Q in the direction of ; and finally, the angle 6_{Ga} between G(TQ) and .

Solution. Substituting the coordinates of point *Q* into the expression for G, we have

$G(TQ) = 5ax - 10ay + 3a_z$

Next we find the scalar component. Using the dot product, we have

$G \cdot a_w = (5ax - 10ay + 3az) \cdot {}^1 (2ax + ay - 2az) = {}^1 (10 - 10 - 6) = -2$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of ,

$(G \cdot a_w)a_w = -(2)\ 3\ (2a_x + a_y - 2a_z) = -1.333a_x - 0.667a_y + 1.333a_z$

The angle between G(TQ) and a_w is found from

$G \cdot a_w = |G| \cos \theta_{Ga}$

$—2 = V25 + 100 + 9 \cos_G \theta_a$

and

$\theta_{Ga} = \cos^{-1} \text{-}\bar{}L = 99.9^C$

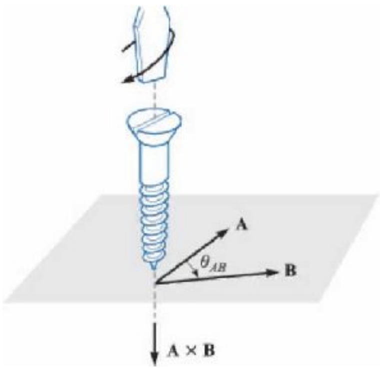
1.7 THE CROSS PRODUCT

Given two vectors A and B, we shall now define the *cross product,or vector product,of* A and B, written with a cross between the two vectors as A x B and read "A cross B." The cross product A x B is a vector; the magnitude of A x B is equal to the product of the magnitudes of A, B, and the sine of the smaller angle between A and B; the direction of A x B is perpendicular to the plane containing A and B and is along that one of the two possible perpendiculars which is in the direction of advance of a right-handed screwas A is turned into B. This direction is illustrated in Fig. 1.5. Remember that either vector may be moved about at will, maintaining its direction constant, until the two vectors have a "common origin." This determines the plane containing both. However, in most of our applications we shall be concerned with vectors defined at the same point. As an equation we can write

$A \times B = |A||B| \sin \theta_{BA}$

where an additional statement, such as that given above, is still required to explain the direction of the unit vector . The subscript stands for "normal."

Reversing the order of the vectors A and B results in a unit vector in the opposite direction, and we see that the cross product is not commutative, for B x A = — (A x B). If the definition of the cross product is applied to the unit



]RE 1.5
lirection of A x B is in the direction of
nce of a right-handed screwas A is turned into B.

vectors a_x and a_y, we find $a_x \times a_y = a_z$, for each vector has unit magnitude, the two vectors are perpendicular, and the rotation of ax into ay indicates the positive z direction by the definition of a right-handed coordinate system. In a similar way $a_y \times a_z = a_x$, and $a_z \times a_x = a_y$. Note the alphabetic symmetry. As long as the three vectors ax, ay, and az are written in order (and assuming that ax follows a_z, like three elephants in a circle holding tails, so that we could also write ay, az, a_x or a_z, a_x, a_y), then the cross and equal sign may be placed in either of the two vacant spaces. As a matter of fact, it is nowsimpler to define a right-handed cartesian coordinate system by saying that ax x ay = az.

A simple example of the use of the cross product may be taken from geometry or trigonometry. To find the area of a parallelogram, the product of the lengths of two adjacent sides is multiplied by the sine of the angle between them. Using vector notation

for the two sides, we then may express the (scalar) area as the *magnitude* of A x B, or |A x B|.
The cross product may be used to replace the right-hand rule familiar to all electrical engineers. Consider the force on a straight conductor of length L, where the direction assigned to L corresponds to the direction of the steady current /, and a uniform magnetic field of flux density B is present. Using vector notation, we may write the result neatly as F = /L x B. The evaluation of a cross product by means of its definition turns out to be more work than the evaluation of the dot product from its definition, for not only must we find the angle between the vectors, but we must find an expression for the unit vector . This work may be avoided by using cartesian components for the two vectors A and B and expanding the cross product as a sum of nine simpler cross products, each involving two unit vectors,

Thus, if $A = 2a_x — 3a_y + a_z$ and $B = —4a_x — 2a_y + 5a_z$, we have $a_x \times a_y = a_z$

A x B = 2 —3 1

—4 —2 5

= [(—3)(5) — (1(—2)]ax — [(2)(5) — 4)]aY, + [(2)(—2) — (—3)(—4)]az = $—13a_x$ — $14a_y$ — $16a_{zz}$

1.8 OTHER COORDINATE SYSTEMS: CIRCULAR CYLINDRICAL COORDINATES

The circular cylindrical coordinate system is the three-dimensional version of the polar coordinates of analytic geometry. In the two-dimensional polar coordinates, a point was located in a plane by giving its distance p from the origin, and the angle 0 between the line from the point to the origin and an arbitrary radial line, taken as 0 = 0.[3] A three-dimensional coordinate system, circular cylindrical coordinates, is obtained by also specifying the distance z of the point from an arbitrary z = 0 reference plane which is perpendicular to the line p = 0. For simplicity, we usually refer to circular cylindrical coordinates simply as cylindrical coordinates. This will not cause any confusion in reading this book, but it is only fair to point out that there are such systems as elliptic cylindrical coordinates, hyperbolic cylindrical coordinates, parabolic cylindrical coordinates, and others.
We no longer set up three axes as in cartesian coordinates, but must instead consider any point as the intersection of three mutually perpendicular surfaces. These surfaces are a circular cylinder (p = constant), a plane (0 = constant), and another plane (z = constant). This corresponds to the location of a point in a cartesian coordinate system by the intersection of three planes (x = constant, y = constant, and z = constant). The three surfaces of circular cylindrical coordinates are shown in Fig. 1.6a. Note that three such surfaces may be passed through any point, unless it lies on the z axis, in which case one plane suffices.

Three unit vectors must also be defined, but we may no longer direct them along the "coordinate axes," for such axes exist only in cartesian coordinates. Instead, we take a broader view of the unit vectors in cartesian coordinates and realize that they are directed toward increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant (i.e., the unit vector ax is normal to the plane x = constant and points toward larger values of x). In a corresponding way we may now define three unit vectors in cylindrical coordinates, a_p, a^, and a_z.

The unit vector a_p at a point $P(p_1, 0_1, z_1)$ is directed radially outward, normal to the

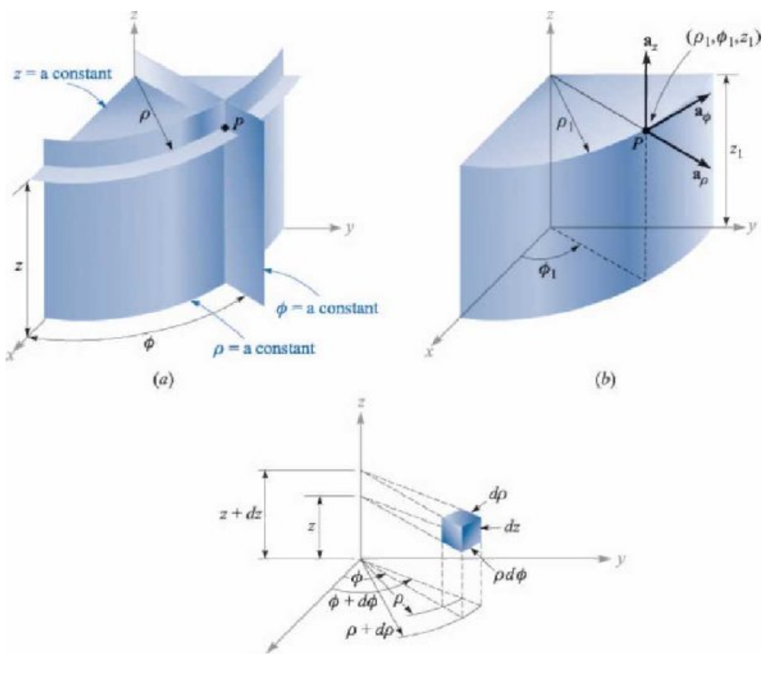


FIGURE 1.6
(aa) The three mutually perpendicular surfaces of the circular cylindrical coordinate system. (b) The three unit vectors of the circular cylindrical coordinate system. (c) The differential volume unit in the circular cylindrical coordinate system; dp, pdc/>, and dz are all elements of length.

cylindrical surface $p = p_1$. It lies in the planes $0 = 0_1$ and $z = z_1$. The unit vector a0 is normal to the plane $0 = 0_1$, points in the direction of increasing 0, lies in the plane $z = z_1$, and is tangent to the cylindrical surface $p = p_1$. The unit vector a_z is the same as the unit vector a_z of the cartesian coordinate system. Fig. 1.6ft shows the three vectors in cylindrical coordinates.

In cartesian coordinates, the unit vectors are not functions of the coordinates. Two of the unit vectors in cylindrical coordinates, a_p and a0, however, do vary with the coordinate 0, since their directions change. In integration or differentiation with respect to 0, then, a_p and a0 must not be treated as constants.

The unit vectors are again mutually perpendicular, for each is normal to one of the three mutually perpendicular surfaces, and we may define a right-handed cylindrical coordinate system as one in which a_p x a0 = a_z, or (for those who have flexible fingers) as one in which the thumb, forefinger, and middle finger point in the direction of increasing p, 0, and z, respectively.

A differential volume element in cylindrical coordinates may be obtained by increasing p, 0, and z by the differential increments dp, $d0$, and dz. The two cylinders of radius p and p + dp, the two radial planes at angles 0 and 0 + d0, and the two "horizontal" planes at "elevations" z and z + dz nowenclose a small volume, as shown in Fig. 1.6c, having the shape of a truncated wedge. As the volume element becomes very small, its shape approaches that of a rectangular parallelepiped having sides of length dp, pd0 and dz. Note that dp and dz are dimensionally lengths, but d0 is not; pd0 is the length. The surfaces have areas of p dp d0, dp dz, and p d0 dz, and the volume becomes p dp d0 dz. The variables of the rectangular and cylindrical coordinate systems are easily related to each other. With reference to Fig. 1.7, we see tha

$$x = p \cos \Phi$$
$$y = p \sin \Phi \qquad (10)$$
$$Z = z$$

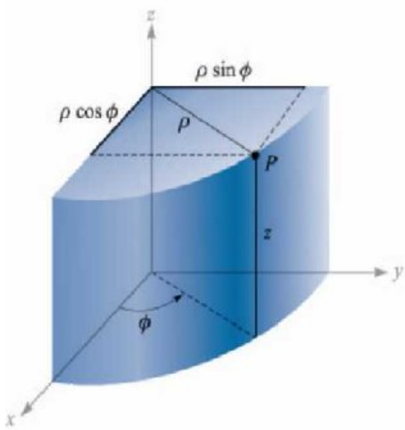


FIGURE 1.7

The relationship between the cartesian variables x, *y*, *z* and the cylindrical coordinate variables p, 0, z. There is no change in the variable *z* between the two systems.

From the other viewpoint, we may express the cylindrical variables in terms of x, y, and z:

$p = \sqrt{x^2 + y^2} \quad (p > 0)$

$\Phi = \tan^{-1} y/x$ (11)

$z = z$

We shall consider the variable p to be positive or zero, thus using only the positive sign for the radical in (11). The proper value of the angle Φ is determined by inspecting the signs of x and y. Thus, if x = —3 and y = 4, we find that the point lies in the second quadrant so that p = 5 and Φ = 126.9°. For x = 3 and y = —4, we have Φ = —53.1° or 306.9°, whichever is more convenient. Using (10) or (11), scalar functions given in one coordinate system are easily transformed into the other system.
A vector function in one coordinate system, however, requires two steps in order to transform it to another coordinate system, because a different set of component vectors is generally required. That is, we may be given a cartesian vector

since az • ap and az • a Φ are zero.
In order to complete the transformation of the components, it is necessary to knowthe dot products $a_x \cdot a_p$, $a_y \cdot a_p$, $a_x \cdot a_\Phi$ and $a_y \cdot a_\Phi$. Applying the definition of the dot product, we see that since we are concerned with unit vectors, the result is merely the cosine of the angle between the two unit vectors in question. Referring to Fig. 1.7 and thinking mightily, we identify the angle between a_x and

TABLE 1.1
Dot products of unit vectors in cylindrical and cartesian coordinate systems

a_P	a_Φ	a_z
cos Φ	— sin Φ	0
sin Φ	cos Φ	0
0	0	1

a_p as p, and thus $a_x \cdot a_p = \cos \Phi$, but the angle between a_y and a_p is 90° — Φ , and $a_y \cdot a_p = \cos (90° - \Phi) = \sin \Phi$. The remaining dot products of the unit vectors are found in a similar manner, and the results are tabulated as functions of Φ in Table 1.1
Transforming vectors from cartesian to cylindrical coordinates or vice versa is therefore accomplished by using (10) or (11) to change variables, and by using the dot products of the unit vectors given in Table 1.1 to change components. The two steps may be taken in either order.

Example 1.3

Transform the vector $B = ya_x - xa_y + za_z$ into cylindrical coordinates.

Solution. The newcomponents are

$B_p = B \cdot a_p = y(a_x \cdot a_p) - x(a_y \cdot a_p)$

$= y \cos \Phi - x \sin \Phi = p \sin \Phi \cos \Phi - p \cos \Phi \sin \Phi = 0$ Bp = B • ap =
y(ax • ap) — x(ay • ap)

$= -y \sin p - x \cos p = -p \sin^2 p - p \cos^2 p = -p$

Thus,

B = — pap + zaz

1.9 THE SPHERICAL COORDINATE SYSTEM

We have no two-dimensional coordinate system to help us understand the three-dimensional spherical coordinate system, as we have for the circular cylindrical coordinate system. In certain respects we can draw on our knowledge of the latitude-and-longitude system of locating a place on the surface of the earth, but usually we consider only points on the surface and not those below or above ground.

Let us start by building a spherical coordinate system on the three cartesian axes (Fig. 1.8a). We first define the distance from the origin to any point as r. The surface r = constant is a sphere.

The second coordinate is an angle θ between the z axis and the line drawn from the origin to the

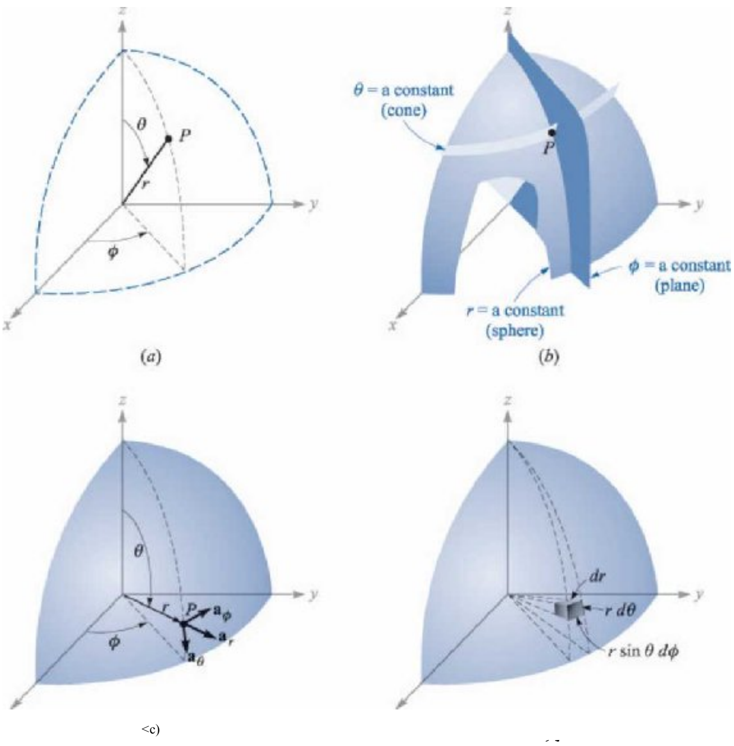


FIGURE 1.8
(a) The three spherical coordinates. (b) The three mutually perpendicular surfaces of the spherical coordinate system. (c) The three unit vectors of spherical coordinates: a_r x a_z = a0. (r f) The differential volume element in the spherical coordinate system.

point in question. The surface θ = constant is a cone, and the two surfaces, cone and sphere, are everywhere perpendicular along their intersection, which is a circle of radius r sin θ. The coordinate θ corresponds to latitude, except that latitude is measured from the equator and θ is measured from the "North Pole."

The third coordinate Φ is also an angle and is exactly the same as the angle Φ of cylindrical coordinates. It is the angle between the x axis and the projection in the z = 0 plane of the line drawn from the origin to the point. It corresponds to the angle of longitude, but the angle Φ increases to the "east." The surface Φ = constant is a plane passing through the Φ = 0 line (or the z axis).

We should again consider any point as the intersection of three mutually perpendicular surfaces—a sphere, a cone, and a plane—each oriented in the manner described above. The three surfaces are shown in Fig. 1.8ft.

Three unit vectors may again be defined at any point. Each unit vector is perpendicular to one ofthe three mutually perpendicular surfaces and oriented in that direction in which the coordinate increases. The unit vector a_r is directed radially outward, normal to the sphere r = constant, and lies in the cone 6 = constant and the plane p = constant. The unit vector a_6 is normal to the conical surface, lies in the plane, and is tangent to the sphere. It is directed along a line of "longitude" and points "south." The third unit vector ap is the same as in cylindrical coordinates, being normal to the plane and tangent to both the cone and sphere. It is directed to the "east."

The three unit vectors are shown in Fig. 1.8c. They are, of course, mutually perpendicular, and a right-handed coordinate system is defined by causing a_r x a_6 = ap. Our system is right-handed, as an inspection of Fig. 1.8c will show, on application of the definition of the cross product. The right-hand rule serves to identify the thumb, forefinger, and middle finger with the direction of increasing r, 6, and p, respectively. (Note that the identification in cylindrical coordinates was with p, p, and z, and in cartesian coordinates with x, y, and z). A differential volume element may be constructed in spherical coordinates by increasing r, 6, and p by dr, d6, and *d*p, as shown in Fig. 1.8*d*. The distance between the two spherical surfaces of radius r and r + dr is dr; the distance between the two cones having generating angles of 6 and 6 + d6 is rd6; and the distance between the two radial planes at angles p and p + dp is found to be r sin 6dp, after a fewmoments of trigonometric thought. The surfaces have areas of rdrd6, r sin 6 dr dp, and r^2 sin 6 d6 dp, and the volume is r^2 sin 6 dr d6 dp.

The transformation of scalars from the cartesian to the spherical coordinate system is easily made by using Fig. 1.8a to relate the two sets of variables:

$$x = r \sin\theta \cos\Phi$$

$$y = r \sin\theta \sin\Phi \qquad (15)$$

TABLE 1.2

Dot products of unit vectors in spherical and cartesian coordinate systems

	a.	a6	a0
a*' a_z '	sin 9 cos 0 sin 9 sin 0 cos9	cos 9 cos 0 cos 9 sin 0 - sin 9	- sin 0 cos0 0

$$z \quad r \cos\theta$$

The transformation in the reverse direction is achieved with the help of

$$r = -Jx^2 + y^2 + z^2 \qquad (r > 0)$$

$$\theta = \cos^{-1} \frac{z}{Vx^2 + y^2 + z^2} = \qquad (0° < \theta < 180°) \qquad (16)$$

$$\Phi = \tan^{-1} y/x$$

The radius variable r is nonnegative, and 9 is restricted to the range from 0° to 180°, inclusive. The angles are placed in the proper quadrants by inspecting the signs of x, y, and z.

The transformation of vectors requires the determination of the products of the unit vectors in cartesian and spherical coordinates. We work out these products from Fig. 1.8c and a pinch of trigonometry. Since the dot product of any spherical unit vector with any cartesian unit vector is the component of the spherical vector in the direction of the cartesian vector, the dot products with a_z are found to be

$a_z \cdot a_r = \cos 9$ $a_z \cdot a_9 = — \sin 9$ $az \cdot a0 = 0$

The dot products involving a_x and a_y require first the projection of the spherical unit vector on the xy plane and then the projection onto the desired axis. For example, $a_r \bullet a_x$ is obtained by projecting a_r onto the xy plane, giving sin 9, and then projecting sin 9 on the x axis, which yields sin 9 cos 0. The other dot products are found in a like manner, and all are shown in Table 1.2.

Problems:

1.1 Given the vectors M = — $10a_x + 4a_y$ — $8a_z$ and N = $8a_x + 7a_y$ — $2a_z$, find: (a) a unit vector in the direction of —M + 2N;(b) the magnitude of $5a_x$ + N — 3M;(c) |M||2N|(M + N).
1.2 Given three points, ,4(4, 3, 2), B(—2, 0, 5), and C(7, —2, 1):(a) specify the vector A extending from the origin to point A;(b) give a unit vector extending from the origin toward the midpoint of line *AB;(c)* calculate the length of the perimeter of triangle *ABC*.
1.3 The vector from the origin to point A is given as $6a_x$ — $2a_y$ — $4a_z$, and the unit vector directed from the origin toward point B is (|, — ^{2},3). If points A and B are 10 units apart, find the coordinates of point B.
1.4 Given points A(8, —5, 4) and B(—2, 3, 2), find: (a) the distance from A to B;(b) a unit vector directed from A towards B;(c) a unit vector directed from the origin toward the midpoint of the line *AB;(d)* the coordinates of the point on the line connecting A to B at which the line intersects the plane z = 3.
1.5 A vector field is specified as G = $24xya_x + 12(x^2 + 2)a_y + 18z^2a_z$. Given two points, P(1, 2, —1) and Q(—2, 1, 3), find: (a) G at *P; (b)* a unit vector in the direction of G at Q;(c) a unit vector directed from Q toward *P; (d)* the equation of the surface on which | G|=60.
1.6 Two vector fields are F = — $10a_x + 20x(y$ — $1)a_y$ and G = $2x^2ya_x$— $4a_y + za_z$. For the point P(2, 3, —4), find: (a) |F|;(b) |G|;(c) a unit vector in the direction of F — G;(<f) a unit vector in the direction of F + G.
1.7 Use the definition of the dot product to find the interior angles at A and B of the triangle defined by the three points: A(1, 3, —2), B(—2, 4, 5), and C(0, —2, 1).
1.8 Given points ,4(10, 11, —6), 5(16, 8, —1), C(8, 1, 4), and D(—1, —5, 8), determine: (a) the vector projection of $R_{AB} + R_{BC}$ on R_{AD};(b) the vector projection of $R_{AB} + R_{BC}$ on R_{DC};(c) the angle between R_{DA} and R_{DC}.
1.9 (a) Find the vector component of F = $10a_x$ — $6a_y + 5a_z$ that is parallel to G = $0.1a_x + 0.1a_y + 0.3a_z$.(b) Find the vector component of F that is perpendicular to G.(c) Find the vector component of G that is perpendicular to F.
1.10 press in cartesian components: (a) the vector at A(p = 4, p = 40°, z = — 1) that extends to B(p = 5, p = —110°, z = 1); (b) a unit vector at B directed toward A;(c) a unit vector at B directed toward the origin.

Chapter Two

COULOMB'S LAW AND ELECTRIC FIELD INTENSITY

2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term "electricity," derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces offluffand stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force."

Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect and in 1600 stated that glass, sulfur, amber, and other materials which he named would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil."

Shortly thereafter a colonel in the French Army Engineers, Colonel Charles Coulomb, a precise and orderly minded officer, performed an elaborate series of experiments using a delicate torsion balance, invented by himself, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result is now known to many high school students and bears a great similarity to Newton's gravitational law (discovered about a hundred years earlier). Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them, Units[4] (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as

$$K = 1 / 4\pi\varepsilon$$

The factor will appear in the denominator of Coulomb's law but will not appear in the more useful equations (including Maxwell's equations) which we shall obtain with the help of Coulomb's law. The new constant e_0 is called the *permittivity of free space* and has the magnitude, measured in farads per meter (F/m),

$$\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \quad \text{F/m} \tag{1}$$

The quantity e_0 is not dimensionless, for Coulomb's law shows that it has the label $C^5/N \bullet m^6$. We shall later define the farad and show that it has the dimensions $C^7/N \bullet m$; we have anticipated this definition by using the unit F/m in (1) above.

Coulomb's law is now

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction. We might equally well have written

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$

Coulomb's law is linear, for if we multiply Q_1 by a factor w, the force on Q_2 is also multiplied by the same factor w. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

$$F\ \underline{Q_1Q_2}\ 4n€_0R^2 \tag{2}$$

Not all SI units are as familiar as the English units we use daily, but they are now standard in electrical engineering and physics. The newton is a unit of force that is equal to 0.2248 lby, and is the force required to give a 1-kilogram (kg) mass an acceleration of1 meter per second per second (m/s^2). The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in mks units as 1.602 x 10^{-19} C; hence a negative charge of one coulomb represents about 6 x 10^{18} electrons.[8] Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9 x 10^9 N, or about one million tons. The electron has a rest mass of 9.109 x 10^{-31} kg and has a radius of the order of magnitude of 3.8 x 10^{-15}m. This does not mean that the electron is spherical in shape, but merely serves to describe the size of the region in which a slowly moving electron has the greatest probability of being found. All other

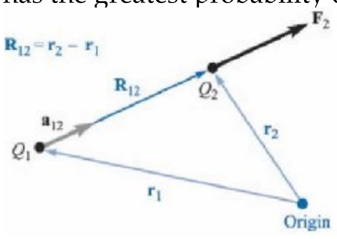


FIGURE 2.1
If Q_1 and Q_2 have like signs, the vector force F_2 on Q_2 is in the same direction as the vector R_{12}.

known charged particles, including the proton, have larger masses, and larger radii, and occupy a probabilistic volume larger than does the electron.

In order to write the vector form of (2), we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges and is repulsive if the charges are alike in sign and attractive if they are of opposite sign. Let the vector **ri** locate Q_1 while $\mathbf{r_2}$ locates Q_2. Then the vector $\mathbf{R_{12}} = \mathbf{r_2} - \mathbf{r_i}$ represents the directed line segment from Q_i to g_2, as shown in Fig. 2.1. The vector $\mathbf{F_2}$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$F_2 = a \text{ ---------- } nT \tag{3}$$

where $\mathbf{a_{12}}$ = a unit vector in the direction of i?$_{12}$,or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{{}^\wedge 12} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

Example 2.1

Let us illustrate the use of the vector form of Coulomb's law by locating a charge of 61 = 3 x 10^{-4} Cat M(1, 2, 3) and a charge of 62 = -10^{-4} Cat JV(2, 0, 5) in a vacuum. We desire the force exerted on 6_2 by *6l*

Solution. We shall make use of (3) and (4) to obtain the vector force. The vector $\mathbf{R_{12}}$ is

$$\mathbf{R}12 = \mathbf{r}2 - \mathbf{r}1 = (2 - 1)\mathbf{a_x} + (0 - 2)\mathbf{a_7} + (5 - 3)\mathbf{a_z} =$$

$a_x - 2\mathbf{a}, + 2\mathbf{a_z}$ leading to $|\mathbf{R_{12}}| = 3$, and the unit vector, $\mathbf{a_{12}} = 3(\mathbf{a_x} - 2\mathbf{a}, + 2\mathbf{a_z})$. Thus, The magnitude of the force is 30 N (or about 7 lby), and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on 62 may also be considered as three component forces,

2.2 ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say Q_1, and move a

(5)

second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field.* Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

- *a* n

4n€oR\t'

Writing this force as a force per unit charge gives

$$\frac{Q_1}{4n€oR\backslash_t} \quad \mathbf{a}_{1r} \quad 11 \tag{6}$$

The quantity on the right side of (6) is a function only of Q_1 and the directed line segment from Q_1 to the position of the test charge. This describes a vector field and is called the *electric field intensity.*

We define the electric field intensity as the vector force on a unit positive test charge. We would not *measure* it experimentally by finding the force on a 1-C test charge, however, for this would probably cause such a force on Q_1 as to change the position of that charge.

Electric field intensity must be measured by the unit newtons per coulomb—the force per unit charge. Again anticipating a new dimensional quantity, the volt (F), to be presented in Chap. 4 and having the label of joules per coulomb (J/C) or newton-meters per coulomb (N-m/C), we shall at once measure electric field intensity in the practical units of volts per meter (V/m). Using a capital letter **E** for electric field intensity, we have finally

$$4jTe_0R^2_u$$

Equation (7) is the defining expression for electric field intensity, and (8) is the expression for the electric field intensity due to a single point charge Q_2 in a vacuum. In the succeeding sections we shall obtain and interpret expressions for the electric field intensity due to more complicated arrangements of charge, but now let us see what information we can obtain from (8), the field of a single point charge.

First, let us dispense with most of the subscripts in (8), reserving the right to use them again any time there is a possibility of misunderstanding:

e q (9)

We should remember that *R* is the magnitude of the vector **R,** the directed line segment from the point at which the point charge *Q* is located to the point at which **E** is desired, and $\mathbf{a}_R$ is a unit vector in the **R** direction.[9]

Let us arbitrarily locate Q_2 at the center of a spherical coordinate system. The unit vector $\mathbf{a}_R$ then becomes the radial unit vector $\mathbf{a}_r$, and R is r. Hence

E = »r (10)

or

*Q*2

4it€o r[1]

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

Writing these expressions in cartesian coordinates for a charge Q at the origin, we have $\mathbf{R} = \mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ and $\mathbf{a}_R = \mathbf{a}_r = (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)/\sqrt{x^2 + y^2 + z^2}$; therefore,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0(x^2+y^2+z^2)}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_x + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_y + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_z\right) \quad (11)$$

This expression no longer shows immediately the simple nature of the field, and its complexity is the price we pay for solving a problem having spherical symmetry in a coordinate system with which we may (temporarily) have more familiarity.

Without using vector analysis, the information contained in (11) would have to be expressed in three equations, one for each component, and in order to obtain the equation we would have to break up the magnitude of the electric field intensity into the three components by finding the projection on each coordinate axis. Using vector notation, this is done automatically when we write the unit vector.

If we consider a charge which is not at the origin of our coordinate system, the field no longer possesses spherical symmetry (nor cylindrical symmetry, unless the charge lies on the z axis), and we might as well use cartesian coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Fig. 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3} = \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \quad (12)$$

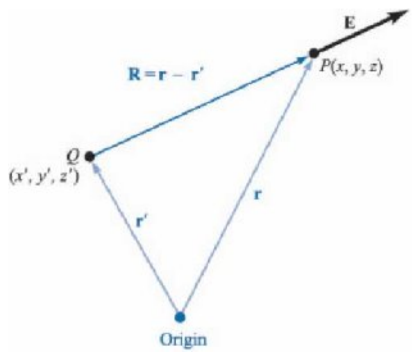


FIGURE 2.2
The vector **r**' locates the point charge Q, the vector **r** identifies the general point in space *P(x, y, z)*, and the vector **R** from Q to *P(x, y, z)* is then **R** = **r** − **r**'.

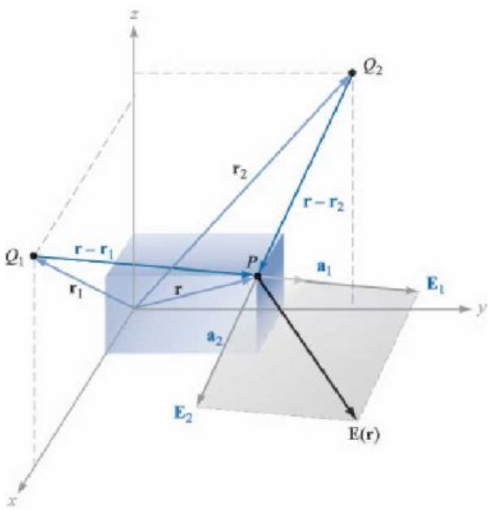


FIGURE 2.3
The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting **E** be symbolized in functional notation by **E(r).**

Equation (11) is merely a special case of (12), where x' = *y*' = *z*' = 0.

Since the coulomb forces are linear, the electric field intensity due to two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

E(r)= Q ____ 47r<? ^____Q2 **r**2|2 32
-------2 »1 H ------
r11² 4^eo|**r**

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in the direction of **(r — r**1**) r**2**)**, respec- and **(r** tively. The vectors **r, r**1; **r**2**, r — r**1**, r — r**2**, a**1**,** and **a**2 are shown in Fig. 2.3

If we add more charges at Q Q other positions, the field due to n point charges is

E(r)= r **a**1 + ■ 4^eo|**r** — **r**11² **a**2 + Qn 47r<?o|**r** — **r**m |² an (13)
47r<?o|**r** — **r**2|² . . . +

This expression takes up less space when we use a summation sign ^ and a summing integer m which takes on all integral values between 1 and n,

m= — am (14)
E(r)= 4^eo|**r** — **r**m |²

When expanded, (14) is identical with (13), and students unfamiliar with summation signs should check that result.

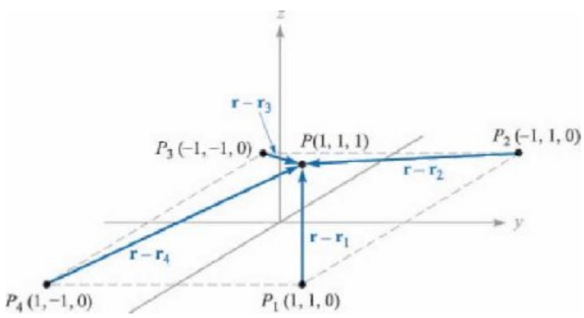


FIGURE 2.4
A symmetrical distribution of four identical 3-nC point charges produces a field at P, **E** $6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

Example 2.2

In order to illustrate the application of (13) or (14), let us find **E** at P(1, 1, 1) caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1,0)$, $P_2(-1, 1,0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Fig. 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| =$ v^, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| =$ \/5- Since g/4jre0 = 3 x 10^{-9}/(4лx 8.854 x 10^{-12}) = 26.96 V • m, we may now use (13) or (14) to obtain

E = 26.96[T h+^ ~+

1 2ax -l- az 1
1 1^2 + sT5 (V5)· $2\mathbf{a}_x$ +
$2\mathbf{a}_y + \mathbf{a}_z$ 1 + $2\mathbf{a}_y + \mathbf{a}_z$ 1
3 3^2 5 ($5)^2$

E

or

$$6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$$

V/m

2.3 FIELD DUE TO A CONTINUOUS VOLUME CHARGE DISTRIBUTION

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, such as the space between the control grid and the cathode in the electron-gun assembly of a cathode-ray tube operating with space charge, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density,* just as we describe water as having a density of 1 g/cm^3 (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. We are able to do this only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.[10]

10

We denote volume charge density by p_v, having the units of coulombs per cubic meter (C/m^3).
The small amount of charge AQ in a small volume Av is

$$AQ = PvAv \qquad (15)$$

and we may define p_v mathematically by using a limiting process on (15),

$$Pv \lim_{Av \to o} \frac{AQ}{Av} \qquad (16)$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$\boxed{Q = \int_{vol} \rho_v \, dv} \qquad (17)$$

Only one integral sign is customarily indicated, but the differential *dv* signifies integration throughout a volume, and hence a triple integration. Fortunately, we may be content for the most part with no more than the indicated integration, for multiple integrals are very difficult to evaluate in all but the most symmetrical problems.

Example 2.3

As an example of the evaluation of a volume integral, we shall find the total charge contained in a 2-cm length of the electron beam shown in Fig. 2.5.

Solution. From the illustration, we see that the charge density is

$p_{„} = -5 \times 10^{-}V^{105} \text{"}^{z}$ C/m^2 The volume differential in cylindrical coordinates is given in Sec. 1.8; therefore,

•0.04 |.2ir f0.01

$$Q = -5 \times 10^{-6} e^{-10^5 pz} p dp d0 dz$$

J 0.02 J 0 J0

We integrate first with respect to 0 since it is so easy,

$$Q \qquad J_0^{0.01} -10^{-5} jr e^{-10^5 pz} p \, d \, p \, dz$$

and then with respect to z, because this will simplify the last integration with respect to p,

$$Q \qquad \int_{0}^{0.01} \frac{-10^{-5}?r}{-10^5 p} \left. e^{-10^5 pz} \right|_{z=0.02}^{z=0.04} p \, dp$$

$$-10^{-5} jr(e^{-2000p} - e^{-4000p}) d \, p$$

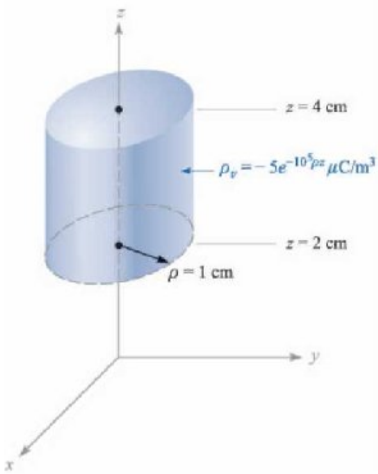


FIGURE 2.5
The total charge contained within the right circular cylinder may be obtained by evaluating

$$Q = \int_{vol} \rho_v \, dv.$$

Finally,

Q = -1o-1ojr g 2ooop g-4ooop\
0 ■ 01

-2ooo - -4oooJo

Q = -1o-1oK 2ooo - 4^) = 4? = o'o785pC

where pC indicates picocoulombs.

Incidentally, we may use this result to make a rough estimate of the electron-beam current. If we assume these electrons are moving at a constant velocity of 1o percent of the velocity of light, this 2-cm-long packet will have moved 2 cm in |ns, and the current is about equal to the product,

AQ _ -(jr/4Q)1Q-12 "AT = (2/3)1o-9

or approximately 118 |j,A.

The incremental contribution to the electric field intensity at **r** produced by an incremental charge AQ at **r**' is

AQ **r** - **r**' pvAv **r** - **r**'
A**E(r)** =
4jreo|**r** - **r**'|2 |**r** - **r**'| 4jreo|**r** - **r**'|2 |**r** - **r**'|

If we sum the contributions of all the volume charge in a given region and let the volume element Av approach zero as the number of these elements becomes infinite, the summation becomes an integral,

f pv(r')dv' r - r' n8.

This is again a triple integral, and (except in the drill problem that follows) we shall do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (18) might stand a little review. The vector **r** from the origin locates the field point where **E** is being determined, while the vector **r**' extends from the origin to the source point where pv(**r**')dv' is located. The scalar distance between the source point and the field point is |**r** - **r**'|, and the fraction (**r** - **r**')/|**r** - **r**'| is a unit vector directed from source point to field point. The variables of integration are x', y', and z' in cartesian coordinates.

2.4 FIELD OF A LINE CHARGE

Up to this point we have considered two types of charge distribution, the point charge and charge distributed throughout a volume with a density pv C/m3.Ifwe now consider a filamentlike distribution of volume charge density, such as a very fine, sharp beam in a cathode-ray tube or a charged conductor of very small radius, we find it convenient to treat the charge as a line charge of density Pi C/m. In the case of the electron beam the charges are in motion and it is true that we do not have an electrostatic problem. However, if the electron motion is steady and uniform (a dc beam) and if we ignore for the moment the magnetic field which is produced, the electron beam may be considered as composed of stationary electrons, for snapshots taken at any time will show the same charge distribution.

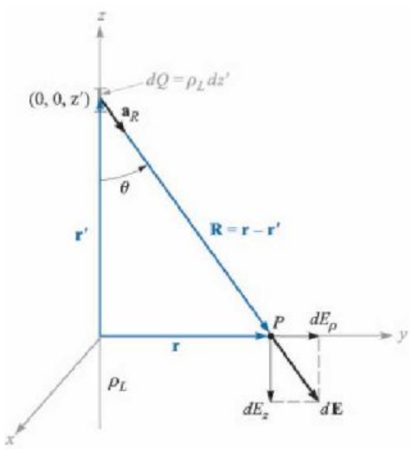
Let us assume a straight line charge extending along the z axis in a cylindrical coordinate system from —oo to oo, as shown in Fig. 2.6. We desire the electric field intensity E at any and every point resulting from a *uniform* line charge density p^.

Symmetry should always be considered first in order to determine two specific factors: (1) with which coordinates the field does *not* vary, and (2) which components of the field are not present. The answers to these questions then tell us which components are present and with which coordinates they *do* vary.

Referring to Fig. 2.6, we realize that as we move around the line charge, varying 0 while keeping p and z constant, the line charge appears the same from every angle. In other words, azimuthal symmetry is present, and no field component may vary with 0.

Again, if we maintain p and 0 constant while moving up and down the line charge by changing z, the line charge still recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields which are not functions of z.

?E 2 6
ntribution d**E** = di;$_p$**a**$_p$ + d£$_z$**a**$_z$ to
ctric field intensity produced by
nent of charge dQ = p^dz'
l a distance z' from the origin.
iear charge density is uniform
tends along the entire z axis



If we maintain <p and z constant and vary p, the problem changes, and Coulomb's law leads us to expect the field to become weaker as p increases. Hence, by a process of elimination we are led to the fact that the field varies only with p.

Now, which components are present? Each incremental length of line charge acts as a point charge and produces an incremental contribution to the electric field intensity which is directed away from the bit of charge (assuming a positive line charge). No element of charge produces a < component of electric intensity; *E*< is zero. However, each element does produce an E_p and an E_z component, but the contribution to E_z by elements of charge which are equal distances above and below the point at which we are determining the field will cancel.

We therefore have found that we have only an E_p component and it varies only with p. Now to find this component.

We choose a point P(o, y, 0) on the *y* axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with < and z. Applying (12) to find the incremental field at P due to the incremental charge dQ = $p_L dz$we have

d **E** =

where

r = y**a**y = p**a**p **r**' = z '$\mathbf{a}_z$

and

$$\mathbf{r} - \mathbf{r}' = p\mathbf{a}p - z'\mathbf{a}z$$

Therefore,

$$d\mathbf{E} \quad \frac{p_L dz'(p\mathbf{a}p - z'\mathbf{a}z)}{47reo(p^2 + z'^2)^{3/2}}$$

Since only the **Ep** component is present, we may simplify:

? + *z* '2)3/2

˛47reo(p2 + z'

pLp*dz*'

Integrating by integral tables or change of variable, z' = p cot 6, we have

E PL / 1 *z*< ___________________________ \

—oc

and

pi 2^60 p (19)

This is the desired answer, but there are many other ways of obtaining it. We might have used the angle 6 as our variable of integration, for z' = p cot 6 from Fig. 2.6 and dz' = — p csc^{2}6 d6. Since R = p csc 6, our integral becomes, simply,

p$_L$dz' 47T€0 p sin 6
47T<?$_0$R^2

£p=- pi pi sin 6 d6 pi cos 6
pi sin 47T€0
6d6 p

27T€0 p

Here the integration was simpler, but some experience with problems of this type is necessary before we can unerringly choose the simplest variable of integration at the beginning of the problem.

We might also have considered (18) as our starting point,

E p^du '**(r — r')** .vol
47re0 **|r — r'|**^{3}

letting p„ du' = p$_i$ dz' and integrating along the line which is now our "volume" containing all the charge. Suppose we do this and forget everything we have learned from the symmetry of the problem. Choose point P now at a general location (p, 0, z) (Fig. 2.7) and write

r = p**a**p + z**a**z **r**' = z '**a**z

R = r — r ' = p**a**p +(z z ')**a**z

aR

p^2 +(z — z

p**a**p + (z — z'))**a**z

p^2 + (z — z ')^2

E pidz '[p**a**p + (z — z >z]
47T60[p^2 + (z — z ')^2]$^{3/2}$

pi p dz '**a**p (z — z') dz '**a**z
47160 [p^2 + (z — z ')^2])[p^2 + (z — z ')^2]

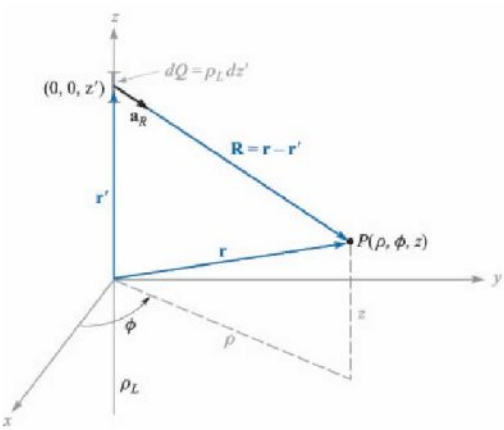


FIGURE 2.7
The geometry of the problem for the field about an infinite line charge leads to more difficult integrations when symmetry is ignored.

Before integrating a vector expression, we must know whether or not a vector under the integral sign (here the unit vectors $\mathbf{a}_\rho$ and $\mathbf{a}_z$) varies with the variable of integration (here dz'). If it does not, then it is a constant and may be removed from within the integral, leaving a scalar which may be integrated by normal methods. Our unit vectors, of course, cannot change in magnitude, but a change in direction is just as troublesome. Fortunately, the direction of $\mathbf{a}_\rho$ does not change with z' (nor with ρ, but it does change with φ), and $\mathbf{a}_z$ is constant always.

Hence we remove the unit vectors from the integrals and again integrate with tables or by changing variables,

$$\mathbf{E} = \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\rho\, dz'}{[\rho^2 + (z - z')^2]^{3/2}}\mathbf{a}_\rho - \frac{\rho_L}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{(z - z')\, dz'}{[\rho^2 + (z - z')^2]^{3/2}}\mathbf{a}_z$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\rho\,\mathbf{a}_\rho\left[\frac{1}{\rho^2}\frac{-(z - z')}{\sqrt{\rho^2 + (z - z')^2}}\right]_{-\infty}^{\infty} - \frac{\rho_L}{4\pi\epsilon_0}\mathbf{a}_z\left[\frac{1}{\sqrt{\rho^2 + (z - z')^2}}\right]_{-\infty}^{\infty}$$

$$= \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$$

Again we obtain the same answer, as we should, for there is nothing wrong with the method except that the integration was harder and there were two integrations to perform. This is the price we pay for neglecting the consideration of symmetry and plunging doggedly ahead with mathematics. Look before you integrate.

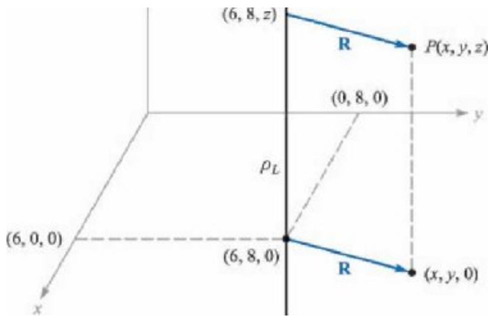
Other methods for solving this basic problem will be discussed later after we introduce Gauss's law and the concept of potential.

Now let us consider the answer itself,

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho \qquad (20)$$

We note that the field falls off inversely with the distance to the charged line, as compared with the point charge, where the field decreased with the *square* of the distance. Moving ten times as far from a point charge leads to a field only 1 percent the previous strength, but moving ten times as far from a line charge only reduces the field to 10 percent of its former value. An analogy can be drawn with a source of illumination, for the light intensity from a point source of light also falls off inversely as the square of the distance to the source. The field of an infinitely long fluorescent tube thus decays inversely as the first power of the radial distance to the tube, and we should expect the light intensity about a finite-length tube to obey this law near the tube. As our point recedes farther and farther from a finite-length tube, however, it eventually looks like a point source and the field obeys the inverse-square relationship.

Before leaving this introductory look at the field of the infinite line charge, we should recognize the fact that not all line charges are located along the z axis. As an example, let us consider an infinite line charge parallel to the z axis at x = 6, y = 8, Fig. 2.8. We wish to find **E** at the general field point P(x, y, z).



RE 2.8
it P(x, y, z) is identified
n infinite uniform line
located at x = 6, y = 8.

We replace p in (20) by the radial distance between the line charge and point, P, $R = J(x - 6)^2 + (y - 8)^2$, and let $\mathbf{a_p}$ be $\mathbf{a_R}$. Thus,

E PL

2neoJ (x — 6)2 + (y — 8)2

where

- = (x — 6K + (y – 8)*y 3R |R| Mx
— 6)2 + (y — 8)2

Therefore,

(x — 6)ax + (y — 8)ay
2 2
2^eo (x — 6)2 + (y – 8)2

We again note that the field is not a function of z.

In Sec. 2.6 we shall describe how fields may be sketched and use the field of the line charge as one example.

2.5 FIELD OF A SHEET OF CHARGE

Another basic charge configuration is the infinite sheet of charge having a uniform density of *ps* C/m2. Such a charge distribution may often be used to approximate that found on the conductors of a strip transmission line or a parallel-plate capacitor. As we shall see in Chap. 5, static charge resides on conductor surfaces and not in their interiors; for this reason, p_s is commonly known as *surface charge density.* The charge-distribution family now is complete—point, line, surface, and volume, or Q, p_L, p_s, and p_v.

Let us place a sheet of charge in the yz plane and again consider symmetry (Fig. 2.9). We see first that the field cannot vary with y or with z, and then that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we wish the field will cancel. Hence only E_x is present, and this component is a function of x alone. We are again faced with a choice of many methods by which to evauate this component, and this time we shall use but one method and leave the others as exercises for a quiet Sunday afternoon.

Let us use the field of the infinite line charge (19) by dividing the infinite sheet into differential-width strips. One such strip is shown in Fig. 2.9. The line charge density, or charge per unit length, is $p_L = p_s dy$and the

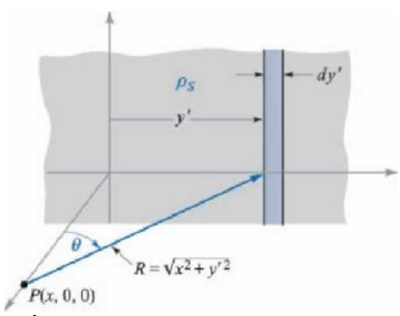


FIGURE 2.9
An infinite sheet of charge in the yz plane, a general point P on the x axis, and the differential-width line charge used as the element in determining the field at P by dE = Psdy 'a_R/(2jrE0R).

distance from
X

this line charge to our general point P on the x axis is R = $x^2 + y'^2$. The contribution to E_x at P from this differential-width strip is then

dEx Psdy' : cos 9 Ps xdy' 27reo
$x^2 + y'^2$

Adding the effects of all the strips,

Ex Ps '°° xdy' Ps 2eo
2T-T6G9J+ y '2

------------- tan 1
—
27reo x

If the point P were chosen on the negative x axis, then

Ex = -Ps-

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector , which is normal to the sheet and directed outward, or away from it. Then

Ps
E = — a_N (21)
2<?o

This is a startling answer, for the field is constant in magnitude and direction. It is just as strong a million miles away from the sheet as it is right off the surface. Returning to our light analogy, we see that a uniform source of light on the ceiling of a very large room leads to just as much illumination on a square foot on the floor as it does on a square foot a few inches below the ceiling. If you desire greater illumination on this subject, it will do you no good to hold the book closer to such a light source.

If a second infinite sheet of charge, having a *negative* charge density — p_s ,is located in the plane x = *a, we* may find the total field by adding the contribution of each sheet. In the region x > a,

E+= 7ps **a**x **E—** = — —pps **a**x **E** = **E+**+ **E—** = 0

and for x < 0,

E+ = — —pps *ax* *E* = ^ **a**x **E** = **E+**+ **E—** = 0

and when 0 < x < a,

Ps Ps
E+=— **a**x **E—** = — **a**x
2<?o
2
eo

and

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \rho_s \, \mathbf{a}_x \qquad (22)$$

This is an important practical answer, for it is the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation and provided also that we are considering a point well removed from the edges. The field outside the capacitor, while not zero, as we found for the ideal case above, is usually negligible.

2.6 STREAMLINES AND SKETCHES OF FIELDS

We now have vector equations for the electric field intensity resulting from several different charge configurations, and we have had little difficulty in interpreting the magnitude and direction of the field from the equations. Unfortunately, this simplicity cannot last much longer, for we have solved most of the simple cases and our new charge distributions must lead to more complicated expressions for the fields and more difficulty in visualizing the fields through the equations. However, it is true that one picture would be worth about a thousand words, if we just knew what picture to draw. Consider the field about the line charge,

Fig. 2.10a shows a cross-sectional view of the line charge and presents what might be our first effort at picturing the field—short line segments drawn here and there having lengths proportional to the magnitude of **E** and pointing in the direction of **E.** The figure fails to show the symmetry with respect to p, so we try again in Fig. 2.10ft with a symmetrical location of the line segments. The real trouble now appears—the longest lines must be drawn in the most crowded region, and this also plagues us if we use line segments of equal length but of a thickness which is proportional to **E** (Fig. 2.10c). Other schemes which have been suggested include drawing shorter lines to represent stronger fields (inherently misleading) and using intensity of color to represent stronger fields (difficult and expensive).

For the present, then, let us be content to show only the *direction* of **E** by drawing continuous lines from the charge which are everywhere tangent to **E.** Fig. 2.10d shows this compromise. A symmetrical distribution of lines (one every 45°) indicates azimuthal symmetry, and arrowheads should be used to show direction.

These lines are usually called *streamlines,* although other terms such as flux lines and direction lines are also used. A small positive test charge placed at any point in

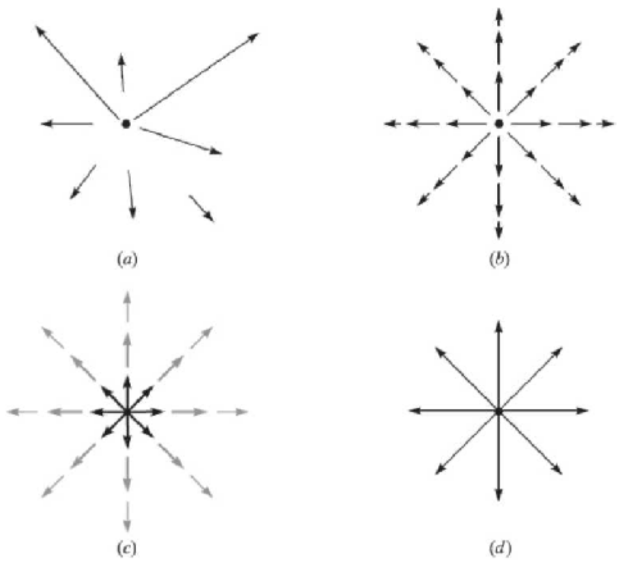


FIGURE 2.10 ^ | y | e-Text Main Menu | Textbook Table of Contents

(a) One very poor sketch, (b) and (c) two fair sketches, and (d) the usual form of streamline sketch. In the last form, the arrows show the direction of the field at every point along the line, and the spacing of the lines is inversely proportional to the strength of the field.

this field and free to move would accelerate in the direction of the streamline passing through that point. If the field represented the velocity of a liquid or a gas (which, incidentally, would have to have a source at p = o), small suspended particles in the liquid or gas would trace out the streamlines.

We shall find out later that a bonus accompanies this streamline sketch, for the magnitude of the field can be shown to be inversely proportional to the spacing of the streamlines for some important special cases. The closer they are together, the stronger is the field. At that time we shall also find an easier, more accurate method of making that type of streamline sketch.

If we attempted to sketch the field of the point charge, the variation of the field into and away from the page would cause essentially insurmountable difficulties; for this reason sketching is usually limited to two-dimensional fields.

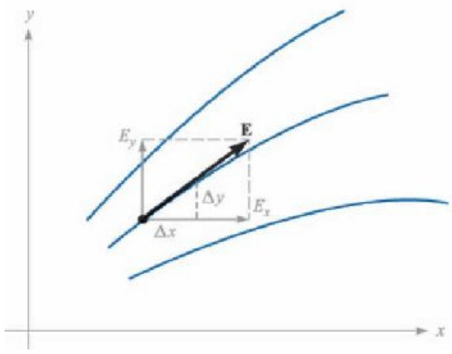
In the case of the two-dimensional field let us arbitrarily set E_z = o. The streamlines are thus confined to planes for which z is constant, and the sketch is the same for any such plane. Several streamlines are shown in Fig. 2.11, and the E_x and E_y components are indicated at a general point. Since it is apparent from the geometry that

$$\boxed{\quad \_\ \frac{dy}{dx}} \qquad (23)$$

a knowledge of the functional form of Ex and Ey (and the ability to solve the resultant differential equation) will enable us to obtain the equations of the streamlines.

As an illustration of this method, consider the field of the uniform line charge with $pi = 2ne_o$,

$$\mathbf{E} = \frac{}{\rho}\mathbf{a}_\rho$$



RE 2.11
uation of a streamline is ed by solving the differential on E_y/E_x =

In cartesian coordinates,

$$\mathbf{E} = \frac{x}{x^2 - y^2}\mathbf{a}_x + \frac{y}{x^2 + y^2}$$

Thus we form the differential equation

$$\frac{dy}{dx} = \frac{E_y}{} = \frac{y}{x} \quad \text{or} \quad \frac{dy}{y} \quad dx$$

Therefore,

$$\ln y = \ln x + C_1 \quad \text{or} \quad \ln y = \ln x + \ln C$$

from which the equations of the streamlines are obtained,

$$y \quad Cx$$

If we want to find the equation of one particular streamline, say that one passing through P(−2, 7, 10), we merely substitute the coordinates of that point into our equation and evaluate C. Here, 7 = C(−2), and C = —3.5, so that y =— 3.5x.

Each streamline is associated with a specific value of C, and the radial lines shown in Fig. 2.10d are obtained when C = 0, 1, −1, and 1/C = 0.

The equations of streamlines may also be obtained directly in cylindrical or spherical coordinates. A spherical coordinate example will be examined in Sec. 4.7.

PROBLEMS

2.1 A charge Q_1 = o.1 uC is located at the origin in free space, while Q_2 = o.2 uCisat 4(o.8, —o.6, o). Find the locus of points in the z = o plane at which the x-component of the force on a third positive charge is zero.
2.2 Point charges of 5onC each are located at ,4(1, o, o), B(−1, o, o), C(o, 1, o), and £)(o, — 1, o) in free space. Find the total force on the charge at 4.
2.3 Let Q_1 = 8 uC be located at P_1 (2, 5, 8) while Q_2 = −5 u C i s a t P_2(6, 15, 8). Let e = e_o. (a) Find $\mathbf{F_2}$, the force on Q_2. (b) Find the coordinates of P_3 if a charge Q_3 experiences a total force $\mathbf{F_3}$ = oat P_3.
2.4 Let a point charge Q_1 = 25 nC be located at P_1(4, −2, 7) and a charge Q2 = 6onC be at P2(−3, 4, −2). (a) If e = eo, find **E** at P(1, 2, 3). (b) At what point on the y axis is E_x = o?
2.5 Point charges of 12onC are located at 4(o, o, 1) and 5(o, o, −1) in free space. (a) Find **E** at P(o.5, o, o). (b) What single charge at the origin would provide the identical field strength?
2.6 Given point charges of −1 uCat P_1(o, o, o.5) and P_2(o, o, —o.5), and a charge of 2 uC at the origin, find **E** at P(o, 2, 1) in spherical components. Assume e = e_o.
2.7 A 1oo-nC point charge is located at 4(− 1, 1, 3) in free space. (a) Find the locus of all points P(x, y, z) at which E_x = 5ooV/m. (b) Find y_1 if P(−2, y1, 3) lies on that locus.

2.10 Charges of 2o and −2o nC are located at (3, o, o) and (−3, o, o), respectively. Let e = e_o. (a) Determine |**E**| at P(o, y, o). (b) Sketch |**E**| vs y at P.
2.11 A charge Qo, located at the origin in free space, produces a field for which E_z = 1kV/m at point P(−2, 1, — 1). (a) Find Q_o. Find **E** at M(1, 6, 5) in: (b) cartesian coordinates; (c) cylindrical coordinates; (d) spherical coordinates.
2.12 The volume charge density p„ = $p_o e^{-|x|-|y|-|z|}$ exists over all free space. Calculate the total charge present.
2.13 A uniform volume charge density of o.2 uC/m^2 is present throughout the spherical shell extending from r = 3 cm to r = 5 cm. If p„ = o elsewhere, find: (a)

the total charge present within the shell, and (b) r_1 if half the total charge is located in the region 3 cm < r < r_1.

2.15 A spherical volume having a 2-urn radius contains a uniform volume charge density of 10^{15}C/m^3. (a) What total charge is enclosed in the spherical volume? (b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3 mm on a side, and that there is no charge between the spheres. What is the average volume charge density throughout this large region?

2.16 A uniform line charge of 16nC/m is located along the line defined by y = − 2, z = 5. If e = e_0: (a) find **E** at P(1, 2, 3); (b) find **E** at that point in the z = 0 plane where the direction of **E** is given by

2.17 Uniform line charges of 0.4 uC/m and −0.4 uC/m are located in the x = 0 plane at y = − 0.6 and y = 0.6m, respectively. Let e = e_0. Find **E** at: (a) P(x, 0, z); (b) g(2, 3, 4).

2.18 A uniform line charge of 2 uC/m is located on the z axis. Find **E** in cartesian coordinates at P(1 , 2, 3) if the charge extends from: (a) z = − oo to z = oo; (b) z = − 4to z = 4.

2.19 Uniform line charges of 120 nC/m lie along the entire extent of the three coordinate axes. Assuming free space conditions, find **E** at P(−3, 2, − 1).

2.20 Two identical uniform line charges, with p_E = 75 nC/m, are located in free space at x = 0, y = ±0.4m. What force per unit length does each line charge exert on the other?

2.21 A uniform surface charge density of 5nC/m^2 is present in the region x = 0, −2 < y < 2, all z. I f e = e_0, find **E** at: (a) (3, 0, 0); (b) P«(0, 3, 0).

2.22 Given the surface charge density, = 2 uC/m^2 in the region p < 0.2 m, z = 0, and is zero elsewhere, find **E** at: (a) P^(p = 0, z = 0.5); (b) P_B(p = 0, z = − 0.5).

2.23 Surface charge density is positioned in free space as follows: 20 nC/m^2 at x = −3, −30nC/m^2 at y = 4, and 40nC/m^2 at z = 2. Find the magnitude of **E** at: (a) P^(4, 3, −2); (b) P_{fi}(−2, 5, −1); (c) PC(0, 0, 0).

2.24 Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC, at P(2, 0, 6); uniform line charge density, 3 nC/m, at x = −2, y = 3; uniform surface charge density, 0.2nC/m^2,at x = 2.

2.25 A uniform line charge density of 5 nC/m is at y = 0, z = 2 m in free space, while −5 nC/m is located at y = 0, z =−2m. A uniform surface charge density of 0.3 nC/m^2 is at y = 0.2m, and −0.3 nC/m^2 is at y = − 0.2m. Find |**E**| at the origin.

2.26 Given the electric field **E** = (4x − 2y)$\mathbf{a}_x$ − (2x + 4y)$\mathbf{a}_y$, find: (a) the equation of that streamline passing through the point P(2, 3, −4); (b) a unit vector $\mathbf{a}_E$ specifying the direction of **E** at g(3, −2, 5).

2.27 Let **E** = $5x^3$**a**$_x$ — $15x^2y$**a**$_y$, and find: (a) the equation of the streamline that passes through P(4, 2, 1); (b) a unit vector **a**$_E$ specifying the direction of **E** at Q(3, −2, 5); (c) a unit vector = ((, m, o) that is perpendicular to **a**$_E$ at Q.

2.28 If **E** = 2oe^{-5y}(cos 5x**a**$_x$ — sin5x**a**$_y$), find: (a)|**E**| at P(TT/6, o.1, 2); (b) a unit vector in the direction of **E** at P; (c) the equation of the direction line passing through P.

2.29 Given the electric field intensity, **E** = 4ooy**a**$_x$ + 4oox**a**$_y$ V/m, find: (a) the equation of the streamline passing through point 4(2, 1, −2); (b) the equation of the surface on which |**E**|=8oo V. (c) Sketch the streamline of part a. (d) Sketch the trace produced by the intersection of the z = o plane and the surface of part b.

2.30 In cylindrical coordinates with **E**(p, 0) = **E**$_p$(p, 0)**a**$_p$ + E^(p, 0)**a**0, the differential equation describing the direction lines is E$_p$/E^ = dp/(p d0) in any z = constant plane. Derive the equation of the line passing through point P(p = 4, 0 = 1o°, z = 2) in the field **E** = 2p^2 cos 30**a**$_p$+ 2p^2 sin 30**a**0.

Chapter Three

ELECTRIC FLUX DENSITY, GAUSS'S LAW, AND DIVERGENCE

3.1 ELECTRIC FLUX DENSITY

About 1837 the Director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been botheringhim duringthe past ten years when he was experimentingin his now famous work on induced electromotive force, which we shall discuss in Chap. 10. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulatingmaterial (or dielectric material, or simply dielectric) which would occupy the entire volume between the concentric spheres. We shall not make immediate use of his findings about dielectric materials, for we are restrict-ingour attention to fields in free space until Chap. 5. At that time we shall see that the materials he used will be classified as ideal dielectrics.
His experiment, then, consisted essentially of the followingsteps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, usingtools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as displacement, displacement flux, or simply electric flux.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leadingto a direct proportionality between the electric flux and the charge on the inner sphere. The constant of proportionality is dependent on the system of units involved, and we are fortunate in our use of SI units, because the constant is unity. If electric flux is denoted by * (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$* = Q$$

and the electric flux * is measured in coulombs.

We can obtain more quantitative information by consideringan inner sphere of radius a and an outer sphere of radius b, with charges of Q and — Q, respectively (Fig. 3.1). The paths of electric flux * extendingfrom the inner sphere to the outer sphere are indicated by the symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of the inner sphere, * coulombs of electric flux are produced by the charge Q(= *) coulombs distributed uniformly over a surface having an area of 47 ra^2m^2. The density of the flux at this surface is */47 ra^2 or Q/4na² C/m^2, and this is an important new quantity.

Electric flux density, measured in coulombs per square meter (sometimes described as "lines per square meter," for each line is due to one coulomb), is given the letter **D,** which was originally chosen because of the alternate names of displacement flux density or displacement density. Electric flux density is more descriptive, however, and we shall use the term consistently.

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric field intensity **E.** The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossinga surface normal to the lines divided by the surface area.

Referring again to Fig. 3.1, the electric flux density is in the radial direction and has a value of

$$\mathbf{D} \quad 47\ ra^2 \quad \mathbf{a_r} \quad \text{(inner sphere)}$$

$$\mathbf{D} \quad \mathbf{r=b} \quad \frac{Q}{4*b^2}\mathbf{a_r} \quad \text{(outer sphere)}$$

and at a radial distance r, where a < r < b,

$$4nr^2\ r$$

If we now let the inner sphere become smaller and smaller, while still retaininga charge of Q, it becomes a point charge in the limit, but the electric flux density at a point r meters from the point charge is still given by

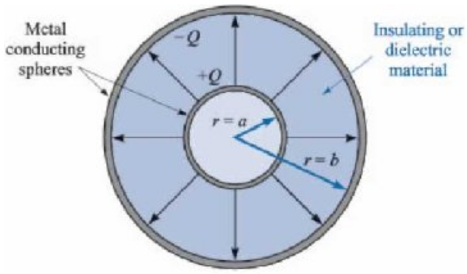


FIGURE 3.1
The electric flux in the region between a pair of charged concentric spheres. The direction and magnitude of **D** are not functions of the dielectric between the spheres.

D Q 4itr^2

for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area 47 rr^2.

This result should be compared with Sec. 2.2, Eq. (10), the radial electric field intensity of a point charge in free space,

In free space, therefore,

(free space only) (2)

Although (2) is applicable only to a vacuum, it is not restricted solely to the field of a point charge. For a general volume charge distribution in free space

E J vol p$_v$dv 4it€$_0$R^2 (free space only) (3)

where this relationship was developed from the field ofa single point charge. In a similar manner, (1) leads to

D vol p$_v$ dv 4*ttR*^{2} (4)

and (2) is therefore true for any free-space charge configuration; we shall consider (2) as defining **D** in free space.

As a preparation for the study of dielectrics later, it might be well to point out now that, for a point charge embedded in an infinite ideal dielectric medium, Faraday's results show that (1) is still applicable, and thus so is (4). Equation (3) is not applicable, however, and so the relationship between **D** and **E** will be slightly more complicated than (2).

Since **D** is directly proportional to **E** in free space, it does not seem that it should really be necessary to introduce a new symbol. We do so for several reasons. First, **D** is associated with the flux concept, which is an important new idea. Second, the **D** fields we obtain will be a little simpler than the corresponding **E** fields, since e$_0$ does not appear. And, finally, it helps to become a little familiar with **D** before it is applied to dielectric materials in Chap. 5.

Let us consider a simple numerical example to illustrate these new quantities and units.

Example 3.1

(1)

We wish to find D in the region about a uniform line charge of 8 nC/m lying along the z axis in free space.

Solution. The E field is

pL _ 8 x 1(t[9] _143.8
E " 2^p A" " 2i(8.854 x 10—12)p = ~/m

At p = 3 m, E = 47.9AP V/m.
Associated with the E field, we find
PL 8 x IP'9 1.273 x IP"9 2
2itp D = -- Ap = ------------Ap = ----------- Ap C/m
2itp p
The value at p = 3 m is D = 0.424AP nC/m.
The total flux leaving a 5-m length of the line charge is equal to the total charge on that length, or * = 40 nC.

\/ D3.1. Given a point charge located at the origin, find the total electric flux passingthrough: *(a)* that portion of the sphere r = 26 cm bounded by 0 < *6* 2 and
U < 0 < 2; (b) the closed surface defined by p = 26 cm and z = ±26 cm; (c) the plane
z 26 cm.

Ans. *7.5* |j,C; 60 |j,C; 30 |j,C

\/ D3.2. Calculate D in rectangular coordinates at point P(2, —3, 6) produced by: *(a)* a point charge Q_a = 55 mC at Q(—2, 3, —6); (b) a uniform line charge *pLs* = 20mC/m on the *x* axis; (c) a uniform surface charge density *psc* = 120 |j.C/m[2] on the plane z = — 5 m.

ANS.6.38AX — 9.57A7 + 19.14Az ^C/m[2]; —212A7 + 424Az nC/m[2];60Az ^C/m[2]

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by statingthat the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, since one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same. Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but +Q coulombs on any inner conductor would produce an induced charge of — Q coulombs on the surroundingsphere. Goingone step further, we could now replace the two outer hemispheres by an empty (but completely closed) soup can. Q coulombs on the brass door key would produce ^ = Q lines of electric flux and would induce — Q coulombs on the tin can.[1]
These generalizations of Faraday's experiment lead to the following statement, which is known as Gauss's law:

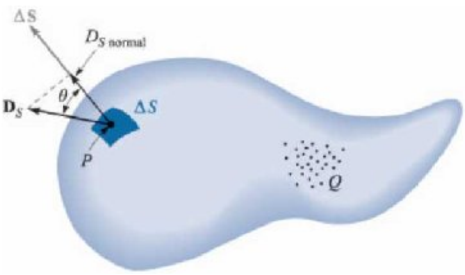
The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in statingthe law as we have above, but in providinga mathematical form for this statement, which we shall now obtain.
Let us imagine a distribution of charge, shown as a cloud of point charges in Fig. 3.2, surrounded by a closed surface of any shape. The closed surface may be the surface of some

(1)

real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector D will have some value D_S, where the subscript S merely reminds us that D must be evaluated at the surface, and D_S will in general vary in magnitude and direction from one point on the surface to another.

We must now consider the nature of an incremental element of the surface. An incremental element of area AS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude AS but also of its orientation in space. In other words, the incremental surface element is a vector quantity. The only unique direction which may be associated with AS is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two



RE 3.2
LECTRIC FLUX DENSITY D_S DUE TO CHARGE Q. THE FLUX PASSINGTHROUGH
D_S • AS

such normals, and the ambiguity is removed by specifying the outward normal whenever the

WERE A PERFECT INSULATOR, THE SOUP COULD EVEN BE LEFT IN THE CAN

surface is closed and "outward" has a specific meaning.

At any point P consider an incremental element of surface AS and let $\mathbf{D_S}$ make an angle 9 with AS, as shown in Fig. 3.2. The flux crossing A**S** is then the product of the normal component of $\mathbf{D_S}$ and A**S,**

$$A^* = \text{flux crossing } A\mathbf{S} = D_{Sj,,orm}A = D_S S \cos 9\ AS = \mathbf{D_S} \cdot A\mathbf{S}$$

where we are able to apply the definition of the dot product developed in Chap. 1.

The total flux passingthrough the closed surface is obtained by addingthe differential contributions crossingeach surface element A**S,**

i'
$$* = \quad d^* = \rho \quad \mathbf{D_S} \cdot d\mathbf{S}$$
closed surface

The resultant integral is a closed surface integral, and since the surface element d**S** always involves the differentials of two coordinates, such as dx dy, p d<p dp , o r r^2 sin 9 d9 d<p, the integral is a double integral. Usually only one integral sign is used for brevity, and we shall always place an S below the integral sign to indicate a surface integral, although this is not actually necessary since the differential d**S** is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a closed surface. Such a surface is often called a gaussian surface. We then have the mathematical formulation of Gauss's law,

$$* \quad c\!: \quad \mathbf{D_S} \cdot d\mathbf{S} = \text{charge enclosed} = Q \qquad (5)$$
S

The charge enclosed might be several point charges, in which case

Q = SQ«

or a line charge,

$$Q = \int_L \rho_L dL$$

or a surface charge,

$$Q = \int_S \rho_S dS \qquad \text{(not necessarily a closed surface)}$$

or a volume charge distribution,

$$Q = \int_{vol} \rho_v dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understandingGauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D}_s \cdot d\mathbf{S} = \int_{vol} \rho_v \, dv \qquad (6)$$

a mathematical statement meaningsimply that the total electric flux through any closed surface is equal to the charge enclosed.

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placinga point charge Q at the origin of a spherical coordinate system (Fig. 3.3) and by choosing our closed surface as a sphere of radius a. The electric field intensity of the point charge has been found to be

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \mathbf{a}_r$$

and since $\mathbf{D} = \epsilon_0 \mathbf{E}$

$$\mathbf{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r$$

we have, as before,

$$\frac{Q}{4\pi a^2}$$

At the surface of the sphere,

$$\mathbf{D}_s \qquad \mathbf{a}_r$$

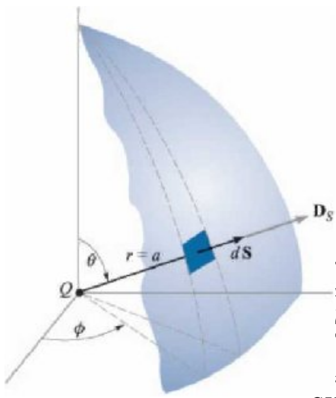


FIGURE 3.3
APPLICATION OF GAUSS'S LAW TO THE FIELD OF A POINT CHARGE Q ON A SPHERICAL CLOSED SURFACE OF RADIUS a. THE ELECTRIC FLUX DENSITY D IS EVERYWHERE NORMAL TO THE SPHERICAL SURFACE AND HAS A CONSTANT MAGNITUDE AT EVERY POINT ON IT.

The differential element of area on a spherical surface is, in spherical coordinates from Chap. 1,

dS = r2 sin 9 d9 dp = a2 sin 9 d9 dp dS = a2 sin 9 d9 dp

The integrand is

$\mathbf{D_S} \cdot d\mathbf{S} =$ Q a2 sin 9 d9 dp$\mathbf{a_r} \cdot \mathbf{a_r}$ = — sin 9 d9 dp

47ra2 41

leadingto the closed surface integral

J

|-9=jr

sin 9 d9 d(

J 0=0 J 9=0

where the limits on the integrals have been chosen so that the integration is carried over the entire surface of the sphere once.[2] Integrating gives

— (- cos 9)"d 0 f2jr —

0 41 - d 0 = Q

Jo 2TT

and we obtain a result showingthat Q coulombs of electric flux are crossingthe surface, as we should since the enclosed charge is —coulombs.

The followingsection contains examples of the application of Gauss's law to problems of a simple symmetrical geometry with the object of finding the elelctric field intensity.

3.3 APPLICATION OF GAUSS'S LAW: SOME SYMMETRICAL CHARGE DISTRIBUTIONS

Let us now consider how we may use Gauss's law,

$$Q = \quad D_s \cdot dS$$

to determine $\mathbf{D_s}$ if the charge distribution is known. This is an example of an integral equation in which the unknown quantity to be determined appears inside the integral.
The solution is easy if we are able to choose a closed surface which satisfies two conditions:

1. **D**s is everywhere either normal or tangential to the closed surface, so that **D**s • d**S** becomes either D_s dS or zero, respectively.
2. On that portion of the closed surface for which $\mathbf{D_s}$ • dS is not zero, D_s = constant.

This allows us to replace the dot product with the product of the scalars D_s and dS and then to bring D_s outside the integral sign. The remaining integral is then $\int_s$ dS over that portion of the closed surface which $\mathbf{D_s}$ crosses normally, and this is simply the area of this section of that surface.
Only a knowledge of the symmetry of the problem enables us to choose such a closed surface, and this knowledge is obtained easily by remembering that the electric field intensity due to a positive point charge is directed radially outward from the point charge.
Let us again consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements listed above. The surface in question is obviously a spherical surface, centered at the origin and of any radius r. $\mathbf{D_s}$ is everywhere normal to the surface; D_s has the same value at all points on the surface.
Then we have, in order,

$$Q \qquad o\ \mathbf{D_s} \cdot \mathbf{dS} = o \qquad \Gamma$$

$$D_s \qquad {}_{sph}\ o\ dS = D_s \ {}_{J\,0=0} \qquad {}_{9=0} \quad r^2 \sin 9\ d\ 9\ d\ 0$$

and hence

$$_{sph}\ dS = D_s \quad J\ 0=0$$

$$4jrr^2 D_s$$

$$\frac{Q}{47rr^2}$$

Since r may have any value and since $\mathbf{D_s}$ is directed radially outward,

$$\mathbf{D} \quad \frac{Q}{4itr^2} \qquad \mathbf{E} \quad \underline{Q}$$

which agrees with the results of Chap. 2. The example is a trivial one, and the objection could be raised that we had to know that the field was symmetrical and directed radially outward before we could obtain an answer. This is true, and that leaves the inverse-square-law relationship as the only check obtained from Gauss's law. The example does, however, serve to illustrate a method which we may apply to other problems, includingseveral to which Coulomb's law is almost incapable of supplyingan answer.
Are there any other surfaces which would have satisfied our two conditions? The student should determine that such simple surfaces as a cube or a cylinder do not meet the requirements.

As a second example, let us reconsider the uniform line charge distribution ρ_L lyingalong the z axis and extendingfrom —oo to +00. We must first obtain a knowledge of the symmetry of the field, and we may consider this knowledge complete when the answers to these two questions are known:

1. With which coodinates does the field vary (or of what variables is D a function)?

2. Which components of **D** are present?

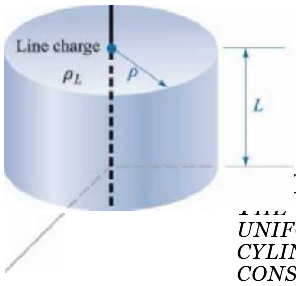
These same questions were asked when we used Coulomb's law to solve this problem in Sec. 2.5. We found then that the knowledge obtained from answering them enabled us to make a much simpler integration. The problem could have been (and was) worked without any consideration of symmetry, but it was more difficult.
In usingGauss's law, however, it is not a question of usingsymmetry to simplify the solution, for the application of Gauss's law depends on symmetry, and if we cannot show that symmetry exists then we cannot use Gauss's law to obtain a solution. The two questions above now become "musts."
From our previous discussion of the uniform line charge, it is evident that only the radial component of D is present, or

D =

and this component is a function of p only.

Dp = f(p)

The choice of a closed surface is now simple, for a cylindrical surface is the only surface to which D_p is everywhere normal and it may be closed by plane surfaces normal to the z axis. A closed right circular cylindrical of radius p extendingfrom z = O t o z = L is shown in Fig. 3.4.
We apply Gauss's law,



JRE 3.4
GAUSSIAN SURFACE FOR AN INFINITE UNIFORM LINE CHARGE IS A RIGHT CIRCULAR CYLINDER OF LENGTH L AND RADIUS P. D IS CONSTANT IN MAGNITUDE AND EVERYWHERE PERPENDICULAR TO THE CYLINDRICAL SURFACE; D IS PARALLEL TO THE END FACES.

f Q = o Ds ■ d**S** = Ds dS + f 0 f dS + 0
S J cyl J sides J top J bottom d

p d<p dz = Ds 2itpL
0=0

and obtain Ds = D p =
2itpL

In terms of the charge density pL, the total charge enclosed is

Q =
pLL

pL
2itp

giving

2lt€o p

D

p

or

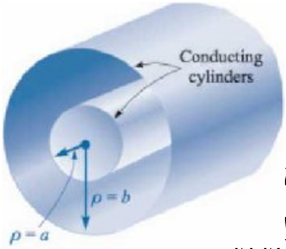
E

p

Comparison with Sec. 2.4, Eq. (20), shows that the correct result has been obtained and with much less work. Once the appropriate surface has been chosen, the integration usually amounts only to writing down the area of the surface at which **D** is normal.

The problem of a coaxial cable is almost identical with that of the line charge and is an example which is extremely difficult to solve from the standpoint of Coulomb's law. Suppose that we have two coaxial cylindrical conductors, the inner of radius a and the outer of radius b, each infinite in extent (Fig. 3.5). We shall assume a charge distribution of p_s on the outer surface of the inner conductor.

Symmetry considerations show us that only the Dp component is present and that it can be a function only of p. A right circular cylinder of length L and



URE 3.5
TWO COAXIAL CYLINDRICAL CONDUCTORS
MINGA COAXIAL CABLE PROVIDE AN
ELECTRIC FLUX DENSITY WITHIN THE
CYLINDERS, GIVEN BY $D_p = aps/p$.

radius p, where a < p < b, is necessarily chosen as the gaussian surface, and we quickly have

Q = D_s liipL

The total charge on a length L of the inner conductor is

t'2n

Q $p_s$adcp dz = 2itaLp_s

z=0 J 0=0

from which we have

a p s p
Ds

aps

D =

----------------- a

p

p

(a < p < b)

This result might be expressed in terms of charge per unit length, because the inner conductor has 2nap$_s$ coulombs on a meter length, and hence, letting pL = 2naps,

$$\boxed{\mathbf{D} = \frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho}$$

and the solution has a form identical with that of the infinite line charge.

Since every line of electric flux startingfrom the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

Qouter cyl = —2naLps .inner cyl and the surface charge on the outer cylinder is found as

2ltbLps,outer cyl = — 2ltaLps,inner cyl

a

or ps. outer cyl b ps .inner cyl

What would happen if we should use a cylinder of radius p, p > b, for the gaussian surface? The total charge enclosed would then be zero, for there are equal and opposite charges on each conducting cylinder. Hence

0 = D_s2itpL (p > b)

D_s = 0 (p > b)

An identical result would be obtained for p < a. Thus the coaxial cable or capacitor has no external field (we have proved that the outer conductor is a "shield"), and there is no field within the center conductor.

Our result is also useful for a finite length of coaxial cable, open at both ends, provided the length L is many times greater than the radius b so that the unsymmetrical conditions at the two ends do not appreciably affect the solution. Such a device is also termed a coaxial capacitor. Both the coaxial cable and the coaxial capacitor will appear frequently in the work that follows.

Perhaps a numerical example can illuminate some of these results.

Example 3.2

Let us select a 50-cm length of coaxial cable having an inner radius of 1 mm and an outer radius of 4 mm. The space between conductors is assumed to be filled with air. The total charge on the inner conductor is 30 nC. We wish to know the charge density on each conductor, and the E and D fields.

Solution. We begin by finding the surface charge density on the inner cylinder,

Sinner cyl 30 x 10 9 2

Ps,innercyl =^aOLL = ~~2^(10-3)(0.5)~~ = 9,55 ^C/m The negative charge density on the inner surface of the outer cylinder is

gouter cyl -30 x 10-9 2

ps,outer cyl = = 2^(4 x 10-3)(0.5) = -2.39 MC/m

The internal fields may therefore be calculated easily:

Dp = aPs = 10-3(9.55 x 10-6) = 955 nC/m2

P P P

Dp 9.55 x 10^{-9} 1079

and Ep = ^ = ~~8.854 x 10-12p~~ = — V/m

Both of these expressions apply to the region where 1 < p < 4 mm. For p < 1mm or p > 4 mm, E and D are zero.

3.4 APPLICATION OF GAUSS'S LAW: DIFFERENTIAL VOLUME ELEMENT

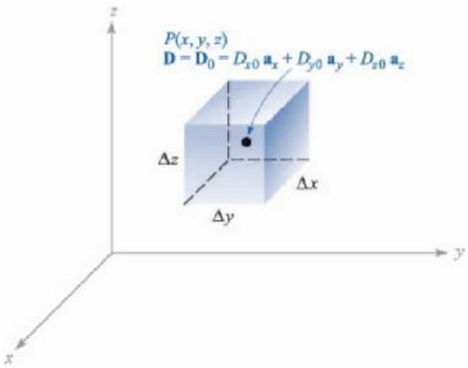
We are now going to apply the methods of Guass's law to a slightly different type of problem—one which does not possess any symmetry at all. At first glance it might seem that our case is hopeless, for without symmetry a simple gaussian surface cannot be chosen such that the normal component of **D** is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties, and that is to choose such a very small closed surface that **D** is almost constant over the surface, and the small change in **D** may be adequately represented by usingthe first two terms of the Taylor's-series expansion for **D.** The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

This example also differs from the precedingones in that we shall not obtain the value of **D** as our answer, but instead receive some extremely valuable information about the way **D** varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.

Let us consider any point P, shown in Fig. 3.6, located by a cartesian coordinate system. The value of **D** at the point P may be expressed in cartesian components, $\mathbf{D_0} = D_{x0}a_x + D_{y0}a_y + D_{z0}\mathbf{a_z}$. We choose as our closed surface the small rectangular box, centered at P, havingsides of lengths Ax, Ay, and Az, and apply Gauss's law,

D • d **S** = Q
s

RE 2.6
'ERENTIAL-SIZED GAUSSIAN
CE ABOUT THE POINT *P* IS
TO INVESTIGATE THE SPACE
OF CHANGE OF *D* IN THE
BORHOOD OF P



In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

r r i' i' i' i' o**D** • d**S** = + + + + +
JS *J* front *J* back *J* left *J* right *J* top *J*
bottom

Consider the first of these in detail. Since the surface element is very small, **D** is essentially constant (over this portion of the entire closed surface) and

= Dfront " ASfront
front

= **D**front • Ay Az ax

= Ax,front Ay Az

where we have only to approximate the value of D_x at this front face. The front face is at a distance of Ax/2 from P, and hence

Ax

Dx,front= Dxo + — x rate of change of Dx with x

Ax dD_x
= ^{D}x0 + -7r- -7— 2 dx

where Dxo is the value of Dx at P, and where a partial derivative must be used to express the rate of change of D_x with x, since D_x in general also varies with y and z. This expression could have been obtained more formally by usingthe constant term and the term involvingthe first derivative in the Taylor's-series expansion for D_x in the neighborhood of P. We have now

[(Ax dD_x\
= Dxo + — Ay Az
front 2 dx

Consider now the integral over the back surface,

[== **D**back • A**S**back
back

Dback • (-Ay Az **a**x) - Dx, back Ay Az

Ax dDx
and Dx,back = Dxo . .
2 dx

[(
giving = -Dxo + Ax dD_x -t—) Ay Az
back 2 dx

If we combine these two integrals, we have

J front *J* back 3x *J* b + = -^j-Ax Ay Az

By exactly the same process
we find that

* * 3D
\+ = Ax Ay Az
J right *J* le
J right J left 3y

f *f* -D_z
and + = ---- Ax Ay Az
* top * bottom 3z

and these results may be collected to yield

f /3D_x 3D_y 3D_z\
o **D** • d**S** = + + — 1) Ax Ay Az
s -x -y -z

or

*I*D •dS = Q = +f+1) Au (7)

The expression is an approximation which becomes better as Au becomes smaller, and in the followingsection we shall let the volume Au approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element Au and have as a result the approximation (7) statingthat

$$\text{Charge enclosed in volume } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \text{volum}$$

Example 3.3

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $D = e^{-x} \sin y\ a_x - e^{-x} \cos y\ a_y + 2z a_z$ C/m^2.

Solution. We first evaluate the three partial derivatives in (8):

-*Dx* —*x* ■
---------- = — *e* ^{x} sin *y*
3*x*

—— = *e* ^{x} sin *y*

3*z*

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately 2Au.If Au is $10^{-9}$$m^3$, then we have enclosed about 2nC.

3.5 DIVERGENCE

We shall now obtain an exact relationship from (7), by allowingthe volume element Av to shrink to zero. We write this equation as

dDx dDy dDz\ . §s **D** • d**S** Q

dx 3y dz) Av Av

or, as a limit

\ dx dy dz) Av^o Av Av^o Av

where the approximation has been replaced by an equality. It is evident that the last term is the volume charge density pv, and hence that

^ + ^ + ^ = Aimo = P,

(9)

dx dy dz / Av^o Av

This equation contains too much information to discuss all at once, and we shall write it as two separate equations,

lim ^---------------------- (10)

Av^0 Av

and *+ 1?* + ^

where we shall save (11) for consideration in the next section.

Equation (10) does not involve charge density, and the methods of the previous section could have been used on any vector **A** to find |s **A** • d**S** for a small closed surface, leadingto

^ + ^ + ^ = lim *i*s**A_** dS

(12

)

dx dy dz / Av^0 Av

where **A** could represent velocity, temperature gradient, force, or any other vector field.

This operation appeared so many times in physical investigations in the last century that it received a descriptive name, divergence. The divergence of **A** is defined as

$$\text{Divergence of A} = \text{div A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \qquad (13)$$

and is usually abbreviated div A. The physical interpretation ofthe divergence of a vector is obtained by describingcarefully the operations implied by the right-hand side of (13), where we shall consider A as a member of the flux-density family of vectors in order to aid the physical interpretation.

The divergence of the vector flux density A is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

The physical interpretation of divergence afforded by this statement is often useful in obtainingqualitative information about the divergence of a vector field without resortingto a mathematical investigation. For instance, let us consider the divergence of the velocity of water in a bathtub after the drain has been opened. The net outflow of water through any closed surface lyingentirely within the water must be zero, for water is essentially incompressible and the water enteringand leavingdifferent regions of the closed surface must be equal. Hence the divergence of this velocity is zero.
If, however, we consider the velocity of the air in a tire which has just been punctured by a nail, we realize that the air is expandingas the pressure drops, and that consequently there is a net outflow from any closed surface lyingwithin the tire. The divergence of this velocity is therefore greater than zero.
A positive divergence for any vector quantity indicates a source of that vector quantity at that point. Similarly, a negative divergence indicates a sink. Since the divergence of the water velocity above is zero, no source or sink exists.[11] The expandingair, however, produces a positive divergence of the velocity, and each interior point may be considered a source.
Writing(10) with our new term, we have

[11] HAVINGCHOSEN A DIFFERENTIAL ELEMENT OF VOLUME WITHIN THE WATER, THE GRADUAL DECREASE IN WATER LEVEL WITH TIME WILL EVENTUALLY CAUSE THE VOLUME ELEMENT TO LIE ABOVE THE SURFACE OF THE WATER. AT THE INSTANT THE SURFACE OF THE WATER INTERSECTS THE VOLUME ELEMENT, THE DIVERGENCE IS POSITIVE AND THE SMALL VOLUME IS A SOURCE. THIS COMPLICATION IS AVOIDED ABOVE BY SPECIFYINGAN INTEGRAL POINT.

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \tag{14}$$

This expression is again of a form which does not involve the charge density. It is the result of applyingthe definition of divergence (13) to a differential volume element in cartesian coordinates.

If a differential volume unit p dp d<p dz in cylindrical coordinates, or r^2 sin 6 drd6 d<p in spherical coordinates, had been chosen, expressions for diver

gence involving the components of the vector in the particular coordinate system and involvingpartial derivatives with respect to the variables of that system would have been obtained. These expressions are obtained in Appendix A and are given here for convenience:

$$\text{div}\,\mathbf{D} = \frac{\mathbf{b}d_x}{dx}\ 1\ \frac{dD_y}{dy}\ H\ \frac{\mathbf{b}d_z}{dz} \qquad \text{(cartesian)} \qquad (15)$$

$$\text{div}\,\mathbf{D} \quad \frac{1}{P}\frac{d}{dp}(pD_P)+\text{-}_P \qquad p\,dcf) \quad dz \qquad \text{(cylindrical)} \qquad (16)$$

$$\boxed{\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \qquad \text{(spherical)}} \qquad (17)$$

These relationships are also shown inside the back cover for easy reference.

It should be noted that the divergence is an operation which is performed on a vector, but that the result is a scalar. We should recall that, in a somewhat similar way, the dot, or scalar, product was a multiplication of two vectors which yielded a scalar product.

For some reason it is a common mistake on meetingdivergence for the first time to impart a vector quality to the operation by scatteringunit vectors around in the partial derivatives. Divergence merely tells us how much flux is leavinga small volume on a per-unit-volume basis; no direction is associated with it.

We can illustrate the concept of divergence by continuing with the example at the end of the previous section.

Example 3.4

Find div D at the origin if $D = e^{x} \sin y\, A_X — e^{x} \cos y\, A_Y + 2zA_Z$.

Solution. We use (14) or (15) to obtain

$$\text{div D}|z \quad dD_y \quad dD_z\, dx$$

$= —e^{—x} \sin y + e^{—x} \sin y + 2 = 2$

The value is the constant 2, regardless of location.

If the units of **D** are C/m^2, then the units of div **D** are C/m^3. This is a volume charge density, a concept discussed in the next section.

3.6 MAXWELL'S FIRST EQUATION (ELECTROSTATICS)

We now wish to consolidate the gains of the last two sections and to provide an interpretation of the divergence operation as it relates to electric flux density. The expressions developed there may be written as

<L **D** • d **S**
div **D** = lim ^ ----------------- (18)
Ai>^0 Av

div **D** = aDx , dDy dDz ---- + - (19) (20)

and

9y

div **D**

The first equation is the definition of divergence, the second is the result of applyingthe definition to a differential volume element in cartesian coordinates, giving us an equation by which the divergence of a vector expressed in cartesian coordinates may be evaluated, and the third is merely (11) written usingthe new term div **D.** Equation (20) is almost an obvious result if we have achieved any familiarity at all with the concept of divergence as defined by (18), for given Gauss's law,

{ **A** • d**S** = Q

s

per unit volume

§s A ^ d S
A „.
Av

Av

As the volume shrinks to zero,

lim ^ --= lim —
Av^0 Av Av^0 Av

we should see div **D** on the left and volume charge density on the right,

$$\text{div } \mathbf{D} = p_v \qquad (20)$$

This is the first of Maxwell's four equations as they apply to electrostatics and steady magnetic fields, and it states that the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there. This equation is aptly called the point form of Gauss's law. Gauss's law relates the flux leavingany closed surface to the charge enclosed, and Maxwell's first equation makes an identical statement on a per-unit-volume basis for a vanishingly small volume, or at a point. Remembering that the divergence may be expressed as the sum of three partial derviatves, Maxwell's first equation is also described as the differential-equation form of Gauss's law, and conversely, Gauss's law is recognized as the integral form of Maxwell's first equation.

As a specific illustration, let us consider the divergence of **D** in the region about a point charge Q located at the origin. We have the field

$47rr^{2\ r}$

and make use of (17), the expression for divergence in spherical coordinates given in the previous section:

div **D** = i d (r^2D_r) + -^- d- (D, sin -) + -±- ^
r^2 dr r sin - d- r sin - d(p

Since D_e and are zero, we have

1 *a*Q
div D=r* srl^j=0 (if r = 0)

Thus, p_v = 0 everywhere except at the origin where it is infinite.
The divergence operation is not limited to electric flux density; it can be applied to any vector field. We shall apply it to several other electromagnetic fields in the comingchapters.

3.7 THE VECTOR OPERATOR V AND THE DIVERGENCE THEOREM

If we remind ourselves again that divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find somethingwhich may be dotted formally with **D** to yield the scalar

$$\frac{dD_x}{dx} + \frac{dDy}{dy} + \frac{dD_z}{dz}$$

Obviously, this cannot be accomplished by usinga dot product; the process must be a dot operation.

With this in mind, we define the del operator V as a vector operator,

$$V = \frac{V}{dx}\mathbf{a}x + \frac{V}{dy}\mathbf{a}y + \frac{V}{dz}\mathbf{a}z \qquad (21)$$

Similar scalar operators appear in several methods of solvingdifferential equations where we often let D replace d/dx, D^2 replace d^2/dx^2, and so forth.[4] We agree on defining V (pronounced "del") that it shall be treated in every way as an ordinary vector with the one important exception that partial derivatives result instead of products of scalars. Consider V • **D,** signifying

$$V \cdot \mathbf{D} = \left(\frac{d}{gx}\,\mathbf{a}x + \frac{d}{dy}\,\mathbf{a}y + \frac{d}{dz}\right) \cdot (Dx\mathbf{a}x + Dy\mathbf{a}y + Dz\mathbf{a}z)$$

We first consider the dot products of the unit vectors, discardingthe six zero terms and havingleft

$$V \cdot \mathbf{D} = \frac{}{dx}(Dx) + \frac{}{dy}(Dy) + \frac{}{dz}(Dz)$$

where the parentheses are now removed by operatingor differentiating:

$$V \cdot \mathbf{D} = \frac{dDx}{dx} + \frac{dDy}{dy} + \frac{dDz}{dz}$$

This is recognized as the divergence of **D,** so that we have

$$\text{div } \mathbf{D} = V \cdot \mathbf{D} = \frac{dDx}{dx} + \frac{dDy}{dy} + \frac{dDz}{dz}$$

The use of V • **D** is much more prevalent than that of div **D,** although both usages have their advantages. Writing V • **D** allows us to obtain simply and quickly the correct partial derivatives, but only in cartesian coordinates, as we shall see below. On the other hand, div **D** is an excellent reminder of the physical interpretation of divergence. We shall use the operator notation V • **D** from now on to indicate the divergence operation.

The vector operator V is used not only with divergence, but willl appear in several other very important operations later. One of these is Vu, where u is any scalar field, and leads to

THIS SCALAR OPERATOR D, *WHICH WILL NOT APPEAR AGAIN, IS NOT TO BE CONFUSED WITH THE ELECTRIC FLUX DENSITY.*

/ 3 3 3 \ du du du

\3x 3y 3z J Vu = — a_x + — a_y + — a_z u = — a_x + — a_y + — a_z

dx 3y dz

The V operator does not have a specific form in other coordinate systems. If we are considering D in cylindrical coordinates, then V • D still indicates the divergence of D, or

^ 1 3 x 1 3D.* 3£>z

\> P

p dp P 30 dz

where this expression has been taken from Sec. 3.5. We have no form for V itself to help us obtain this sum of partial derivatives. This means that Vu, as yet unnamed but easily written above in cartesian coordinates, cannot be expressed by us at this time in cylindrical coordinates. Such an expression will be obtained when Vu is defined in Chap. 4.

We shall close our discussion of divergence by presenting a theorem which will be needed several times in later chapters, the divergence theorem. This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easiest for us to develop it for the electric flux density. We have actually obtained it already and now have little more to do than point it out and name it, for startingfrom Gauss's law,

■

o D • dS = Q

is

and letting

Q = $p_v$dv

J vol

and then replacing p_v by its equal,

V • D = pv

c
we have o D • dS = Q = pvdv = c V • D dv
s vol vol

The first and last expressions constitute the divergence theorem,

(22) lS = *f* V • D dv

which may be stated as follows:

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

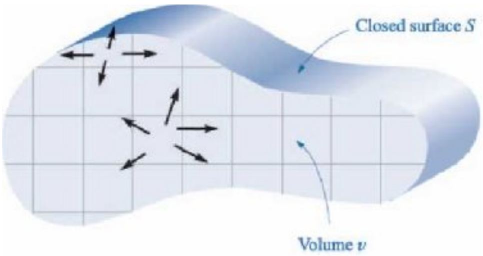


FIGURE 3.7
The divergence theorem states that the total flux crossingthe closed surface is equal to the integral of the divergence of the flux density throughout the enclosed volume. The volume is shown here in cross section.

Again, we emphasize that the divergence theorem is true for any vector field, although we have obtained it specifically for the electric flux density **D,** and we shall have occasion later to apply it to several different fields. Its benefits derive from the fact that it relates a triple integration throughout some volume to a double integration over the surface of that volume. For example, it is much easier to look for leaks in a bottle full of some agitated liquid by an inspection of the surface than by calculatingthe velocity at every internal point.

The divergence theorem becomes obvious physically if we consider a volume v, shown in cross section in Fig. 3.7, which is surrounded by a closed surface S. Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell enters,or converges on, the adjacent cells unless the cell contains a portion of the outer surface. In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossingthe enclosingsurface.

Let us consider an example to illustrate the divergence theorem.

Example 3.5

Evaluate both sides of the divergence theorem for the field $D = 2xyA_X + x^2a_y$ C/m^2 and the rectangular parellelepiped formed by the planes x = 0 and 1, y = 0 and 2, and z = 0 and 3.

Solution. Evaluatingthe surface integral first, we note that D is parallel to the surfaces at $z = 0$ and $z = 3$, so D • DS = 0 there. For the remainingfour surfaces we have

$$\oint_S D\, dS - (D)_{x=0} \cdot (-dydz\, a_x) + (D)_{x=1} \cdot (dydz\, a_x)$$

$$(D)_{Y=0} \cdot (-dx\, dz\, A_y) + (D)_{Y=2} \cdot (dx\, dz\, A_y)$$

$$(AO\hat{}o \; dydz + (D_X)_{X=x} dydz$$

$$(Dy)_y = {}_0 dx \; dz + (D_y)_{y=1} dx \; dz$$

However, $(D_x)_{x=o} = 0$, and $(D_y)_{y=o} = (D_y)_{y=2}$, which leaves only

$$D \, d S \qquad (D_x)_{x=l} dydz : \qquad 2ydy \; dz$$

$$4 \, dz = 12$$

Since

the volume integral becomes

$$v \cdot D = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

$$\int V\text{-}D \, dv. \quad \int_3 \int_2 \int 2ydxdydz = \int_3 \int 2ydy \; dz$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

PROBLEMS

3.1 An empty metal paint can is placed on a marble table, the lid is removed, and both parts are discharged (honorably) by touching them to ground. An insulatingnylon thread is glued to the center of the lid, and a penny, a nickel, and a dime are glued to the thread so that they are not touching each other. The penny is given a charge of +5nC, and the nickel and dime are discharged. The assembly is lowered into the can so that the coins hangclear of all walls, and the lid is secured. The outside of the can is again touched momentarily to ground. The device is carefully disassembled with insulatinggloves and tools. (a) What charges are found on each of the five metallic pieces? (b) If the penny had been given a charge of +5 nC, the dime a charge of —2 nC, and the nickel a charge of — 1 nC, what would the final charge arrangement have been?

3.2 A point charge of 12nC is located at the origin. Four uniform line charges are located in the x = 0 plane as follows: 80 nC/m at y = —1 and —5m, — 50nC/m at y = —2 and —4m. (a) Find **D** at P(0, —3, 2). (b) How much electric flux crosses the plane y = —3, and in what direction? (c) How much electric flux leaves the surface of a sphere, 4 m in radius, centered at C(0, —3, 0)?

3.3 The cylindrical surface p = 8 cm contains the surface charge density, $p_S = 5e^{-20|z|}$ nC/m^2. (a) What is the total amount of charge present? (b) How much electric flux leaves the surface p = 8 cm, 1 cm < z < 5cm, 30° < < 90°?

3.4 The cylindrical surfaces p = 1, 2, and 3 cm carry uniform surface charge densities of 20, —8, and 5 nC/m^2, respectively. (a) How much electric flux passes through the closed surface p = 5 cm, 0 < z < 1m? (b) Find **D** at
cm, 2 cm, 3 cm).

3.5 Let **D** = 4xy$\mathbf{a_x}$ + 2(x^2 + z^2)$\mathbf{a_y}$ + 4yz$\mathbf{a_z}$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped 0 < x < 2, 0 < y < 3, 0 < z < 5m.

3.6 Two uniform line charges, each 20 nC/m, are located at y = 1, z = ±1m. Find the total electric flux leavingthe surface of a sphere havinga radius of 2m, if it is centered at: (a) ,4(3, 1, 0); (b) 5(3, 2, 0).

3.7 Volume charge density is located in free space as $p_v = 2e^{-1000r}$ nC/m^3 for 0 < r < 1 mm, and p_v = 0 elsewhere. (a) Find the total charge enclosed by the spherical surface r = 1 mm. (b) By usingGauss's law, calculate the value of £)$_r$ on the surface r = 1 mm.

3.8 Uniform line charges of 5 nC/m are located in free space at x = 1, z = 1, and at y = 1, z = 0. (a) Obtain an expression for **D** in cartesian coordinates at P(0, 0, z). (b) Plot |**D**| versus z at P, -3 < z < 10.

3.9 A uniform volume charge density of 80 uC/m^3 is present throughout the region 8 mm < r < 10 mm. Let p_v = 0 for 0 < r < 8 mm. (a) Find the total charge inside the spherical surface r = 10 mm. (b) Find D_r at r = 10 mm. (c) If there is no charge for r > 10 mm, find D_r at r = 20 mm.

3.10 Let p_s = 8 uC/m^2 in the region where x = 0 and —4 < z < 4 m, and let p_s = 0 elsewhere. Find **D** at P(x, 0, z), where x > 0.

3.11 In cylindrical coordinates, let p_v = 0 for p < 1mm, p_v = 2 sin 2000 ip nC/m^3 for 1mm < p < 1.5 mm, and p_v = 0 for p > 1.5 mm. Find **D** everywhere.

3.12 A nonuniform volume charge density, p_v = 120rC/m^3, lies within the spherical surface r = 1m, and p_v = 0 elsewhere. (a) Find D_r everywhere. (b) What surface charge density ps2 should be on the surface r = 2 s o that $D_r i_{r=2-}$ = $2D_r i_{r=2+}$? (c) Make a sketch of D_r vs r for 0 r < 5 with both distributions present.

3.13 Spherical surfaces at r = 2, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, —4 nC/m^2, and p_{s0}, respectively. (a) Find **D** at r = 1, 3, and 5 m. (b) Determine p_{s0} such that **D** = 0 at r = 7 m.
3.14 If $p_v = 5nC/m^3$ for 0 < p < 1 mm and no other charges are present: (a) find D_p for p < 1 mm; (b) find D_p for p > 1 mm. (c) What line charge p_L at p = 0 would give the same result for part b?
3.15 Volume charge density is located as follows: p_v = 0 for p < 1 mm and for p > 2 mm, p_v = 4p uC/m^3 for 1 < p < 2 mm. (a) Calculate the total charge in the region, 0 < p < p_1, 0 < z < L, where 1 < p_1 < 2 mm. (b) Use Gauss's law to determine D_p at p = p_1. (c) Evaluate D_p at p = 0.8mm, 1.6mm, and 2.4mm.
3.16 Given the electric flux density, **D** = $2xy\mathbf{a_x} + x^2\mathbf{a_y} + 6z^3\mathbf{a_z}$ C/m^2: (a) use Gauss's law to evaluate the total charge enclosed in the volume 0 < x, y, z < a; (b) use Eq. (8) to find an approximate value for the above charge. Evaluate the derivatives at P(a/2, a/2, a/2). (c) Show that the results of parts a and b agree in the limit as a — 0.
3.18 Let a vector field be given by **G** = $5x^4y^4z^4\mathbf{a_y}$**.** Evaluate both sides of Eq. (8) for this **G** field and the volume defined by x = 3 and 3.1, y = 1 and 1.1, and z = 2 and 2.1. Evaluate the partial derivatives at the center of the volume.
3.19 A spherical surface of radius 3 mm is centered at P(4, 1, 5) in free space. Let **D** = x$\mathbf{a_x}$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leavingthe spherical surface.
3.20 A cube of volume a^3 has its faces parallel to the cartesian coordinate surfaces. It is centered at P(3, —2, 4). Given the field **D** = $2x^3\mathbf{a_x}$ C/m^2:
(a) calculate div **D** at P; (b) evaluate the fraction in the rightmost side of Eq. (13) for a = 1m, 0.1m, and 1 mm.

3.22 Let **D** = 8psin<$\mathbf{a_p}$ + 4pcos<**a**<C/m^2. (a) Find div **D.** (b) Find the volume charge density at P(2.6, 38°, —6.1). (c) How much charge is located inside the region defined by 0 < p < 1 .8, 20° < < < 70°, 2.4 < z < 3.1?
3.23 (a) A point charge Q lies at the origin. Show that div **D** = 0 everywhere except at the origin. (b) Replace the point charge with a uniform volume charge density p_{v0} for 0 < r < a. Relate p_{v0} to Q and a so that the total charge is the same. Find div **D** everywhere.
3.24 Inside the cylindrical shell, 3 < p < 4 m, the electric flux density is given as $5(p — 3)^3\mathbf{a_p}$ C/m^2. (a) What is the volume charge density at p = 4m?
(b) What is the electric flux density at p = 4m? (c) How much electric flux leaves the closed surface: 3 < p < 4, 0 < < < 27 r, —2.5 < z < 2.5?
(d) How much charge is contained within the volume 3 < p < 4, 0 < < < 2TT, —2.5 < z < 2.5?
3.25 Within the spherical shell, 3 < r < 4 m, the electric flux density is given as **D** = $5(r — 3)^3\mathbf{a_r}$ C/m^2. (a) What is the volume charge density at r = 4? (b) What is the electric flux density at r = 4? (c) How much electric flux leaves the sphere r = 4? (d) How much charge is contained within the sphere r = 4? 5sin 6 cos <
3.26let **D** = $5r^2\mathbf{a_r}$ mC/m^2 for r < 0.08 m, and **D** = $0.1\mathbf{a}_r/r^2 C/m^2$ for r > 0.08 m. (a) Find p_v for r = 0.06 m. (b) Find p_v for r = 0.1m. (c) What surface charge density could be located at r = 0.08 m to cause **D** = 0 for r > 0.08 m?
3.27 The electric flux density is given as **D** = $20p^3\mathbf{a_p}$ C/m^2 for p < 100 urn, and k$\mathbf{a}_p$/p for p > 100 urn. (a) Find k so that **D** is continuous at p = 100 urn. (b) Find and sketch p_v as a function of p.

CHAPTER FOUR

ENERGY AND POTENTIAL

4.1 ENERGY EXPENDED IN MOVING A POINT CHARGE IN AN ELECTRIC FIELD

The electric field intensity was defined as the force on a unit test charge at that point at which we wish to find the value of this vector field. If we attempt to move the test charge against the electric field, we have to exert a force equal and opposite to that exerted by the field, and this requires us to expend energy, or do work. If we wish to move the charge in the direction of the field, our energy expenditure turns out to be negative; we do not do the work, the field does.

Suppose we wish to move a charge Q a distance $d\mathbf{L}$ *in an electric field* $\mathbf{E}$. *The force on Q due to the electric field is*

$$= Q\mathbf{E} \qquad (1)$$

where the subscript reminds us that this force is due to the field. The component of this force in the direction $d\mathbf{L}$ *which we must overcome is*

$$F_{EL} = \mathbf{F} \cdot \mathbf{a}_L = Q\mathbf{E} \cdot \mathbf{a}_L$$

where $\mathbf{a}_L$ *= a unit vector in the direction of* $d\mathbf{L}$.

The force which we must apply is equal and opposite to the force due to the field,

$$F_{appl} = -Q\mathbf{E} \cdot \mathbf{a}_L$$

and our expenditure of energy is the product of the force and distance. That is, Differential work done by external source moving $Q = -Q\mathbf{E}\cdot\mathbf{a}_L dL = -Q\mathbf{E}\cdot d\mathbf{L}$

or

$$dW = -Q\mathbf{E}\cdot d\mathbf{L} \qquad (2)$$

where we have replaced $\mathbf{a}_L dL$ *by the simpler expression* $d\mathbf{L}$.

This differential amount of work required may be zero under several conditions determined easily from (2). There are the trivial conditions for which $\mathbf{E}$, Q, *or* $d\mathbf{L}$ *is zero, and a much more important case in which* $\mathbf{E}$ *and* $d\mathbf{L}$ *are perpendicular. Here the charge is moved always in a direction at right angles to the electric field. We can draw on a good analogy between the electric field and the gravitational field, where, again, energy must be expended to move against the field. Sliding a mass around with constant velocity on a frictionless surface is an effortless process if the mass is moved along a constant elevation contour; positive or negative work must be done in moving it to a higher or lower elevation, respectively.*

Returning to the charge in the electric field, the work required to move the charge a finite distance must be determined by integrating,

$$\boxed{W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}} \qquad (3)$$

where the path must be specified before the integral can be evaluated. The charge is assumed to be at rest at both its initial and final positions.

This definite integral is basic to field theory, and we shall devote the following section to its interpretation and evaluation.

4.2 THE LINE INTEGRAL

The integral expression for the work done in moving a point charge Q *from one position to another, Eq. (3), is an example of a line integral, which in vector-analysis notation always takes the form of the integral along some prescribed path of the dot product of a vector field and a differential vector path length* $d\mathbf{L}$. *Without using vector analysis we should have to write*

$$W = -Q\int_{\text{init}}^{\text{final}} E_L\, dL$$

where E_L = *component of* $\mathbf{E}$ *along* $d\mathbf{L}$.

A line integral is like many other integrals which appear in advanced analysis, including the surface integral appearing in Gauss's law, in that it is essentially descriptive. We like to look at it much more than we like to work it out. It tells us to choose a path, break it up into a large number of very small segments, multiply the component of the field along each segment by the length of the segment, and then add the results for all the segments. This is a summation, of course, and the integral is obtained exactly only when the number of segments becomes infinite.

This procedure is indicated in Fig. 4.1, where a path has been chosen from an initial position B *to a final position*[1] A *and a* ***uniform electric field*** *selected for simplicity. The path is divided into six segments,* $\Delta L_1, \Delta L_2, \ldots, \Delta L_6$, *and the components of* $\mathbf{E}$ *along each segment denoted by* $E_{L1}, E_{L2}, \ldots, E_{L6}$. *The work involved in moving a charge* Q *from* B *to* A *is then approximately*

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \dots + E_{L6}\Delta L_6)$$

or, using vector notation,

$$W = -Q(\mathbf{E}_1 \cdot \Delta\mathbf{L}_1 + \mathbf{E}_2 \cdot \Delta\mathbf{L}_2 + \dots + \mathbf{E}_6 \cdot \Delta\mathbf{L}_6)$$

and since we have assumed a uniform field,

$$\mathbf{E}_1 = \mathbf{E}_2 = \dots = \mathbf{E}_6$$

$$W = -Q\mathbf{E} \cdot (\Delta\mathbf{L}_1 + \Delta\mathbf{L}_2 + \dots + \Delta\mathbf{L}_6)$$

What is this sum of vector segments in the parentheses above? Vectors add by the parallelogram law, and the sum is just the vector directed from the initial point B *to the final point* A, $\mathbf{L}_{BA}$. *Therefore*

[1] The final position is given the designation A to correspond with the convention for potential difference, as discussed in the following section.

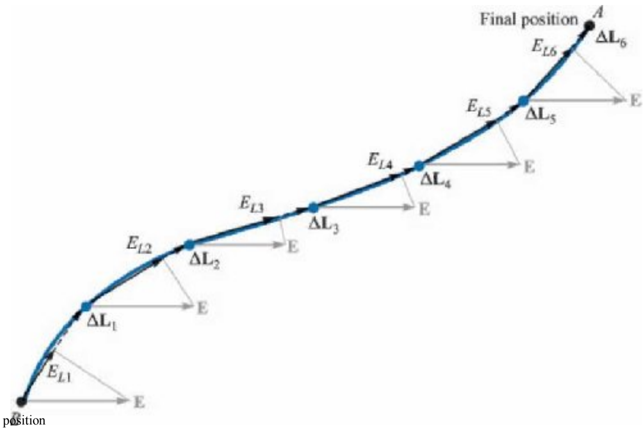


FIGURE 4.1

A graphical interpretation of a line integral in a uniform field. The line integral of E between points B and A is independent of the path selected, even in a nonuniform field; this result is not, in general, true for time-varying fields.

$$W = -Q\mathbf{E} \cdot \mathbf{L}_{BA} \quad (\textit{uniform } \mathbf{E}) \qquad (4)$$

Remembering the summation interpretation of the line integral, this result for the uniform field can be obtained rapidly now from the integral expression

$$W = -Q \int \mathbf{E} \cdot d\mathbf{L} \qquad (5)$$

as applied to a uniform field

W D

= -QE ■ L

where the last integral becomes Lba and (uniform E)

W = - Q E ■ L_ba

For this special case of a uniform electric field intensity, we should note that the work involved in moving the charge depends only on Q, E, *and* L_{BA}*, a vector drawn from the initial to the final point of the path chosen. It does not depend on the particular path we have selected along which to carry the charge. We may proceed from* B *to* A *on a straight line or via the Old Chisholm Trail; the answer is the same. We shall show in Sec. 4.5 that an identical statement may be made for any nonuniform (static)* E *field.*

Let us use several examples to illustrate the mechanics of setting up the line integral appearing in (5).

Example 4.1

We are given the nonuniform field

$$\boldsymbol{E} = y\boldsymbol{a}_x + x\boldsymbol{a}_y + 2\boldsymbol{a}_z$$

and we are asked to determine the work expended in carrying 2C from B(l, 0, 1) to A(0.8, 0.6,1) along the shorter arc of the circle

$$x^2 + y^2 = 1 \quad z = 1$$

Solution. **We use W = — Q J$_B$ *E* ■ d*L*, where *E* is not necessarily constant. Working in cartesian coordinates, the differential path d*L* is dx$\boldsymbol{a}_x$ + dy$\boldsymbol{a}_y$ + dz$\boldsymbol{a}_z$, and the integral becomes**

W = -Q *E* d*L*

2 (y$\boldsymbol{a}_x$ + x$\boldsymbol{a}_y$ + 2$\boldsymbol{a}_z$) ■ (dx $\boldsymbol{a}_x$ + dy $\boldsymbol{a}_y$ + dz $\boldsymbol{a}_z$)

■0.8 ■0.6

2 ydx - 2 xdy — dz
4

where the limits on the integrals have been chosen to agree with the initial and final values of the appropriate variable of integration. Using the equation of the circular path (and selecting the sign of the radical which is correct for the quadrant involved), we have

W = -2 J И - x²dx - 2 J Jl-Fdy - 0

r / -------- -i ~|0-8 r i------------ -i ~|0-6
= -xv 1 - x² + sin ¹ xj - p>v 1 - y² + sin ¹ yj

= -(0.48 + 0.927 - 0 - 1.571)-(0.48 + 0.644 - 0 - 0) = -0.96 J

Example 4.2

Again find the work required to carry 2 C from B to A in the same field, but this time use the straight-line path from B to A.

Solution. **We start by determining the equations of the straight line. Any two of the following three equations for planes passing through the line are sufficient to define the line:**

$$y - y_B = \frac{y_A - y_B}{x_A - x_B}(x - x_B)$$

$$z - z_B = \frac{z_A - z_B}{y_A - y_B}(y - y_B)$$

$$x - x_B = \frac{x_A - x_B}{z_A - z_B}(z - z_B)$$

From the first equation above we have

$$y = -3(x - 1)$$

and from the second we obtain

$$z = 1$$

Thus,

$$W = -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^{1} dz$$

$$= 6\int_1^{0.8}(x - 1)\,dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy$$

$$= -0.96 \text{ J}$$

This is the same answer we found using the circular path between the same two points, and it again demonstrates the statement (unproved) that the work done is independent of the path taken in any electrostatic field.

It should be noted that the equations of the straight line show that dy = -3 dx and dx = -3 dy. These substitutions may be made in the first two integrals above, along with a change in limits, and the *answer may be obtained by evaluating the new integrals. This method is often simpler if the integrand is a function of only one variable.*

Note that the expressions for DL *in our three coordinate systems utilize the differential lengths obtained in the first chapter (cartesian in Sec. 1.3, cylindrical in Sec. 1.8, and spherical in Sec. 1.9):*

=	***dx*** a_x + ***dy*** a_y + DZ a_z	*artesian)*	*(6)*
d L =	D P a***p*** + ***p dp*** ap + ***dz*** a_z	*ylindrical)*	*(7)*
=	***dr*** a_r **+ *rdO*** ag + R *sin* 9 ***dp*** ap	*pherical)*	*(8)*

The interrelationships among the several variables in each expression are determined from the specific equations for the path.

As a final example illustrating the evaluation of the line integral, let us investigate several paths which we might take near an infinite line charge. The field has been obtained several times and is entirely in the radial direction,

E = ***E***p ***P*L** 27160 P

ap

Let us first find the work done in carrying the positive charge Q *about a circular path of radius* P* *centered at the line charge, as illustrated in Fig. 4.2*A. *Without lifting a pencil, we see that the work must be nil, for the path is always perpendicular to the electric field intensity, or the force on the charge is always exerted at right angles to the direction in which we are moving it. For practice, however, let us set up the integral and obtain the answer.*

*The differential element **d**L is chosen in cylindrical coordinates, and the circular path selected demands that* DP *and* DZ *be zero, so* DL = P1D***p*** ap. *The work is then*

W **=-*Q*** (

■final

PL

i

nit 2760P1

■2ji ap ■

pi d p ap

0 ***2iT€o*** ***d p*** a_p ■ ap = *0*

*Let us now carry the charge from **p** 4.2ft). Here* DL = DP a_p *and* A *to* P = B *along a radial path (Fig.*

p1

2it€op

* p1 DP

a p ■ dp a p = - Q

, 2tt6o ***p***

or ***W*** *Qpl In b*

2tx€0 ***a***

Since b is larger than a,In (b/a) is positive, and we see that the work done is negative, indicating that the external source that is moving the charge receives energy.

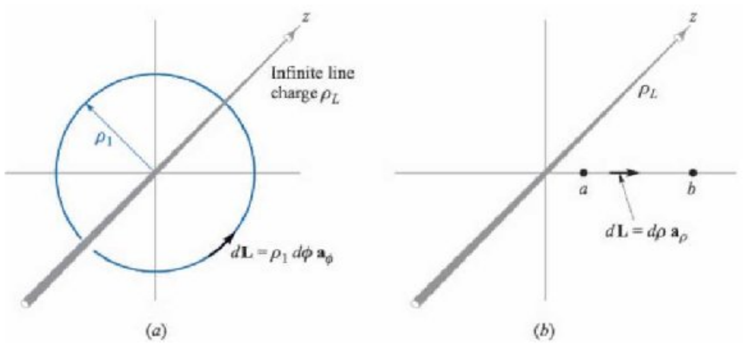


FIGURE 4.2
(a) A circular path and (b) a radial path along which a charge of Q is carried in the field of an infinite line charge. No work is expected in the former case.

One of the pitfalls in evaluating line integrals is a tendency to use too many minus signs when a charge is moved in the direction of a ***decreasing*** *coordinate value. This is taken care of completely by the limits on the integral, and no misguided attempt should be made to change the sign of* $d\mathbf{L}$. *Suppose we carry* Q *from* B *to* A *(Fig. 4.2b). We still have* $d\mathbf{L} = d\rho\, \mathbf{a}_\rho$ *and show the different direction by recognizing* $\rho = b$ *as the initial point and* $\rho = a$ *as the final point,*

$$W = -Q\int_b^a \frac{\rho_L}{2\pi\epsilon_0 \rho}\, d\rho = \frac{Q\rho_L}{2\pi\epsilon_0}\ln\frac{b}{a}$$

This is the negative of the previous answer and is obviously correct.

4.3 DEFINITION OF POTENTIAL DIFFERENCE AND POTENTIAL

We are now ready to define a new concept from the expression for the work done by an external source in moving a charge Q from one point to another in an electric field $\mathbf{E}$,

$$\boxed{W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}}$$

In much the same way as we defined the electric field intensity as the force on a ***unit*** *test charge, we now define* ***potential difference V*** *as the work done (by an external source) in moving a* ***unit*** *positive charge from one point to another in an electric field,*

$$\boxed{\text{Potential difference} = V = -\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}} \qquad (9)$$

We shall have to agree on the direction of movement, as implied by our language, and we do this by stating that V_{AB} *signifies the potential difference between points* A *and* B *and is the work done in moving the unit charge from*

B *(last named) to* A *(first named). Thus, in determining* V_{AB}, B *is the initial point and* A *is the final point. The reason for this somewhat peculiar definition will become clearer shortly, when it is seen that the initial point* B *is often taken at infinity, whereas the final point* A *represents the fixed position of the charge; point* A *is thus inherently more significant.*

Potential difference is measured in joules per coulomb, for which the ***volt*** *is defined as a more common unit, abbreviated as V. Hence the potential difference between points* A *and* B *is*

$$\boxed{Vab = — \Big| DL \quad V \quad b} \qquad (10)$$

and V_{AB} *is positive if work is done in carrying the positive charge from* B *to* A.

From the line-charge example of the last section we found that the work done in taking a charge Q *from* P ***= b*** *to* ***p = a*** *was*

W

27rEo ***a***

Thus, the potential difference between points at P ***= a*** *and* ***p = b*** *is*

Fab = 77 = ----- *In-* *(11)*

We can try out this definition by finding the potential difference between points A *and* B *at radial distances* R_A *and* R_B *from a point charge* Q. *Choosing an origin at* Q,

47rEo R^2

and jL = JR a_r

we have Fab E • JL: = — E • J ***Jr*** = t!H- – H *(12)*

L = —

Jb

If $R_B > R_A$, *the potential difference* F_{ab} *is positive, indicating that energy is expended by the external source in bringing the positive charge from* R_B *to* R_A. *This agrees with the physical picture showing the two like charges repelling each other.*

It is often convenient to speak of the ***potentiator absolute potential,ofa*** *point, rather than the potential difference between two points, but this means only that we agree to measure every potential difference with respect to a specified reference point which we consider to have zero potential. Common agreement must be reached on the zero reference before a statement of the potential has any significance. A person having one hand on the deflection plates of a cathode-ray tube which are "at a potential of 50 V" and the other hand on the cathode terminal would probably be too shaken up to understand that the cathode is not the zero reference, but that all potentials in that circuit are customarily measured with respect to the metallic shield about the tube. The cathode may be several thousands of volts negative with respect to the shield.*

Perhaps the most universal zero reference point in experimental or physical potential measurements is "ground," by which we mean the potential of the surface region of the earth itself. Theoretically, we usually represent this surface by an infinite plane at zero potential, *although some large-scale problems, such as those involving propagation across the Atlantic Ocean, require a spherical surface at zero potential.*

Another widely used reference "point" is infinity. This usually appears in theoretical problems approximating a physical situation in which the earth is relatively far removed from the region in which we are interested, such as the static field near the wing tip of an airplane that has acquired a charge in flying through a thunderhead, or the field inside an atom. Working with the ***gravitational*** *potential field on earth, the zero reference is normally taken at sea level; for an interplanetary mission, however, the zero reference is more conveniently selected at infinity.*

A cylindrical surface of some definite radius may occasionally be used as a zero reference when cylindrical symmetry is present and infinity proves inconvenient. In a coaxial cable the outer conductor is selected as the zero reference for potential. And, of course, there are numerous special problems, such as those for which a two-sheeted hyperboloid or an oblate spheroid must be selected as the zero-potential reference, but these need not concern us immediately.

If the potential at point A *is* V_a *and that at* B *is* V_b*, then*

$$V_{ab} = V_a - V_b \tag{13}$$

where we necessarily agree that V_a *and* V_b *shall have the same zero reference point.*

4.4 THE POTENTIAL FIELD OF A POINT CHARGE

In the previous section we found an expression (12) for the potential difference between two points located at R **=** ***r_A*** *and* R **=** ***r_B*** *in the field of a point charge* Q *placed at the origin,*

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B \tag{14}$$

It was assumed that the two points lay on the same radial line or had the same 9 *and* **<p** *coordinate values, allowing us to set up a simple path on this radial line along which to carry our positive charge. We now should ask whether different* 9 *and* **<p** *coordinate values for the initial and final position will affect our answer and whether we could choose more complicated paths between the two points without changing the results. Let us answer both questions at once by choosing two general points* A *and* B *(Fig. 4.3) at radial distances of* R_A *and* R_B*, and any values for the other coordinates.*

The differential path length **d**L *has* R, 9, *and* **<p** *components, and the electric field has only a radial component. Taking the dot product then leaves us only*

$$V_{AB} = -\int_{r_B}^{r_A} E_r\,dr = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

We obtain the same answer and see, therefore, that the potential difference between two points in the field of a point charge depends only on the distance of

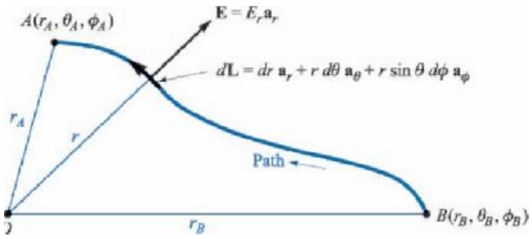


FIGURE 4.3
A general path between general points B and A in the field of a point charge Q at the origin. The potential difference V_{AB} is independent of the path selected.

each point from the charge and does not depend on the particular path used to carry our unit charge from one point to the other.

How might we conveniently define a zero reference for potential? The simplest possibility is to let $V = 0$ at infinity. If we let the point at $r = r_B$ recede to infinity the potential at r_A becomes

$$V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

or, since there is no reason to identify this point with the A subscript,

$$V = \frac{Q}{4\pi\epsilon_0 r} \qquad (15)$$

This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference. Returning to a physical interpretation, we may say that $Q/4\pi\epsilon_0 r$ joules of work must be done in carrying a 1-C charge from infinity to any point r meters from the charge Q.

A convenient method to express the potential without selecting a specific zero reference entails identifying r_A as r once again and letting $Q/4\pi\epsilon_0 r_B$ be a constant. Then

$$V = \frac{Q}{4\pi\epsilon_0 r} + C_1 \qquad (16)$$

and C_1 may be selected so that $V = 0$ at any desired value of r. We could also select the zero reference indirectly by electing to let V be V_0 at $r = r_0$.

*It should be noted that the **potential difference** between two points is not a function of C_1.*

Equation (15) or (16) represents the potential field of a point charge. The potential is a scalar field and does not involve any unit vectors.

*Let us now define an **equipotential surface** as a surface composed of all those points having the same value of potential. No work is involved in moving a unit charge around on an equipotential surface, for, by definition, there is no potential difference between any two points on this surface.*

The equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.

An inspection of the form of the potential field of a point charge shows that it is an inverse-distance field, whereas the electric field intensity was found to be an inverse-square-law relationship. A similar result occurs for the gravitational force field of a point mass (inverse-square law) and the gravitational potential field (inverse distance). The gravitational force exerted by the earth on an object one million miles from it is four times that exerted on the same object two million miles away. The kinetic energy given to a freely falling object starting from the *end of the universe with zero velocity, however, is only twice as much at one million miles as it is at two million miles.*

4.5 THE POTENTIAL FIELD OF A SYSTEM OF CHARGES: CONSERVATIVE PROPERTY

The potential at a point has been defined as the work done in bringing a unit positive charge from the zero reference to the point, and we have suspected that this work, and hence the potential, is independent of the path taken. If it were not, potential would not be a very useful concept.

Let us now prove our assertion. We shall do so by beginning with the potential field of the single point charge for which we showed, in the last section, the independence with regard to the path, noting that the field is linear with respect to charge so that superposition is applicable. It will then follow that the potential of a system of charges has a value at any point which is independent of the path taken in carrying the test charge to that point.

Thus the potential field of a single point charge, which we shall identify as Q_1 *and locate at* $\mathbf{r}_1$, *involves only the distance* $|\mathbf{r} - \mathbf{r}_1|$ *from* Q_1 *to the point at* $\mathbf{r}$ *where we are establishing the value of the potential. For a zero reference at infinity, we have*

V (r) = T-T

r1|

The potential due to two charges, Q_1 *at* $\mathbf{r}_1$ *and* Q_2 *at* $\mathbf{r}_2$, *is a function only of* $|\mathbf{r} - \mathbf{r}_1|$ *and* $|\mathbf{r} - \mathbf{r}_2|$, *the distances from* Q_1 *and* Q_2 *to the field point, respectively.*

V (r) = ---6------+----- 62 ----

4^E0|r - rI | 4^E0|r - r2|

Continuing to add charges, we find that the potential due to N *point charges is*

V(r) = 6i / 4izeo |r , + 62 , + . . . + 6n

or V(r) = £ 4—Qm *(17)*

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_v \Delta v$, *then*

rrt \ Pv(ri)AVi Pv(r2)AV? ,

Pv(rn)AVn

V (r) =-----1------------------------ + . . . +

4^Eo|r - ri | 4^Eo|r - r2| 47r<?o|r - rn |

As we allow the number of elements to become infinite, we obtain the integral expression

r f Pv(r ')DV' *(18)*

We have come quite a distance from the potential field of the single point charge, and it might be helpful to examine (18) and refresh ourselves as to the meaning of each term. The potential V(r) *is determined with respect to a zero reference potential at infinity and is an exact measure of the work done in bringing a unit charge from infinity to the field point at* r *where we are finding the potential. The volume charge density* $\rho_v(\mathbf{r}')$ *and differential volume element* dv' *combine to represent a differential amount* *of charge* $\rho_v(\mathbf{r}')\,dv'$ *located at* $\mathbf{r}'$*. The distance* $|\mathbf{r} - \mathbf{r}'|$ *is that distance from the source point to the field point. The integral is a multiple (volume) integral.*

If the charge distribution takes the form of a line charge or a surface charge, the integration is along the line or over the surface:

$$V(\mathbf{r}) = \int ^1 \qquad (19)$$

$$V(\mathbf{r}) = \int \qquad (20)$$

The most general expression for potential is obtained by combining (17), (18), (19), and (20).

These integral expressions for potential in terms of the charge distribution should be compared with similar expressions for the electric field intensity, such as (18) in Sec. 2.3:

$$\mathbf{E}(\mathbf{r}) = \int_{vol} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

The potential again is inverse distance, and the electric field intensity, inverse-square law. The latter, of course, is also a vector field.

To illustrate the use of one of these potential integrals, let us find V *on the* Z *axis for a uniform line charge* PL *in the form of a ring,* ***p = a, in*** *the* Z *= 0 plane, as shown in Fig. 4.4. Working with (19), we have* DL' ***= ad<p'****,* R — Z3z, R — ASp, |R — R' | = VA^2 + Z^2, *and*

pLad<<

V

Jo *47*rEoVA^2 + Z^2 ***2***it€o*J***a1*** + Z^2 *For a zero reference at infinity, then:*

1. *The potential due to a single point charge is the work done in carrying a unit positive charge from infinity to the point at which we desire the potential, and the work is independent of the path chosen between those two points.*
2. *The potential field in the presence of a number of point charges is the sum of the individual potential fields arising from each charge.*
3. *The potential due to a number of point charges or any continuous charge distribution may therefore be found by carrying a unit charge from infinity to the point in question along any path we choose.*

In other words, the expression for potential (zero reference at infinity),

t A

VA — — ∫ E • DL

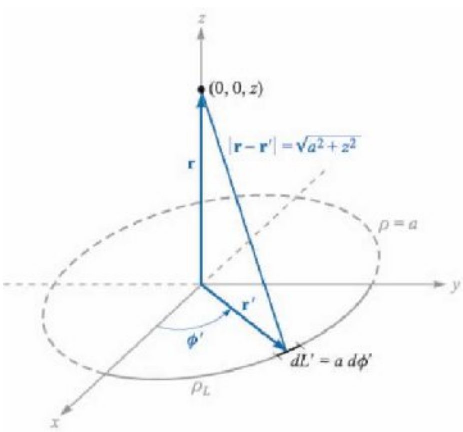


FIGURE 4.4
The potential field of a ring of uniform line charge density is easily obtained from V = ∫pL(r')dL'/ (4πε₀|r — r'|).

or potential difference,

$$V_{ab} = V_A - V_B = - \int_b E \cdot dL$$

is not dependent on the path chosen for the line integral, regardless of the source of the E field.

This result is often stated concisely by recognizing that no work is done in carrying the unit charge around any ***closed path,or***

$$\oint E \cdot dL = 0 \qquad (2i)$$

A small circle is placed on the integral sign to indicate the closed nature of the path. This symbol also appeared in the formulation of Gauss's law, where a closed ***surFace*** *integral was used.*

4.6 POTENTIAL GRADIENT

We now have two methods of determining potential, one directly from the electric field intensity by means of a line integral, and another from the basic charge distribution itself by a volume integral. Neither method is very helpful in determining the fields in most practical problems, however, for as we shall see later, neither the electric field intensity nor the charge distribution is very often known. Preliminary information is much more apt to consist of a description of two equipotential surfaces, such as the statement that we have two parallel conductors of circular cross section at potentials of 100 and — 100 V. Perhaps we wish to find the capacitance between the conductors, or the charge and current distribution on the conductors from which losses may be calculated.

These quantities may be easily obtained from the potential field, and our immediate goal will be a simple method offinding the electric field intensity from the potential.

We already have the general line-integral relationship between these quantities,

$$E \cdot dL \quad V$$

but this is much easier to use in the reverse direction: given E, *find* V.

However, (22) may be applied to a very short element of length AL *along which* E *is essentially constant, leading to an incremental potential difference* AV,

$$AV = -E \cdot AL \qquad (23)$$

Let us see first if we can determine any new information about the relation of V *to* E *from this equation. Consider a general region of space, as shown in Fig. 4.6, in which* E *and* V *both change as we move from point to point. Equation (23) tells us to choose an incremental vector element of length* AL = ***AL*** a_L *and multiply its magnitude by the component of* E *in the direction of* a_L *(one interpretation of the dot product) to obtain the small potential difference between the final and initial points of* VL.

(22)

If we designate the angle between AL *and* E *as* 6, *then*

$$A V = — \textbf{\textit{EAL}} \cos 6$$

We now wish to pass to the limit and consider the derivative DV/DL. *To do this, we need to show that* V *may be interpreted as a* ***function V(x,*** Y, Z). *So far,* V *is merely the result of the line integral (22). If we assume a specified starting point or zero reference and then let our end point be* (X, Y, Z), *we know that the result of the integration is a unique function of the end point* (X, Y, Z) *because* E *is a conservative field. Therefore* V *is a single-valued function* V(X, Y, Z). *We may then pass to the limit and obtain*

$$\frac{DV}{DL} = —E \cos 6$$

In which direction should AL *be placed to obtain a maximum value of* A V? *Remember that* E *is a definite value at the point at which we are working and is independent of the direction of* AL. *The magnitude* AL *is also constant, and our*

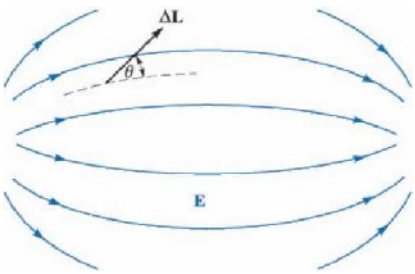


FIGURE 4.6
A vector incremental element of length AL is shown making an angle of 6 with an E field, indicated by its streamlines. The sources of the field are not shown

variable is a_L, *the unit vector showing the direction of* AL. *It is obvious that the maximum positive increment of potential,* A V_{MAX}, *will occur when cos* 6 *is —1, or* AL *points in the direction* ***opposite*** *to* E. *For this condition,*

$$\left.\frac{dV}{dL}\right|_{max} = E$$

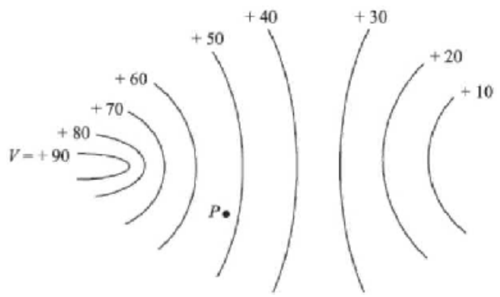
This little exercise shows us two characteristics of the relationship between E *and* V *at any point:*

1. *The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.*
2. *This maximum value is obtained when the direction of the distance increment is opposite to* E *or, in other words, the direction of* E *is* ***opposite*** *to the direction in which the potential is* ***increasing*** *the most rapidly.*

Let us now illustrate these relationships in terms of potential. Fig. 4.7 is intended to show the information we have been given about some potential field. It does this by showing the equipotential surfaces (shown as lines in the two-dimensional sketch). We desire information about the electric field intensity at point P. *Starting at* P, *we lay off a small incremental distance* AL *in various directions, hunting for that direction in which the potential is changing (increasing) the most rapidly. From the sketch, this direction appears to be left and slightly upward. From our second characteristic above, the electric field* *intensity is therefore oppositely directed, or to the right and slightly downward at* ***P.*** *Its magnitude is given by dividing the small increase in potential by the small element of length.*

It seems likely that the direction in which the potential is increasing the most rapidly is perpendicular to the equipotentials (in the direction of ***increasing*** *potential), and this is correct, for if* AL *is directed along an equipotential,* A V *= 0 by our definition of an equipotential surface. But then*

FIGURE 4.7
A potential field is shown by its equipotential surfaces. At any point the E field is normal to the equipotential surface passing through that point and is directed toward the



A V = —E • A L = *0*

and since neither E *nor* AL *is zero,* E *must be perpendicular to this* AL *or perpendicular to the equipotentials.*

Since the potential field information is more likely to be determined first, let us describe the direction of AL *which leads to a maximum increase in potential mathematically in terms of the potential field rather than the electric field intensity. We do this by letting* a_N *be a unit vector normal to the equipotential surface and directed toward the higher potentials. The electric field intensity is then expressed in terms of the potential,*

E ***dV*** *(24)*

max

which shows that the magnitude of E *is given by the maximum space rate of change of* V *and the direction of* E *is* ***normal*** *to the equipotential surface (in the direction of* ***decreasing*** *potential).*

Since $DV/DL|_{MAX}$ *occurs when* AL *is in the direction of* a_v*, we may remind ourselves of this fact by letting*

and DV DL E

D

V

(25)

D

V

a V

Equation (24) or (25) serves to provide a physical interpretation of the process of finding the electric field intensity from the potential. Both are descriptive of a general procedure, and we do not intend to use them directly to obtain quantitative information. This procedure leading from V *to* E *is not unique to this pair of quantities, however, but has appeared as the relationship between a scalar and a vector field in hydraulics, thermodynamics, and magnetics, and indeed in almost every field to which vector analysis has been applied.*

The operation on V *by which* —E *is obtained is known as the* ***gradient,*** *and the gradient of a scalar field* T *is defined as*

$$\boxed{\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\mathbf{a}_N} \qquad (26)$$

where a_V *is a unit vector normal to the equipotential surfaces, and that normal is chosen which points in the direction of increasing values of* T.

Using this new term, we now may write the relationship between V *and* E *as*

E = — *grad* V

Since we have shown that ***V*** *is a unique function of* ***x, y,*** *and* Z, *we may take its total differential*

dV = — ax + ----- ay + -----az
3x 3y dz

But we also have

dV = — E • **= —E_x dx — $E_y$dy — E_z dz**

Since both expressions are true for any ***dx, dy,*** *and* DZ, *then*

_ _3V dx

Ex

Ey = —3V

EZ = —3V

These results may be combined vectorially to yield

dV

(27)

E dV. dV. Ik"
= (28)
—[
—
a x
dx

and comparison of (27) and (28) provides us with an expression which may be used to evaluate the gradient in cartesian coordinates,

grad V = dV/dx ax + dV/dy ay + dV/dz az (29)

The gradient of a scalar is a vector, and old quizzes show that the unit vectors which are often incorrectly added to the divergence expression appear to be those which were incorrectly removed from the gradient. Once the physical interpretation of the gradient, expressed by (26), is grasped as showing the maximum space rate of change of a scalar quantity and **the direction in which this maximum occurs,** the vector nature of the gradient should be self-evident.

The vector operator

V = d/dx ax + d/dy ay + d/dz az

may be used formally as an operator on a scalar, ***T*, *WT*,** producing

dT dT dT WT = ax + — ay + — az dx dy dz

from which we see that

VT = grad T

This allows us to use a very compact expression to relate E and V,

E = —VJ V (30)

The gradient may be expressed in terms of partial derivatives in other coordinate systems through application of its definition (26). These expressions are derived in Appendix A and repeated below for convenience when dealing with problems having cylindrical or spherical symmetry. They

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cartesian)}} \qquad (31)$$

also appear inside the back cover.

dV i dV dV

dp p d<p dz *(cylindrical)* *(32)*

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \quad \text{(spherical)} \quad (33)$$

Note that the denominator of each term has the form of one of the components of dL *in that coordinate system, except that partial differentials replace ordinary differentials; for example,* $r \sin\theta\, d\phi$ *becomes* $r \sin\theta\, \partial\phi$.

Let us now hasten to illustrate the gradient concept with an example.

Example 4.3

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity $\mathbf{E}$, the direction of $\mathbf{E}$, the electric flux density $\mathbf{D}$, and the volume charge density ρ_v.

Solution. **The potential at P(—4, 5, 6) is**

$$V_P = 2(-4)^2(3) - 5(6) = 66 \quad \text{V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

The value of $\mathbf{E}$ at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \quad \text{V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \quad \text{V/m}$$

The direction of $\mathbf{E}$ at P is given by the unit vector

$$\mathbf{a}_{E,P} = (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9$$
$$= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$$

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0\mathbf{E} = -35.4xy\mathbf{a}_x - 17.71x^2\mathbf{a}_y + 44.3\mathbf{a}_z \quad \text{pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla \cdot \mathbf{D} = -35.4y \quad \text{pC/m}^3$$

At P, $\rho_v = -106.2\text{pC/m}^3$.

4.7 THE DIPOLE

The dipole fields which we shall develop in this section are quite important because they form the basis for the behavior of dielectric materials in electric fields, as discussed in part of the following chapter, as well as justifying the use of images, as described in Sec.

5.5 of the next chapter. Moreover, this development will serve to illustrate the importance of the potential concept presented in this chapter.

*An **electric dipole,** or simply a **dipole,** is the name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P at which we want to know the electric and potential fields. The dipole is shown in Fig. 4.9a. The distant point P is described by the spherical coordinates **r , 6 ,** and **<p -** 90°, in view of the azimuthal symmetry. The positive and negative point charges have separation d and cartesian coordinates (0, 0, 5 d) and (0, 0, - 2d), respectively.*

So much for the geometry. What would we do next? Should we find the total electric field intensity by adding the known fields of each point charge? Would it be easier to find the total potential field first? In either case, having found one, we shall find the other from it before calling the problem solved.

If we choose to find E first, we shall have two components to keep track of in spherical coordinates (symmetry shows is zero), and then the only way to find V from E is by use of the line integral. This last step includes establishing a suitable zero reference for potential, since the line integral gives us only the potential difference between the two points at the ends of the integral path.

V

$$\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

Note that the plane z = 0, midway between the two point charges, is the locus of points for which $R_1 = R_2$, and is therefore at zero potential, as are all points at infinity.

For a distant point, $R_1 = R_2$, and the R_1R_2 product in the denominator may be replaced by r^2. The approximation may not be made in the numerator, however, without obtaining the trivial answer that the potential field approaches zero as we go very far away from the dipole. Coming back a little closer to the dipole, we see from Fig. 4.9b that R_2 - R_1 may be approximated very easily if R_1 and R_2 are assumed to be parallel,

$$R_2 - R_1 = d \cos 0$$

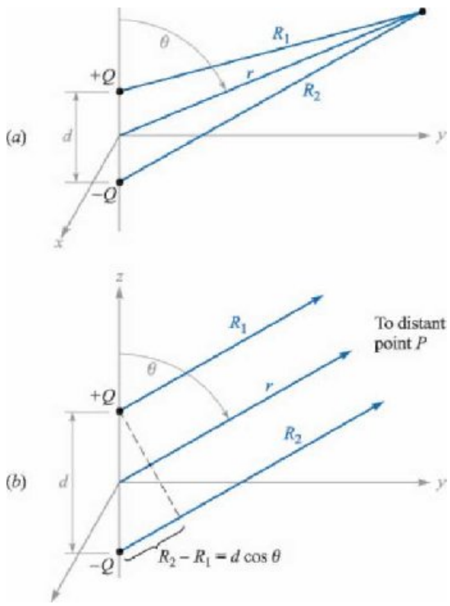


FIGURE 4.9
(a) The geometry of the problem of an electric dipole. The dipole moment p = Qd is in the a_z direction. (b) For a distant point **P** , $\mathbf{R_1}$ is essentially parallel to R_2, and we find that R_2 — R_1 = **d** cos 9.

The final result is then

$$V \quad \frac{Q\,d\,\cos 9}{4it€0\,r^2} \qquad (34)$$

Again we note that the plane ***z*** *= 0 (9 = 90°) is at zero potential. Using the gradient relationship in spherical coordinates,*

$$E = -\nabla V = -\left(\frac{dV}{DV}\,a_r + \frac{1}{r}\frac{\partial V}{\partial 9}\,a_9 + \frac{1}{r\sin 9}\frac{\partial V}{\partial \phi}\,a_\phi\right)$$

we obtain

$$E \quad \frac{Qd\ \cos\ 9}{4it€0\ r^3}\ ', \quad \frac{Qd\ \sin}{4it€0 r^3}$$

(35)

(36)

or

E (2 cos 6 a_r Q sin
D 6 a_6)

These are the desired distant fields of the dipole, obtained with a very small amount of work. Any student who has several hours to spend may try to work the problem in the reverse direction—the authors consider the process too long and detailed to include, even for effect.

To obtain a plot of the potential field, we may choose a dipole such that QD/(4 tt€o) = 1, and then cos 6 = VR^2. The colored lines in Fig. 4.10 indicate equipotentials for which V = 0, + 0.2, + 0.4, + 0.6, + 0.8, and +1, as indicated. The dipole axis is vertical, with the positive charge on the top. The streamlines for the electric field are obtained by applying the methods of Sec. 2.6 in spherical

sin
d 6 coordinates,
2 cos 6

E_r 2cot6d6

or D

from which we

R

r C_1 sin^2
6

The black streamlines shown in Fig. 4.10 are for C_1 = 1, 1.5, 2, and 2.5.

The potential field of the dipole, Eq. (34), may be simplified by making use of the dipole moment. Let us first identify the vector length directed from — Q to +Q as d and then define the ***dipole moment*** as Qd and assign it the symbol p. Thus

i = Qd (37)

The units of p are C • m.

Since d • a_r = D cos 6, we then have

P • »r V (38)
4ne0 r^2

This result may be generalized as

1 r (39)
V
— r'
$47rE_0$|r — r'$|^{2P}$ |r
— r'|

where r locates the field point P, and r' determines the dipole center. Equation (39) is independent of any coordinate system.

(36)

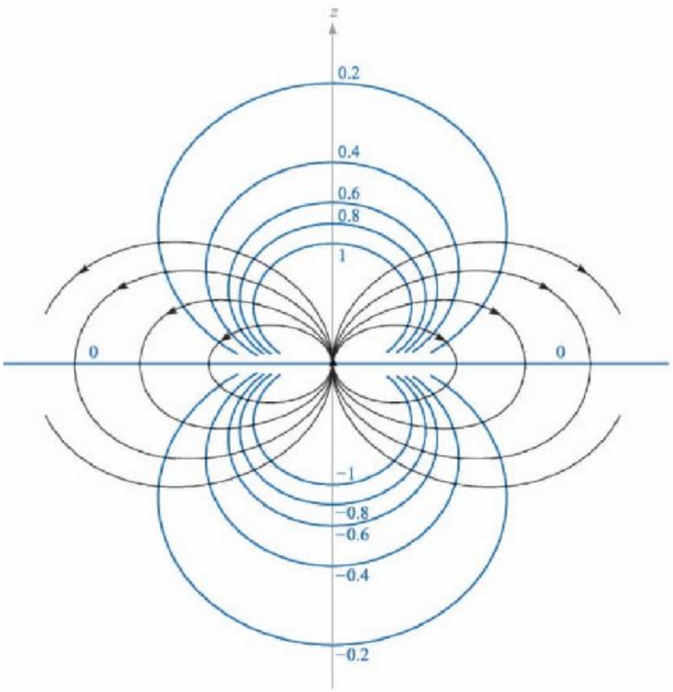


FIGURE 4.10
The electrostatic field of a point dipole with its moment in the $\mathbf{a}_z$ direction. Six equipotential surfaces are labeled with relative values of V.

The dipole moment p *will appear again when we discuss dielectric materials. Since it is equal to the product of the charge and the separation, neither the dipole moment nor the potential will change as **Q** increases and* d *decreases, provided the product remains constant. The limiting case of a **point dipole** is achieved when we let* d *approach zero and **Q** approach infinity such that the product* p *is finite.*

*Turning our attention to the resultant fields, it is interesting to note that the potential field is now proportional to the inverse **square** of the distance, and the electric field intensity is proportional to the inverse **cube** of the distance from the dipole. Each field falls off faster than the corresponding field for the point charge, but this is no more than we should expect because the opposite charges appear to be closer together at greater distances and to act more like a single point charge of 0 C.*

Symmetrical arrangements of larger numbers of point charges produce fields proportional to the inverse of higher and higher powers of R. *These charge distributions are called **multipoles,** and they are used in infinite series to approximate more unwieldy charge configurations.*

4.8 ENERGY DENSITY IN THE ELECTROSTATIC FIELD

We have introduced the potential concept by considering the work done, or energy expended, in moving a point charge around in an electric field, and now we must tie up the loose ends of that discussion by tracing the energy flow one step further.

Bringing a positive charge from infinity into the field of another positive charge requires work, the work being done by the external source moving the charge. Let us imagine that the external source carries the

(36)

charge up to a point near the fixed charge and then holds it there. Energy must be conserved, and the energy expended in bringing this charge into position now represents potential energy, for if the external source released its hold on the charge, it would accelerate away from the fixed charge, acquiring kinetic energy of its own and the capability of doing work.

In order to find the potential energy present in a system of charges, we must find the work done by an external source in positioning the charges.

We may start by visualizing an empty universe. Bringing a charge $Q1$ *from infinity to any position requires no work, for there is no field present.*[12] *The positioning of* $Q2$ *at a point in the field of* $Q1$ *requires an amount of work given by the product of the charge* $Q2$ *and the potential at that point due to* Q_1. *We represent this potential as* V_{21}, *where the first subscript indicates the location and the second subscript the source. That is,* V_{2J1} *is the potential at the location of* Q_2 *due to* Q_1. *Then*

$$\text{Work to position } Q_2 = Q_2 V_{2J1}$$

Similarly, we may express the work required to position each additional charge in the field of all those already present:

$$\text{Work to position } Q_3 = Q_3 V_{31} + Q_3 V_{32}$$

$$\text{Work to position } Q_4 = Q_4 V_{4J1} + Q_4 V_{4J2} + Q_4 V_{4J3}$$

and so forth. The total work is obtained by adding each contribution:

Total positioning work = potential energy of field

$$= We = Q2\ V2{,}1 + Q3\ V3{,}1 + Q3\ V3{,}2 + Q4\ V4{,}1$$
$$+ Q4V4{,}2 + Q4V4{,}3 + \ldots \qquad (40)$$

Noting the form of a representative term in the above equation,

$$Q_3 V_{31} = Q_3 \frac{Q1}{\quad} = Q_1 \frac{Q3}{\quad}$$

where R_{13} *and* R_{31} *each represent the scalar distance between* Q_1 *and* Q_3, *we see that it might equally well have been written as* Q_1V_{13}. *If each term of the total energy expression is replaced by its equal, we have*

$$We = Q1\ V1{,}2 + Q1\ V1{,}3 + Q2\ V2{,}3 + Q1\ V1{,}4 + Q2\ V2{,}4 + Q3\ V3{,}4 + \ldots \qquad (41)$$

Adding the two energy expressions (40) and (41) gives us a chance to simplify the result a little:

$$2We = Q1(V1{,}2 + V1{,}3 + V1{,}4 + \ldots)$$
$$+ Q2(V2{,}1 + V2{,}3 + V2{,}4 + \ldots)$$
$$+ Q3(V3{,}1 + V3{,}2 + V3{,}4 + \ldots)$$
$$+ \ldots$$

Each sum of potentials in parentheses is the combined potential due to all the charges except for the charge at the point where this combined potential is being found. In other words,

$$V1{,}2 + V1{,}3 + V1{,}4 + \ldots = V1$$

[12] However, somebody in the workshop at infinity had to do an infinite amount of work to create the point charge in the first place! How much energy is required to bring two half-charges into coincidence to make a unit charge?

(36)

the potential at the location of Q_1 due to the presence of $Q_2, Q_3, \ldots$ We therefore have

$$W_E = \tfrac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + \ldots) = \tfrac{1}{2}\sum Q_mV_m \tag{42}$$

In order to obtain an expression for the energy stored in a region of continuous charge distribution, each charge is replaced by $\rho_v dv$, and the summation becomes an integral,

$$W_E = \frac{1}{2}\int_{vol} \rho_v V\, dv \tag{43}$$

Equations (42) and (43) allow us to find the total potential energy present in a system of point charges or distributed volume charge density. Similar expressions may be easily written in terms of line or surface charge density. Usually we prefer to use (43) and let it represent all the various types of charge which may have to be considered. This may always be done by considering point charges, line charge density, or surface charge density as continuous distributions of volume charge density over very small regions. We shall illustrate such a procedure with an example shortly.

Before we undertake any interpretation of this result, we should consider a few lines of more difficult vector analysis and obtain an expression equivalent to (43) but written in terms of $\mathbf{E}$ and $\mathbf{D}$.

We begin by making the expression a little bit longer. Using Maxwell's first equation, replace ρ_v by its equal $\nabla \cdot \mathbf{D}$ and make use of a vector identity which is true for any scalar function V and any vector function $\mathbf{D}$,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V) \tag{44}$$

This may be proved readily by expansion in cartesian coordinates. We then have, successively,

$$W_E = \frac{1}{2}\int_{vol} \rho_v V\, dv = \frac{1}{2}\int_{vol} (\nabla \cdot \mathbf{D})V\, dv$$

$$= \frac{1}{2}\int_{vol} [\nabla \cdot (V\mathbf{D}) - \mathbf{D} \cdot (\nabla V)]\, dv$$

Using the divergence theorem from the last chapter, the first volume integral of the last equation is changed into a closed surface integral, where the closed surface surrounds the volume considered. This volume, first appearing in (43), must contain ***every*** charge, and there can then be no charges outside of the volume. We may therefore consider the volume as ***infinite*** in extent if we wish. We have

$$W_E = \frac{1}{2}\oint_S (V\mathbf{D}) \cdot d\mathbf{S} - \frac{1}{2}\int_{vol} \mathbf{D} \cdot (\nabla V)\, dv$$

The surface integral is equal to zero, for over this closed surface surrounding the universe we see that V is approaching zero at least as rapidly as $1/r$ (the charges look like a point charge from there), $\mathbf{D}$ is approaching zero at least as rapidly as $1/r^2$, while the differential element of surface, looking more and more like a portion of a sphere, is increasing only as r^2. The integrand therefore approaches zero at least as rapidly as $1/r$. In the limit the integrand and the integral are zero. Substituting $\mathbf{E} = -\nabla V$ in the remaining volume integral, we have our answer, f

D • E *dv* = 2W ,QE² DV
V
E ol
vo
l

Let us now use this last expression to calculate the energy stored in the electrostatic field of a section of a coaxial cable or capacitor of length L. *We found in Sec. 3.3 of the previous chapter that*

aps

Hence,

^{ap}s
E = --- ap

<?o P

*where p*s *is the surface charge density on the inner conductor, whose radius is* ***a****. Thus,*

i -L f i n eb A²P²S N L fl2P 2 ***b***
€o r-s P ***a p a p az*** =------------- *ln* -
, <?0p 1 €o ***a***

This same result may be obtained from (43). We choose the outer conductor as our zero-potential reference, and the potential of the inner cylinder is then

Va
ep ***a p*** = —
b ^{ap}s ^{ap}s ***b***
---- ***a p*** =------ *ln*-

The surface charge density ***p***s *at* ***p*** = ***a*** *can be interpreted as a volume charge density* ***p****v* = PS*/1, extending from* P = ***a*** — ***11*** *to* P = ***a*** + ²*t, where* ***t*** <§C *a. The integrand in (43) is therefore zero everywhere between the cylinders (where the volume charge density is zero), as well as at the outer cylinder (where the potential is zero). The integration is therefore performed only within the thin cylindrical shell at* P = *a,*

We = ² ***p***v***VdV*** = i L ,ln f$^{a+1/1}$ ***p***s ***p***s, ***b***
J —***a***—P ***a p ap az***
0 Ja—1/1 ***1*** €0 ***a***
vol

from which

We = "s ***a²p²s*** *ln*(B/A) ***

'***nL*** *once again.*

This expression takes on a more familiar form if we recognize the total charge on the inner conductor as ***Q*** = ***InaLps****. Combining this with the potential difference between the cylinders,* VA, *we see that*

We = i ***QV***a

which should be familiar as the energy stored in a capacitor.

The question of where the energy is stored in an electric field has not yet been answered. Potential energy can never be pinned down precisely in terms of physical location. Someone lifts a pencil, and the pencil acquires potential energy. Is the energy stored in the molecules of the pencil, in the gravitational field between the pencil and the earth, or in some obscure place? Is the energy in a capacitor stored in the charges themselves, in the field, or where? No one can offer any

(45)

proof for his or her own private opinion, and the matter of deciding may be left to the philosophers.

Electromagnetic field theory makes it easy to believe that the energy of an electric field or a charge distribution is stored in the field itself, for if we take (45), an exact and rigorously correct expression,

$$W_E = {}^1 \quad D \bullet E \, DV$$
$$\text{vol}$$

and write it on a differential basis,

$$DW_E = {}^1 D \bullet E \, DV$$

or

$$= ID \bullet E \qquad (46)$$
$$DV \quad 2$$

we obtain a quantity 2 D • E, *which has the dimensions of an energy density, or joules per cubic meter. We know that if we integrate this energy density over the entire field-containing volume, the result is truly the total energy present, but we have no more justification for saying that the energy stored in each differential volume element* DV *is* D • E DV *than we have for looking at (43) and saying that the stored energy is* 2 ***p**$_v$* ***Vdv.*** *The interpretation afforded by (46), however, is a convenient one, and we shall use it until proved wrong.*

PROBLEMS

4.1 *The value of* E *at* ***P(p = l,*** P *= 40°,* ***z =*** *3) is given as* E = *100*a_p *— lOOap + 300*a_z*. V/m. Determine the incremental work required to move a iO-uC charge a distance of 6 urn in the direction of:* (A) a_p; ***(b)*** ap; ***(c)*** az; ***(a)*** E; ***(e)*** G = *ia* —* ***3***a_y + *4*az.

4.2 *Let* E = *400a* — 300*a_y *+ 500*a_z *V/m in the neighborhood of point* ***P(6,*** *i, —3). Find the incremental work done in moving a 4-C charge a distance of 1 mm in the direction specified by:* ***(a)*** a* + a_y + a_z; ***(b)***

4.3 *If* E = *1iO*a_p *V/m, find the incremental amount of work done in moving a 50-uC charge a distance of imm from:* ***(a) P(1,*** *i, 3) toward* Q*(i, 1, 4);* ***(b)*** Q*(i, 1, 4) toward* P*(1, i, 3).*

4.4 *Find the amount of energy required to move a 6-C charge from the origin to* P*(3, 1, —1) in the field* E = *i**a* *— 3*Y^1a_y + *4*a_z *V/m along the straight-line path* ******* *= —3*Z, ***y = **** *+ i*Z.

4.5 *Compute the value of* ***j***^{p} G • AL *for* G = *i*Ya* *with* A*(1, — 1, i) and* P*(i, 1, i) using the path:* ***(a)*** *straight-line segments* 4*(1, — 1, i) to* ***B(1,*** *1, i) to* P*(i, 1, i);* ***(b)*** *straight-line segments* 4*(1, — 1, i) to* ***C(i,*** *— 1, i) to* P*(i, 1, i).*

4.6 *Let* G = *4**a* *+ i*Za_y *+ i*Ya_z*. Given an initial point* P*(i, 1, 1) and a final point* Q*(4, 3, 1), f i n d j* G • AL *using the path:* ***(a)*** *straight line:* ***y = * —*** *1,* ***z =*** *1;* ***(b)*** *parabola:* 6Y = **^{i} *+ i,* ***z =*** *1.*

4.7 *Repeat Prob. 6 for* G = 3*Y^{1}a* + *i*Za_y.

4.8 *A point charge* Q_1 *is located at the origin in free space. Find the work done in carrying a charge* Q_I *from:* ***(a) B(r_B,6_B,p_B)*** *to* ***C(r_A,6_B,p_B)*** *with* 6 *and* P *held constant;* ***(b) C(r_A,6_B,p_B)*** *to* ***D(r_A,6_A,p_B)*** *with* ***r*** *and* P *held constant;* ***(c) D(r_A,6_A,p_B)*** *to* ***A(r_A,6_A,p_A)*** *with* ***r*** *and* 6 *held constant.*

4.9 *A uniform surface charge density of iO nC/m*^{i} *is present on the spherical surface* ***r =*** *0.6cm in free space.* ***(a)*** *Find the* *absolute potential at **P(r =** 1cm, 6 = i5°, P = 50°). **(b)** Find* V_{AB}*, given points* A*(icm, 6 = 30°,* P *= 60°) and* B*(3cm, 45°,90°).*

4.10 *Given a surface charge density of 8nC/m^i on the plane line charge density of 30nC/m on the line *** =** 1, **y =** i, and a 1-uC point charge at **P(—**1, —1, i), find **V_{AB}** for points* A*(3, 4, o) and* B*(4, 0, 1).*

4.11 *Let a uniform surface charge density of 5nC/m^i be present at the **z =** 0 plane, a uniform line charge density of 8 nC/m be located at *** =** 0, **z =** 4, and a point charge of i uC be present at* P*(i, 0, 0).If **V =** 0 a t **M(0,** 0, 5), find **V** at **N(1,** i, 3).*

4.12 *Three point charges, 0.4uC each, are located at (0, 0, —1), (0, 0, 0), and (0, 0, 1), in free space. **(a)** Find an expression for the absolute potential as a function of **z** along the line *** =** 0, **y =** 1. **(b)** Sketch **V(z).***

4.13 Three identical point charges of 4pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them?

4.14 *Two 6-nC point charges are located at (1, 0, 0) and (—1, 0, 0) in free space.* (A) *Find* V *at **P(0,** 0,* Z). (B) *Find* V_{MAX}. (C) *Calculate* |DV/DZ| *on the* Z *axis.* (D) *Find* $|DV/DZ|_{MAX}$.

4.15 *Two uniform line charges, 8 nC/m each, are located at* X *= 1,* Z *= 2, and at* X *= —1,* Y *= 2, in free space. If the potential at the origin is 100 V, find* V *at* P*(4, 1 , 3).*

4.16 *Uniform surface charge distributions of 6, 4, and 2 nC/m^2 are present at* R *= 2, 4, and 6 cm, respectively, in free space.* (A) *Assume* V *= 0 at infinity, and find* V(R). (B) *Calculate* V *at* R *= 1, 3, 5, and 7 cm.* (C) *Sketch* V *versus* R *for 1 <* R *< 10 cm.*

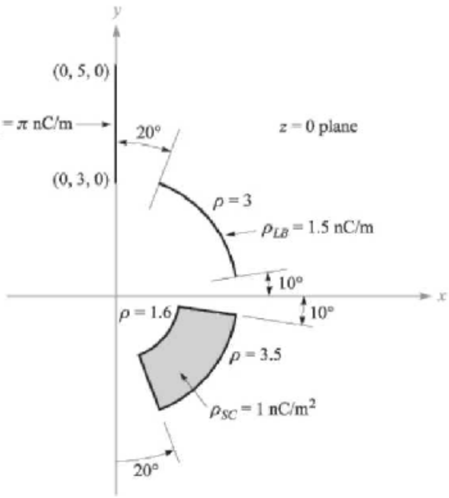
4.17 *Uniform surface charge densities of 6 and 2 nC/m^2 are present at* P *= 2 and 6 cm, respectively, in free space. Assume* V *= 0 a t* P *= 4 cm, and calculate* V *at* P *=* (A) *5 cm;* (B) *7 cm.*

4.18 *The nonuniform linear charge density,* $P_E = 8/(Z^2 + 1)$ *nC/m, lies along the* Z *axis. Find the potential at* P(P *= 1 , 0, 0) in free space if* V *= 0 a t* P = oC

4.19 *The annular surface, 1cm <* P *< 3 cm,* Z *= 0, carries the nonuniform surface charge density = 5*P*nC/m^2. Find* V *at* P*(0, 0, 2cm) if* V *= 0 at infinity.*

4.20 *Fig. 4.11 shows three separate charge distributions in the* Z *= 0 plane in free space.* (A) *Find the total charge for each distribution.* (B) *Find the potential at* P*(0, 0, 6) caused by each of the three charge distributions acting alone.* (C) *Find* V_P.

FIGURE
4 11



4.21 *Let **V = i*y^i z^3** + 3ln(*^l + iy^i + 3z^i) V in free space. Evaluate each of the following quantities at* P*(3, i, —1): **(a)*** V; (b)| ***V***|*; **(c)*** E; ^ *l* E*l;* ***(e)*** a_N*;* ***(f)*** D.

4.22 *It is known that the potential is given as **V =** 8Or*^{O6}*V. Assuming free-space conditions, find: **(a)*** E; ***(b)** the volume charge density at **r** = 0.5m; **(c)** the total charge lying within the surface **r** = 0.6.*

4.23 *It is known that the potential is given as **V =** 8O*P^{O6} *V. Assuming free-space conditions, find: **(a)*** E; ***(b)** the volume charge density at* P *= 0.5m; **(c)** the total charge lying within the closed surface* P *= 0.6, O < **z** < 1.*

4.24 *Given the potential field **V** = 8O*r^i*cos6 and a point* P*(i.5,* 6 *= 30°,* P *= 60°) in free space, find at* P: ***(a)** V; **(b)*** E; ***(c)*** D; ***(a)*** P_V; ***(e)** aV/aN;* ***(f)*** aN

4.25 *Within the cylinder* P *= i, O < **z** < 1, the potential is given by **V =** 1OO **+** 5O*P **+** *15O*P *sin* P *V. **(a)** Find **V**,* E, D, *and* P_V *at* P*(1, 60°, 0.5) in free space. **(b)** How much charge lies within the cylinder?*

4.26 *A dipole having **Qa/(4ne$_O$) =** 1OOV ■ m^i is located at the origin in free space and aligned so that its moment is in the* az *direction. **(a)** Sketch | **V (r** = 1 ,* 6 *,* P *= O)| versus* 6 *on polar graph paper (homemade if you wish). **(b)** Sketch* |E(r *= 1,* 6 *,* P *= O)| versus* 6 *on polar paper.*

4.27 *Two point charges, 1 nC at (O, O, O.1) and — 1nC at (O, O, — O.1), are in free space. **(a)** Calculate **V** at* P*(O.3, O, O.4), **(b)** Calculate* |E| *at* P. ***(c)** Now treat the two charges as a dipole at the origin and find **V** at* P.

4.28 *A dipole located at the origin in free space has a moment* p *= i **x** 1O*$^{-9}$$a_z$ *C ■ m. At what points on the line **y = z,** * = O is: **(a)*** |E$_6$***^** 1mV/m? **(b)^,.^** 1mV/m?*

4.29 *A dipole having a moment* p *= 3*a* *—* 5a_y **+** *1O*a_z *nC ■ m is located at* Q*(1, i, —4) in free space. Find **V** at* P*(i, 3, 4).*

4.30 *A dipole, having a moment of* p *=* iaz *nC ■ m, is located at the origin in free space. Give the magnitude of* E *and its direction* a_E *in cartesian components at **r** = 1OOm,* P *= 90°, and* 6 *=: **(a)** O°; **(b)** 30°; **(c)** 90°.*

4.31 *A potential field in free space is expressed as **V = iO/(*yz) V. (a)** Find the total energy stored within the cube 1 < *, **y, z** < i. **(b)** What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube?*

4.32 *In the region of free space where i < **r** < 3, O.4*N *<* 6 *< O.6*N,

k

O < P < N /i, let E = -it a_r. *(a)* Find a positive value for *k* so that the

ri

total energy stored is exactly 1 J. *(b)* Show that the surface 6 = O.6N is an equipotential surface. *(c)* Find V_{AB} given points A(i, 6 = N/i, P = N/3) and B(3, N/i, N/4).

4.33 A copper sphere of radius 4 cm carries a uniformly distributed total charge of 5 uC on its surface in free space. *(a)* Use Gauss's law to find D external to the sphere. *(b)* Calculate the total energy stored in the electrostatic field. *(c)* Use $W_E = Q^i/(iC)$ to calculate the capacitance of the isolated sphere.

CHAPTER FIVE

CONDUCTORS, DIELECTRICS, AND CAPACITANCE

5.1 CURRENT AND CURRENT DENSITY

Electric charges in motion constitute a **current.** The unit of current is the ampere (A), defined as a rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I, and therefore

Current is thus defined as the motion of positive charges, even though conduction in metals takes place through the motion of electrons, as we shall see shortly.

In field theory we are usually interested in events occurring at a point rather than within some large region, and we shall find the concept of **current density,** measured in amperes per square meter (A/m^2), more useful. Current density is a vector[13] represented by *J.*

The increment of current AI crossing an incremental surface **AS** normal to the current density is

AI = $\mathbf{J}_N$ **AS**

and in the case where the current density is not perpendicular to the surface,

AI = *J*•A*S*

Total current is obtained by integrating,

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}}$$

(2)

Current density may be related to the velocity ofvolume charge density at a point. Consider the element of charge **AQ = p_vAv =** p_v **AS AL,** as shown in Fig. 5.1a. To simplify the explanation, let us assume that the charge element is oriented with its edges parallel to the coordinate axes, and that it possesses only an x component of velocity. In the time interval **At,** the element of charge has moved a distance Ax, as indicated in Fig. **5.1b.** We have therefore moved a charge AQ = p_v **AS Ax** through a reference plane perpendicular to the direction of motion in a time increment **At,** and the resultant current is

$$AI = \frac{AQ}{At} = P_v\, AS \frac{Ax}{At}$$

As we take the limit with respect to time, we have

$$AI = P_v\, ASv_x$$

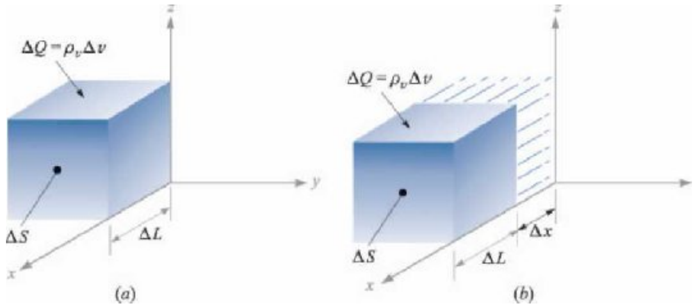


FIGURE 5.1
An increment of charge, AQ = p_vAS AL, which moves a distance Ax in a time At, produces a component of current density in the limit of $J_x = p_v v_x$.

where v_x represents the x component of the velocity *v*.[14] In terms of current density, we find

$$J_x = P_v v_x$$

and in general

$$J = \rho_v v \tag{3}$$

This last result shows very clearly that charge in motion constitutes a current. We call this type of current a **convention current,** and J or $\rho_v v$ is the **convection current density.** Note that the convection current density is related linearly to charge density as well as to velocity. The mass rate of flow of cars (cars per square foot per second) in the Holland Tunnel could be increased either by raising the density of cars per cubic foot, or by going to higher speeds, if the drivers were capable of doing so.

5.2 CONTINUITY OF CURRENT

Although we are supposed to be studying static fields at this time, the introduction of the concept of current is logically followed by a discussion of the conservation of charge and the continuity equation. The principle of conservation of charge states simply that charges can be neither created nor destroyed, although equal amounts of positive and negative charge may be **simultaneously** created, obtained by separation, destroyed, or lost by recombination.

The continuity equation follows fromthis principle when we consider any region bounded by a closed surface. The current through the closed surface is

$$I = \oint_S J \cdot dS$$

and this **outward flow** of positive charge must be balanced by a decrease of positive charge (or perhaps an increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease is $-dQ_i/dt$ and the principle of conservation of charge requires

$$I = \oint_S J \cdot dS = -\frac{dQ_i}{dt} \tag{4}$$

It might be well to answer here an often-asked question. "Isn't there a sign error? I thought $I = dQ/dt$." The presence or absence of a negative sign depends on what current and charge we consider. In circuit theory we usually associate the current flow **into** one terminal of a capacitor with the time rate of increase of charge on that plate. The current of (4), however, is an **outward-flowing** current.

Equation (4) is the integral formof the continuity equation, and the differential, or point, formis obtained by using the divergence theoremto change the surface integral into a volume integral:

$$\oint_S J \cdot dS = \int_{vol} (\nabla \cdot J)\, dv$$

We next represent the enclosed charge Q_i by the volume integral of the charge density,

$$\int_{vol} (\nabla \cdot J)\, dv = -\frac{d}{dt}\int_{vol} \rho_v\, dv$$

If we agree to keep the surface constant, the derivative becomes a partial derivative and may appear within the integral,

$$\int_{vol} (\nabla \cdot J)\, dv = \int_{vol} -\frac{\partial \rho_v}{\partial t}\, dv$$

* *vol* * *vol* ^ t

Since the expression is true for any volume, however small, it is true for an incremental volume,

(V • *J*) Av = — ^ Av
dt

fromwhich we have our point formof the continuity equation,

(V •

J) = dt

Remembering the physical interpretation of divergence, this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

As a numerical example illustrating some of the concepts from the last two sections, let us consider a current density that is directed radially outward and decreases exponentially with time,

J = - e $^{t}a_r$ A/m^2

Selecting an instant of time t = 1 s, we may calculate the total outward current at r = 5 m:

/ = J$_r$S = (i e^{-1}) (4tt5^2) = 23.1 A

At the same instant, but for a slightly larger radius, r = 6 m, we have

/ = J$_R$S = (6 e^{-1})(4jr6^2) = 27.7 A

Thus, the total current is larger at r = 6 than it is at r = 5.
To see why this happens, we need to look at the volume charge density and the velocity. We use the continuity equation first:

$$\frac{\partial \rho_v}{\partial t} = \nabla \cdot \mathbf{J} = \nabla \cdot \left(\frac{1}{r} e^{-t} \mathbf{a}_r\right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r} e^{-t}\right) = \frac{1}{r^2} e^{-t}$$

We next seek the volume charge density by integrating with respect to t. Since ρ_v is given by a partial derivative with respect to time, the "constant" of integration may be a function of r:

$$\rho_v = -\int \frac{1}{r^2} e^{-t}\, dt + K(r) = \frac{1}{r^2} e^{-t} + K(r)$$

If we assume that ρ_v O as t -—oo, then K(r) = 0, and

Pv = r2 e T C/m^3 We may now use J = $\rho_v v$ to find the velocity,

R Pv r

$\frac{1}{r^2} e^{-t}$ m/s

The velocity is greater at r = 6 than it is at r = 5, and we see that some (unspecified) force is accelerating the charge density in an outward direction.

In summary, we have a current density that is inversely proportional to **r**, a charge density that is inversely proportional to r^2, and a velocity and total current that are proportional to r. All quantities vary as $\mathbf{e^{-t}}$.

5.3 METALLIC CONDUCTORS

Physicists today describe the behavior of the electrons surrounding the positive atomic nucleus in terms of the total energy of the electron with respect to a zero reference level for an electron at an infinite distance fromthe nucleus. The total energy is the sumof the kinetic and potential energies, and since energy must be given to an electron to pull it away fromthe nucleus, the energy of every electron in the atom is a negative quantity. Even though the picture has some limitations, it is convenient to associate these energy values with orbits surrounding the nucleus, the more negative energies corresponding to orbits of smaller radius. According to the quantumtheory, only certain discrete energy levels, or energy states, are permissible in a given atom, and an electron must therefore absorb or emit discrete amounts of energy, or quanta, in passing from one level to another. A normal atom at absolute zero temperature has an electron occupying every one of the lower energy shells, starting outward fromthe nucleus and continuing until the supply of electrons is exhausted.

In a crystalline solid, such as a metal or a diamond, atoms are packed closely together, many more electrons are present, and many more permissible energy levels are available because of the interaction forces between adjacent atoms. We find that the energies which may be possessed by electrons are grouped into broad ranges, or "bands," each band consisting of very numerous, closely spaced, discrete levels. At a temperature of absolute zero, the normal solid also has every level occupied, starting with the lowest and proceeding in order until all the electrons are located. The electrons with the highest (least negative) energy levels, the valence electrons, are located in the **valence band.** If there are permissible higher-energy levels in the valence band, or if the valence band merges smoothly into a **conduction band,** then additional kinetic energy may be given to the valence electrons by an external field, resulting in an electron flow. The solid is called a **metallic conductor.** The filled valence band and the unfilled conduction band for a conductor at 0 K are suggested by the sketch

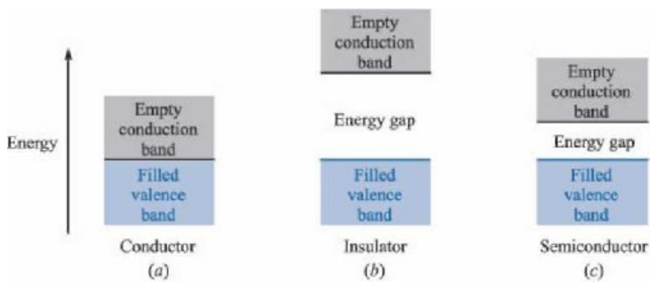


FIGURE 5.2
The energy-band structure in three different types of materials at 0 K. (a) The conductor exhibits no energy gap between the valence and conduction bands. (b) The insulator shows a large energy gap. (c) The semiconductor has only a small energy

in Fig. 5.2a.

If, however, the electron with the greatest energy occupies the top level in the valence band and a gap exists between the valence band and the conduction band, then the electron cannot accept additional energy in small amounts, and the material is an insulator. This band structure is indicated in Fig. 5.2b. Note that if a relatively large amount of energy can be transferred to the electron, it may be sufficiently excited to jump the gap into the next band where conduction can occur easily. Here the insulator breaks down.

An intermediate condition occurs when only a small "forbidden region" separates the two bands, as illustrated by Fig. 5.2c. Small amounts of energy in the formof heat, light, or an electric field may raise the energy of the electrons at the top of the filled band and provide the basis for conduction. These materials are insulators which display many of the properties of conductors and are called **semiconductors.**

Let us first consider the conductor. Here the valence electrons, or **conduc-tion,or free,** electrons, move under the influence of an electric field. With a field *E,* an electron having a charge Q = —e will experience a force

$$F = -eE$$

In free space the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. This velocity v_d is termed the **drift velocity,** and it is linearly related to the electric field intensity by the **mobility** of the electron in the given material. We designate mobility by the symbol /z (mu), so that

$$V_D = -/tEE \qquad (6)$$

where $/z_e$ is the mobility of an electron and is positive by definition. Note that the electron velocity is in a direction opposite to the direction of *E.* Equation (6) alsoshows that mobility is measured in the units of square meters per volt-second; typical values[15] are 0.0012 for aluminum, 0.0032 for copper, and 0.0056 for silver.

For these good conductors a drift velocity of a few inches per second is sufficient to produce a noticeable temperature rise and can cause the wire to melt if the heat cannot be quickly removed by thermal conduction or radiation.

Substituting (6) into Eq. (3) of Sec. 5.1, we obtain

[15] Wert and Thomson, p. 238, listed in the Suggested References at the end of this chapter.

$$J = -\rho_E/X_E E \qquad (7)$$

where ρ_e is the free-electron charge density, a negative value. The total charge density ρ_v is zero, since equal positive and negative charge is present in the neutral material. The negative value of ρ_e and the minus sign lead to a current density *J* that is in the same direction as the electric field intensity *E.*

The relationship between *J* and *E* for a metallic conductor, however, is also specified by the conductivity **a** (sigma),

$$J = aE \qquad (8)$$

[16] per meter (S/m). One siemens (1 S) is the basic unit of conductance in the SI system, and is defined as one ampere per volt. Formerly, the unit of conductance was called the mho and symbolized by an **inverted** Just as the siemens honors the Siemens brothers, the reciprocal unit of resistance which we call the ohm(1 £ is one volt per ampere) honors Georg Simon Ohm, a German physicist who first described the current-voltage relationship implied by (8). We call this equation the **point form of Ohm's law;** we shall look at the more common form of Ohm's law shortly.

First, however, it is informative to note the conductivity of several metallic conductors; typical values (in siemens per meter) are 3.82×10^7 for aluminum, 5.80×10^7 for copper, and 6.17×10^7 for silver. Data for other conductors may be found in Appendix C. On seeing data such as these, it is only natural to assume that we are being presented with **constant** values; this is essentially true. Metallic conductors obey Ohm's law quite faithfully, and it is a **linear** relationship; the conductivity is constant over wide ranges of current density and electric field intensity. Ohm's law and the metallic conductors are also described as **isotropic,** or having the same properties in every direction. A material which is not isotropic is called **anisotropic,** and we shall mention such a material a few pages from now.

[16] This is the family name of two German-born brothers, Karl Wilhelm and Werner von Siemens, who were famous engineer-inventors in the nineteenth century. Karl became a British subject and was knighted, becoming Sir William Siemens.

The conductivity is a function of temperature, however. The resistivity, which is the reciprocal of the conductivity, varies almost linearly with temperature in the region of room temperature, and for aluminum, copper, and silver it increases about 0.4 percent for a 1K rise in temperature.[17] For several metals the resistivity drops abruptly to zero at a temperature of a few kelvin; this property is termed **superconductivity.** Copper and silver are not superconductors, although aluminum is (for temperatures below 1.14 K).

If we now combine (7) and (8), the conductivity may be expressed in terms of the charge density and the electron mobility,

(9)

Fromthe definition of mobility (6), it is now satisfying to note that a higher temperature infers a greater crystalline lattice vibration, more impeded electron progress for a given electric field strength, lower drift velocity, lower mobility, lower conductivity from(9), and higher resistivity as stated.

The application of Ohm's law in point form to a macroscopic (visible to the naked eye) region leads to a more familiar form. Initially, let us assume that J and E are **uniform,** as they are in the cylindrical region shown in Fig. 5.3. Since they are uniform,

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = \mathbf{J}\,\mathbf{S} \qquad (10)$$

[17] **Copious temperature data for conducting materials are available in the "Standard Handbook for Electrical Engineers," listed among the Suggested References at the end of this chapter.**

and **V**ab = E. dL = -E. **dL** = – E. — J

= E. L_{AB} (11)

or

V = EL

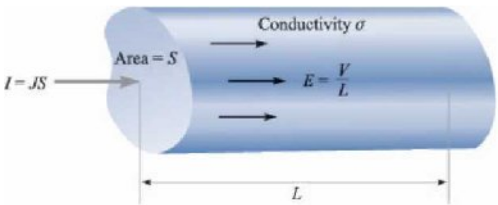


FIGURE 5.3 **Uniformcurrent density *J* and electric field intensity *E* in a cylindrical region of length L and cross-sectional area S. Here V =**

Thus

J 1 E V
S = aE = ° **1**

or

V

1
0
S'

The ratio of the potential difference between the two ends of the cylinder to the current entering the more positive end, however, is recognized from elementary circuit theory as the **resistance** of the cylinder, and therefore

$$\boxed{V = \mathbf{IR}} \qquad (12)$$

where

$$\boxed{R = \frac{1}{\mathbf{aS}}} \qquad (13)$$

Equation (12) is, of course, known as **Ohm's law,** and (13) enables us to compute the resistance R, measured in ohms (abbreviated as £2), of conducting objects which possess uniform fields. If the fields are not uniform, the resistance may still be defined as the ratio of **V** to **I**, where **V** is the potential difference between two specified equipotential surfaces in the material and **I** is the total current crossing the more positive surface into the material. From the general integral relationships in (10) and (11), and from Ohm's law (8), we may write this general expression for resistance when the fields are nonuniform,

R **Kb I**
Js
aE • dS (14)

The line integral is taken between two equipotential surfaces in the conductor, and the surface integral is evaluated over the more positive of these two equi-potentials. We cannot solve these nonuniform problems at this time, but we should be able to solve several of themafter perusing Chaps. 6 and 7.

5.4 CONDUCTOR PROPERTIES AND BOUNDARY CONDITIONS

Once again we must temporarily depart from our assumed static conditions and let time vary for a few microseconds to see what happens when the charge distribution is suddenly unbalanced within a conducting material. Let us suppose, for the sake of the argument, that there suddenly appear a number of electrons in the interior of a conductor. The electric fields set up by these electrons are not counteracted by any positive charges, and the electrons therefore begin to accelerate away fromeach other. This continues until the electrons reach the surface of the conductor or until a number of electrons equal to the number injected have reached the surface.

Here the outward progress of the electrons is stopped, for the material surrounding the conductor is an insulator not possessing a convenient conduction band. No charge may remain within the conductor. If it did, the resulting electric field would force the charges to the surface.

Hence the final result within a conductor is zero charge density, and a surface charge density resides on the exterior surface. This is one of the two characteristics of a good conductor.

The other characteristic, stated for static conditions in which no current may flow, follows directly from Ohm's law: the electric field intensity within the conductor is zero. Physically, we see that if an electric field were present, the conduction electrons would move and produce a current, thus leading to a non-static condition.

Summarizing for electrostatics, no charge and no electric field may exist at any point **within** a conducting material. Charge may, however, appear on the surface as a surface charge density, and our next investigation concerns the fields **external** to the conductor.

We wish to relate these external fields to the charge on the surface of the conductor. The problem is a simple one, and we may first talk our way to the solution with little mathematics.

If the external electric field intensity is decomposed into two components, one tangential and one normal to the conductor surface, the tangential component is seen to be zero. If it were not zero, a tangential force would be applied to the elements of the surface charge, resulting in their motion and nonstatic conditions. Since static conditions are assumed, the tangential electric field intensity and electric flux density are zero.

Gauss's law answers our questions concerning the normal component. The electric flux leaving a small increment of surface must be equal to the charge residing on that incremental surface. The flux cannot penetrate into the conductor, for the total field there is zero. It must then leave the surface normally. Quantitatively, we may say that the electric flux density in coulombs per square meter leaving the surface normally is equal to the surface charge density in coulombs per square meter, or **= p_s.**

If we use some of our previously derived results in making a more careful analysis (and incidentally introducing a general method which we must use later), we should set up a conductor-free space boundary (Fig. 5.4) showing tangential and normal components of D and E on the free-space side of the boundary. Both fields are zero in the conductor. The tangential field may be determined by applying Sec. 4.5, Eq. (21), around the small closed path

$$E \cdot dL = 0$$

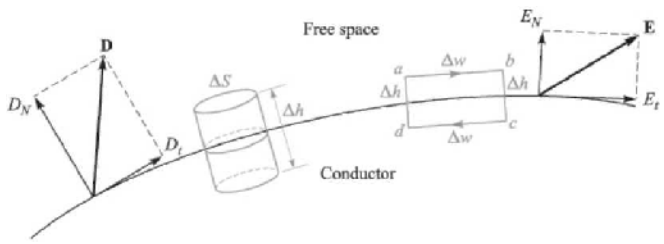


FIGURE 5.4
An appropriate closed path and gaussian surface are used to determine boundary conditions at a conductor-free space boundary; E, = 0 and D« = ρ,.

abcda. The integral must be broken up into four parts

fb a

$$\int_a \quad \int_b \quad \int_c \quad \int_d = 0$$

Remembering that $E = 0$ within the conductor, we let the length from **a** to b or c to d be Aw and from b to c or d to **a** be Ah, and obtain

E,Aw — E^*at b 2* **Ah** + 0
E^*at a 5* **Ah**

As we allow Ah to approach zero, keeping Aw small but finite, it makes no difference whether or not the normal fields are equal at a and b, for Ah causes these products to become negligibly small. Hence

$$E_t Aw = 0$$

and therefore

$$\mathbf{E_t} = 0$$

The condition on the normal field is found most readily by considering D_N rather than $\mathbf{E_N}$ and choosing a small cylinder as the gaussian surface. Let the height be Ah and the area of the top and bottomfaces be AS. Again we shall let Ah approach zero. Using Gauss's law,

$$\oint_S D \cdot dS = Q$$

we integrate over the three distinct surfaces

$$\int_{top} + \int_{bottom} + \int_{sides} = Q$$

and find that the last two are zero (for different reasons). Then

$$D_N \Delta S = Q = \rho_S \Delta S$$

or

$$D_N = \rho_S$$

These are the desired **boundary conditions** for the conductor-free space boundary in electrostatics,

(15)

$$= \epsilon_0 E_N = \rho_S$$

The electric flux leaves the conductor in a direction normal to the surface, and the value of the electric flux density is numerically equal to the surface charge density.

An immediate and important consequence of a zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface. The evaluation of the potential difference between any two points on the surface by the line integral leads to a zero result, because the path may be chosen on the surface itself where $E \cdot dL = 0$.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

Using these three principles, there are a number of quantities that may be calculated at a conductor boundary, given a knowledge of the potential field.

5.5 THE METHOD OF IMAGES

One important characteristic of the dipole field that we developed in the last chapter is the infinite plane at zero potential that exists midway between the two charges. Such a plane may be represented by a vanishingly thin conducting plane that is infinite in extent. The conductor is an equipotential surface at a potential V = 0, and the electric field intensity is therefore normal to the surface. Thus, if we replace the dipole configuration shown in Fig. 5.6a with the single charge and conducting plane shown in Fig. 5.6b, the fields in the upper half of each figure are the same. Below the conducting plane, all fields are zero since we have not provided any charges in that region. Of course, we might also substitute a single negative charge below a conducting plane for the dipole arrangement and obtain equivalence for the fields in the lower half of each region.

If we approach this equivalence fromthe opposite point of view, we begin with a single charge above a perfectly conducting plane and then see that we may maintain the same fields above the plane by removing the plane and locating a negative charge at a symmetrical location below the plane. This charge is called the **image** of the original charge, and it is the negative of that value.

If we can do this once, linearity allows us to do it again and again, and thus **any** charge configuration above an infinite ground plane may be replaced by an

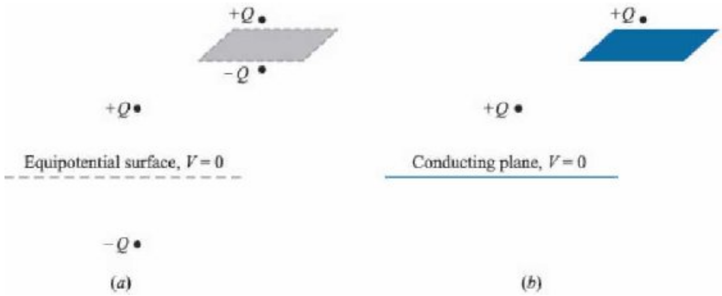


FIGURE 5.6

(a) Two equal but opposite charges may be replaced by (b) a single charge and a conducting plane without affecting the fields above the V = 0 surface.

arrangement composed of the given charge configuration, its image, and no conducting plane. This is suggested by the two illustrations of Fig. 5.7. In many cases, the potential field of the new system is much easier to find since it does not contain the conducting plane with its unknown surface charge distribution.

As an example of the use of images, let us find the surface charge density at P(2, 5, 0) on the conducting plane z = 0 if there is a line charge of 30 nC/m located at x = 0, z = 3, as shown in Fig. 5.8a. We remove the plane and install an image line charge of — 30nC/mat x = 0, z = —3, as illustrated in Fig. 5.8b. The field at P may now be obtained by superposition of the known fields of the line charges. The radial vector fromthe positive line charge to P is $R_+ = 2a_x - 3a_z$, while $R_- = 2a_x + 3a_z$. Thus, the individual fields are

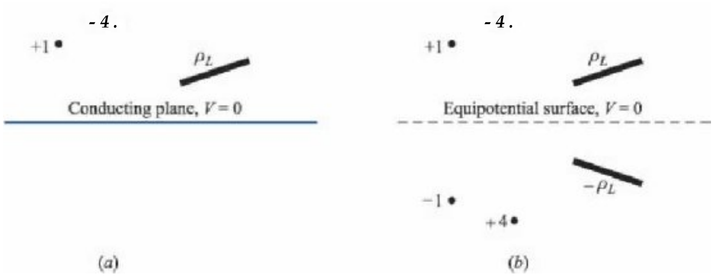


FIGURE 5.7

(a) A given charge configuration above an infinite conducting plane may be replaced by (b) the given charge configuration plus the image configuration,

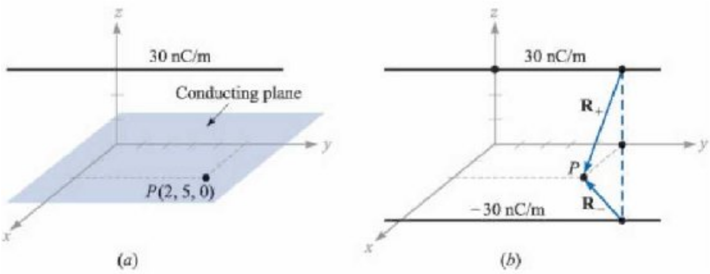


FIGURE 5.8
(*a*) A line charge above a conducting plane. (*b*) The conductor is removed, and the image of the line charge is added.

pL 30 x 10^{-9} $2a$, - $3a_z$

2jt€oR+ $^+$ **2Tze$_0$+J\3**

VT3 30 x 10^{-9} $2a_x$ + $3a_z$

and

E

27reoV13 V13 Adding these results, we have

„ -180 x $10^{-9}a_z$

$E = -_2$ = -249a_z V/m

This then is the field at (or just above) **P** in both the configurations of Fig. 5.8, and it is certainly satisfying to note that the field is normal to the conducting plane, as it must be. Thus, D = e$_0E$ = -2.20a_z nC/m^2, and since this is directed **toward** the conducting plane, **ps** is negative and has a value of -2.20 nC/m^2 at P.

5.6 SEMICONDUCTORS

If we now turn our attention to an intrinsic semiconductor material, such as pure germanium or silicon, two types of current carriers are present, electrons and **holes.** The electrons are those fromthe top of the filled valence band which have received sufficient energy (usually thermal) to cross the relatively small forbidden band into the conduction band. The forbidden-band energy gap in typical semiconductors is of the order of one electronvolt. The vacancies left by these electrons represent unfilled energy states in the valence band which may also move fromatomto atomin the crystal. The vacancy is called a **hole,** and many semiconductor properties may be described by treating the hole as if it had a positive charge of e, a mobility, /u.A, and an effective mass comparable to that of the electron. Both carriers move in an electric field, and they move in opposite directions; hence each contributes a component of the total current which is in the same direction as that provided by the other. The conductivity is therefore a function of both hole and electron concentrations and mobilities,

(17)

For pure, or **intrinsic,** silicon the electron and hole mobilities are 0.12 and 0.025, respectively, while for germanium, the mobilities are, respectively, 0.36 and 0.17. These values are given in square meters per volt-second and range from 10 to 100 times as large as those for aluminum, copper, silver, and other metallic conductors.[6] The mobilities listed above are given for a temperature of 300 K.

The electron and hole concentrations depend strongly on temperature. At 300 K the electron and hole volume charge densities are both 0.0024 C/m^3 in magnitude in intrinsic silicon and 3.0 C/m^3 in intrinsic germanium. These values lead to conductivities of 0.000 35 S/min silicon and 1.6S/min germanium. As temperature increases, the mobilities decrease, but the charge densities increase very rapidly. As a result, the conductivity of silicon increases by a factor of 10 as the temperature increases from 300 to about 330 K and decreases by a factor of 10 as the temperature drops from 300 to about 275 K. Note that the conductivity ofthe intrinsic semiconductor increases with temperature, while that ofa metallic conductor decreases with temperature; this is one ofthe characteristic differences between the metallic conductors and the intrinsic semiconductors.

Intrinsic semiconductors also satisfy the point form of Ohm's law; that is, the conductivity is reasonably constant with current density and with the direction of the current density.

The number of charge carriers and the conductivity may both be increased dramatically by adding very small amounts of impurities. **Donor** materials provide additional electrons and form **n-type** semiconductors, while **acceptors** furnish extra holes and form **p-type** materials. The process is known as **doping,** and a donor concentration in silicon as low as one part in 10^7 causes an increase in conductivity by a factor of 10^5.

The range of value of the conductivity is extreme as we go from the best insulating materials to semiconductors and the finest conductors. In siemens per meter, a ranges from10^{-17} for fused quartz, 10^{-7} for poor plastic insulators, and roughly unity for semiconductors to almost 10^8 for metallic conductors at room temperature. These values cover the remarkably large range of some 25 orders of magnitude.

5.7 THE NATURE OF DIELECTRIC MATERIALS

Although we have mentioned insulators and dielectric materials, we do not as yet have any quantitative relationships in which they are involved. We shall soon see, however, that a dielectric in an electric field can be viewed as a free-space arrangement of microscopic electric dipoles which are composed of positive and negative charges whose centers do not quite coincide.

These are not free charges, and they cannot contribute to the conduction process. Rather, they are bound in place by atomic and molecular forces and can only shift positions slightly in response to external fields. They are called **bound** charges, in contrast to the free charges that determine conductivity. The bound charges can be treated as any other sources of the electrostatic field. If we did not wish to, therefore, we would not need to introduce the dielectric constant as a new parameter or to deal with permittivities different from the permittivity of free space; however, the alternative would be to consider **every charge within a piece of dielectric material.** This is too great a price to pay for using all our previous equations in an unmodified form, and we shall therefore spend some time theorizing about dielectrics in a qualitative way; introducing polarization *P*, permittivity e, and relative permittivity **eR;** and developing some quantitative relationships involving these new quantities.

The characteristic which all dielectric materials have in common, whether they are solid, liquid, or gas, and whether or not they are crystalline in nature, is their ability to store electric energy. This storage takes place by means of a shift in the relative positions of the internal, bound positive and negative charges against the normal molecular and atomic forces.

This displacement against a restraining force is analogous to lifting a weight or stretching a spring and represents potential energy. The source of the energy is the external field, the motion of the shifting charges resulting perhaps in a transient current through a battery which is producing the field.

The actual mechanism of the charge displacement differs in the various dielectric materials. Some molecules, termed **polar** molecules, have a permanent displacement existing between the centers of "gravity" of the positive and negative charges, and each pair of charges acts as a dipole. Normally the dipoles are oriented in a randomway throughout the interior of the material, and the action of the external field is to align these molecules, to some extent, in the same direction. A sufficiently strong field may even produce an additional displacement between the positive and negative charges.

A **nonpolar** molecule does not have this dipole arrangement until after a field is applied. The negative and positive charges shift in opposite directions against their mutual attraction and produce a dipole which is aligned with the electric field.

Either type of dipole may be described by its dipole moment *p*, as developed in Sec. 4.7, Eq. (37),

$$P = Qd \qquad (18)$$

where Q is the positive one of the two bound charges composing the dipole, and *d* is the vector fromthe negative to the positive charge. We note again that the units of *p* are coulomb-meters.

If there are n dipoles per unit volume and we deal with a volume Au, then there are n Au dipoles, and the total dipole moment is obtained by the vector sum,

$$p_{total} = \sum^{N\,AU} p_i$$

If the dipoles are aligned in the same general direction, p_{total} may have a significant value. However, a randomorientation may cause p_{total} to be essentially zero.

We now define the polarization *P* as the **dipole moment per unit volume,**

$$P = \lim 7\text{-y}] \sum_{I=1}^{N\,AU} P_I \qquad (I^9)$$

with units ofcoulombs per square meter. We shall treat *P* as a typical continuous field, even though it is obvious that it is essentially undefined at points within an atomor molecule. Instead, we should think of its value at any point as an average value taken over a sample volume Au—large enough to contain many molecules (n Au in number), but yet sufficiently small to be considered incremental in concept.

Our immediate goal is to show that the bound volume charge density acts like the free volume charge density in producing an external field; we shall obtain a result similar to Gauss's law.

To be specific, let us assume that we have a dielectric containing nonpolar molecules. No molecule has a dipole moment, and *P* = O throughout the material. Somewhere in the interior of the dielectric we select an incremental surface element A*S,* as shown in Fig. 5.9a, and apply an electric field *E.* The electric field produces a moment *p* = Q*d* in each molecule, such that *p* and *d* make an angle 6 with A*S,* as indicated in Fig. **5.9b.**

Now let us inspect the movement of bound charges across A*S*. Each of the charges associated with the creation of a dipole must have moved a distance *ı* **d** cos 6 in the direction perpendicular to A*S*. Thus, any positive charges initially lying below the surface A*S* and within the distance *i* d cos 6 of the surface must have crossed A*S* going upward. Also, any negative charges initially lying above

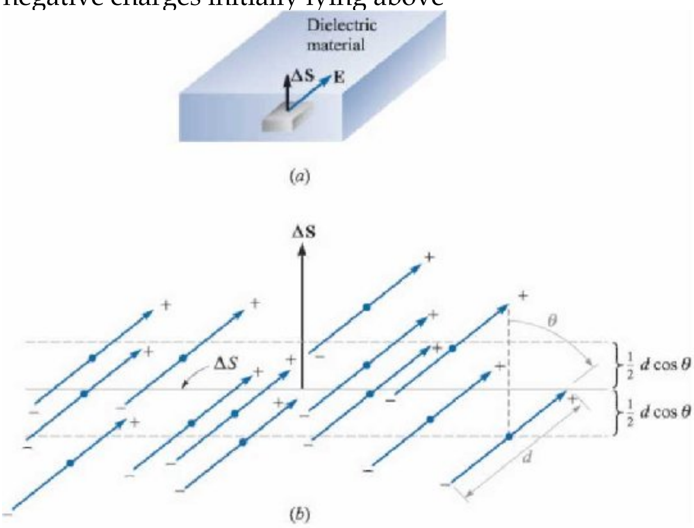


FIGURE 5.9
(*a*) An incremental surface element Δ*S* is shown in the interior of a dielectric in which an electric field **E** is present. (*b*) The nonpolar molecules form dipole moments **p** and a polarization **P**. There is a net transfer of bound charge across Δ**S**.

the surface and within that distance (i d cos 9) from AS must have crossed AS going downward. Therefore, since there are n molecules/m^3, the net total charge which crosses the elemental surface in an upward direction is equal to **nQd** cos 9AS,or

AQb = nQd ■ AS

where the subscript on Q*B* reminds us that we are dealing with a bound charge and not a free charge. In terms of the polarization, we have

AQ*B* = P ■ AS

If we interpret AS as an element of a **closed** surface inside the dielectric material, then the direction of AS is outward, and the net increase in the bound charge **within** the closed surface is obtained through the integral

$$Q_B = -\oint_S \mathbf{P} \cdot d\mathbf{S} \qquad (20)$$

This last relationship has some resemblance to Gauss's law, and we may now generalize our definition of electric flux density so that it applies to media other than free space. We first write Gauss's law in terms of $\epsilon_0 \mathbf{E}$ and Q_T, the **total** enclosed charge, bound plus free:

$$Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \qquad (21)$$

where

$$Q_T = Q_B + Q$$

and Q is the total **free** charge enclosed by the surface S. Note that the free charge appears without subscript since it is the most important type of charge and will appear in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free charge enclosed,

$$Q = Q_T - Q_B = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S} \qquad (22)$$

We may now define $\mathbf{D}$ in more general terms than we did in Chap. 3,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (23)$$

There is thus an added termto D which appears when polarizable material is present. Thus,

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S} \qquad (24)$$

where Q is the free charge enclosed.

Utilizing the several volume charge densities, we have

$$Q_B = \int_V \rho_b \, dv$$

$$Q = \int_V \rho_v \, dv$$

$$Q_T = \int_V \rho_T \, dv$$

With the help of the divergence theorem, we may therefore transform (20), (21), and (24) into the equivalent divergence relationships,

$$\nabla \cdot \mathbf{P} = -\rho_B$$

$$\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T$$

V • **D** = ρ_v

We shall emphasize only (24) and (25), the two expressions involving the free charge, in the work that follows.

In order to make any real use of these new concepts, it is necessary to know the relationship between the electric field intensity *E* and the polarization *P* which results. This relationship will, of course, be a function of the type of material, and we shall essentially limit our discussion to those isotropic materials for which *E* and *P* are linearly related. In an isotropic material the vectors *E* and *P* are always parallel, regardless of the orientation of the field. Although most engineering dielectrics are linear for moderate-to-large field strengths and are also isotropic, single crystals may be anisotropic. The periodic nature of crystalline materials causes dipole moments to be formed most easily along the crystal axes, and not necessarily in the direction of the applied field.

In **ferroelectric** materials the relationship between *P* and *E* is not only nonlinear, but also shows hysteresis effects; that is, the polarization produced by a given electric field intensity depends on the past history of the sample. Important examples of this type of dielectric are barium titanate, often used in ceramic capacitors, and Rochelle salt.

The linear relationship between *P* and *E* is

(26)

where xe (chi) is a dimensionless quantity called the **electric susceptibility** of the material.

Using this relationship in (23), we have

D = eo*E* + X*E*^o*E* = (x*E* + l)eo*E* The expression within the parentheses is now defined as

$$€R = \mathbf{X}e + 1 \tag{27}$$

This is another dimensionless quantity and it is known as the **relative permittivity,** or **dielectric constant** of the material. Thus,

$$D = e\hat{}E = \mathbf{eE} \tag{28}$$

where

(29)

and e is the **permittivity.** The dielectric constants are given for some representative materials in Appendix C.

Anisotropic dielectric materials cannot be described in terms of a simple susceptibility or permittivity parameter. Instead, we find that each component of *D* may be a function of every component of *E*, and *D* = e*E* becomes a matrix equation where *D* and *E* are each 3 x 1 column matrices and e is a 3 x 3 square matrix. Expanding the matrix equation gives

$$D_x - e_{xx}E_x \quad e_{xy}E_y -\backslash\sim e_{xz}E_z$$

$$D_y \quad €_{yx}\mathbf{E}x \sim\sim\backslash\sim \hat{}_{yy}\mathbf{E}y \sim\sim\backslash\sim €_{yz}\mathbf{E_z}$$

(25)

$$D_z = \epsilon_{zx}E_x \quad \epsilon_{zy}E_y \quad \epsilon_{zz}E_z$$

Note that the elements of the matrix depend on the selection of the coordinate axes in the anisotropic material. Certain choices of axis directions lead to simpler matrices.[18]

D and E (and P) are no longer parallel, and although $D = \epsilon_0 E + P$ remains a valid equation for anisotropic materials, we may continue to use $D = \epsilon \mathbf{E}$ only by interpreting it as a matrix equation. We shall concentrate our attention on linear isotropic materials and reserve the general case for a more advanced text.

In summary, then, we now have a relationship between D and E which depends on the dielectric material present,

[18] **A more complete discussion of this matrix may be found in the Ramo, Whinnery, and Van Duzer reference listed at the end of this chapter.**

D (2
— eE 8)

w (2
here 9)

This electric flux density is still related to the free charge by either the point or integral formof Gauss's law:

V (2
5)

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

• D —

(2
4)

The use of the relative permittivity, as indicated by (29) above, makes consideration of the polarization, dipole moments, and bound charge unnecessary. However, when anisotropic or nonlinear materials must be considered, the relative permittivity, in the simple scalar form that we have discussed, is no longer applicable.

Let us now illustrate these new concepts with a numerical example.

5.8 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS

How do we attack a problemin which there are two different dielectrics, or a dielectric and a conductor? This is another example of a **boundary condition,** such as the condition at the surface of a conductor whereby the tangential fields are zero and the normal electric flux density is equal to the surface charge density on the conductor. Now we take the first step in solving a two-dielectric problem, or a dielectric-conductor problem, by determining the behavior of the fields at the dielectric interface.

Let us first consider the interface between two dielectrics having permittivities e_1 and e_2 and occupying regions 1 and 2, as shown in Fig. 5.10. We first examine the tangential components by using

Le-**d***L* = 0

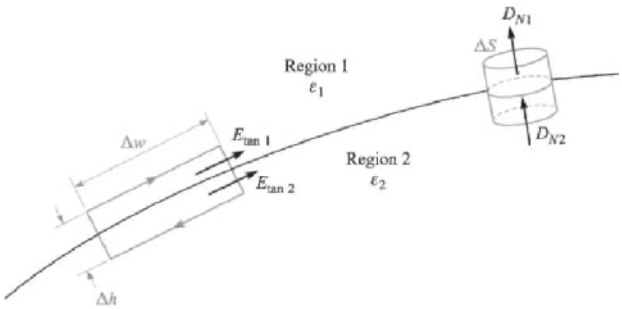


FIGURE 5.10

The boundary between perfect dielectrics of permittivities e_l and e_2. The continuity of D_N is shown by the gaussian surface on the right, and the continuity of E_{tan} by the line integral about the closed path at the

around the small closed path on the left, obtaining

$$E_{tanl}\ AW - E_{tan\ 2}\ Aw = 0$$

The small contribution to the line integral by the normal component of E along the sections of length **Ah** becomes negligible as **Ah** decreases and the closed path crowds the surface- Immediately, then,

$$E_{tan\,l} - E_{tan\ 2} \qquad (30)$$

and we might feel that Kirchhoffs voltage law is still applicable to this case. Certainly we have shown that the potential difference between any two points on the boundary that are separated by a distance A**w** is the same immediately above or below the boundary.
If the tangential electric field intensity is continuous across the boundary, then tangential D is discontinuous, for

$$\frac{D_{tanl}}{} \quad E_{tan\ l} - E_{tan\ 2} \quad \frac{D_{tan2}}{\pounds 2}$$

or

$$\frac{D_{tanl}}{\pounds l} - \frac{D_{tan2}}{e_2} \qquad (31)$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in Fig. 5.l0. The sides are again very short, and the flux leaving the top and bottomsurfaces is the differ-

ence

$$D_n iAS - D_N 2AS - AQ - \mathbf{p}_s AS$$

fromwhich

$$D_{ni} - AV_2 = \rho_S$$

What is this surface charge density? It cannot be a **bound** surface charge density, because we are taking the polarization of the dielectric into effect by using a dielectric constant different fromunity; that is,

(32)

instead of considering bound charges in free space, we are using an increased permittivity. Also, it is extremely unlikely that any **free** charge is on the interface, for no free charge is available in the perfect dielectrics we are considering. This charge must then have been placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, then, we may assume ρ_S is zero on the interface and

$$D_{N1} = D_{N2} \tag{33}$$

or the normal component of D is continuous. It follows that

$$\epsilon_1 E_{N1} = \epsilon_2 E_{N2} \tag{34}$$

and normal E is discontinuous.

These conditions may be combined to show the change in the vectors D and E at the surface. Let D_1 (and E_1) make an angle θ_1 with a normal to the surface (Fig. 5.11). Since the normal components of D are continuous,

$$D_{N1} = D_1 \cos \theta_1 = D_2 \cos \theta_2 = D_{N2} \tag{35}$$

The ratio of the tangential components is given by (31) as

$$\frac{D_{tan1}}{D_{tan2}} = \frac{D_1 \sin \theta_1}{D_2 \sin \theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or

$$\epsilon_2 D_1 \sin \theta_1 = \epsilon_1 D_2 \sin \theta_2 \tag{36}$$

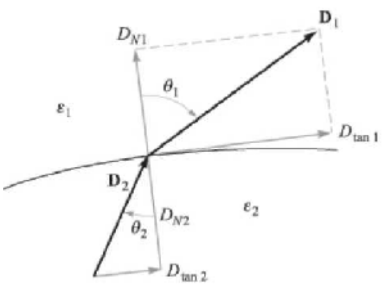


FIGURE 5.11
The refraction of *D* at a dielectric interface. For the

(32)

case shown, $e_1 > e_2$; E_1 and E_2 are directed along D_1 and D_2, with $D_1 > D_2$ and $E_1 < E_2$.

and the division of this equation by (35) gives

$$\frac{\tan 0_1}{\tan 0_2} = \frac{e_1}{e_2} \qquad (37)$$

In Fig. 5.11 we have assumed that $\epsilon_1 > \epsilon_2$, and therefore $0_1 > 0_2$. The direction of E on each side of the boundary is identical with the direction of D, because $D = eE$.

The magnitude of D in region 2 may be found from (35) and (36),

$$D_2 = D_1\sqrt{\cos^2\theta_1 - \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1} \qquad (38)$$

and the magnitude of E_2 is

$$E_2 = E_1\sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1} \qquad (39)$$

An inspection of these equations shows that D is larger in the region of larger permittivity (unless $0_1 = 0_2 = 0°$ where the magnitude is unchanged) and that E is larger in the region of smaller permittivity (unless $0_1 = 0_2 = 90°$, where its magnitude is unchanged).

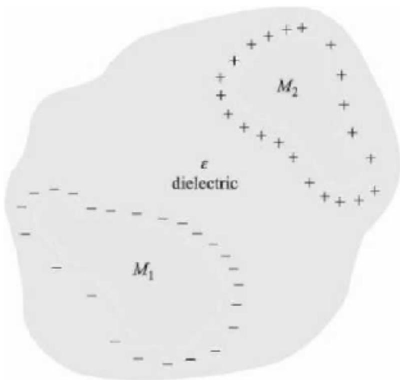
These boundary conditions, (30), (31), (33), and (34), or the magnitude and direction relations derived fromthem, (37) to (39), allow us to find quickly the field on one side of a boundary **if we know the field on the other side.** In the example we began at the end of the previous section, this was the case. Now let's finish up that problem.

5.9 CAPACITANCE

Now let us consider two conductors embedded in a homogeneous dielectric (Fig. 5.13). Conductor M_2 carries a total positive charge Q, and M_1 carries an equal

FIGURE 5.13

Two oppositely charged conductors M_1 and M_2 surrounded by a uniformdielectric. The ratio of the magnitude of the charge on either conductor to the



negative charge. There are no other charges present, and the **total** charge of the systemis zero.

We now know that the charge is carried on the surface as a surface charge density and also that the electric field is normal to the conductor surface. Each conductor is, moreover, an equipotential surface. Since M_2 carries the positive charge, the electric flux is directed from M_2 to M_1, and M_2 is at the more positive potential. In other words, work must be done to carry a positive charge from M_1 to M_2.

Let us designate the potential difference between M_2 and M_1 as V_0. We may now define the **capacitance** of this two-conductor systemas the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors,

$$\boxed{C = \frac{Q}{V_0}} \quad (42)$$

In general terms, we determine Q by a surface integral over the positive conductors, and we find V_0 by carrying a unit positive charge fromthe negative to the positive surface,

$$C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}} \quad (43)$$

The capacitance is independent of the potential and total charge, for their ratio is constant. If the charge density is increased by a factor of N, Gauss's law indicates that the electric flux density or electric field intensity also increases by N, as does the potential difference. The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the homogeneous dielectric.

Capacitance is measured in **farads** (F), where a farad is defined as one coulomb per volt. Common values of capacitance are apt to be very small fractions of a farad, and consequently more practical units are the microfarad (/zF), the nanofarad (nF), and the picofarad (pF).

We can apply the definition of capacitance to a simple two-conductor systemin which the conductors are identical, infinite parallel planes with separation d (Fig. 5.14). Choosing the lower conducting plane at 2 = 0 and the upper one at 2 = d, a uniformsheet of surface charge $\pm\rho_s$ on each conductor leads to the uniformfield [Sec. 2.5, Eq. (22)]

e

where the permittivity of the homogeneous dielectric is e, and

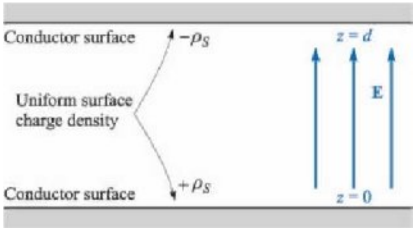


FIGURE 5.14
The problemof the parallel-plate capacitor. The capacitance per square meter of surface area is e/d.

$$D = \mathbf{p_s}\, a_z$$

The charge on the lower plane must then be positive, since *D* is directed upward, and the normal value of *D*,

$$\mathbf{D_N} = \mathbf{D}_z = \mathbf{P}_S$$

is equal to the surface charge density there. On the upper plane,

$$\mathbf{D}_N = -\mathbf{D}_z$$

and the surface charge there is the negative of that on the lower plane. The potential difference between lower and upper planes is

$$V = -\int_{\text{upper}}^{\text{lower}} E \bullet dL = - \quad — dz = — d$$

Since the total charge on either plane is infinite, the capacitance is infinite. A more practical answer is obtained by considering planes, each of area S, whose linear dimensions are much greater than their separation **d.** The electric field and charge distribution are then almost uniform at all points not adjacent to the edges, and this latter region contributes only a small percentage of the total capacitance, allowing us to write the familiar result

$$Q = \mathbf{P}_S\mathbf{S}$$

$$V = \frac{P_S}{\epsilon}\, d$$

$$o = — d\, e$$

More rigorously, we might consider (44) as the capacitance of a portion of the infinite-plane arrangement having a surface area S. Methods of calculating the effect of the unknown and nonuniformdistribution near the edges must wait until we are able to solve more complicated potential problems.

5.10 SEVERAL CAPACITANCE EXAMPLES

As a first brief example we choose a coaxial cable or coaxial capacitor of inner radius a, outer radius b, and length L. No great derivational struggle is required, because the potential difference is given as Eq. (11) in Sec. 4.3, and we find the capacitance very simply by dividing this by the total charge $\rho_L L$ in the length L. Thus,

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(b/a)}}$$

(4
6)

Next we consider a spherical capacitor formed of two concentric spherical conducting shells of radius **a** and b, b > a. The expression for the electric field was obtained previously by Gauss's law,

Q
4jter²

where the region between the spheres is a dielectric with permittivity e. The expression for potential difference was found fromthis by the line integral [Sec. 4.3, Eq. (12)]. Thus,

Vab Q_
4ite

Here Q represents the total charge on the inner sphere, and the capacitance becomes

C **4ne** 1
1
Q_
Vab

(4
7)

a b

If we allow the outer sphere to become infinitely large, we obtain the capacitance of an isolated spherical conductor,

C
= 47rea

(4
8)

For a diameter of 1 cm, or a sphere about the size of a marble,

C = 0.556 pF

in free space.

Coating this sphere with a different dielectric layer, for which e = e1, extending from r = **a** to r = **r1**,

Q

4itr²

E, (a < r <

47160,²

and the potential difference is

Qdr

4it€ır²

4it€or²

$\mathbf{V_a \sim V_x}$

1 (I _ 1) + _L

a r i) _ori 4n

Therefore, (4 9)

Q ri) (ri < r)

In order to look at the problem of multiple dielectrics a little more thoroughly, let us consider a parallel-plate capacitor of area **S** and spacing d, with the usual assumption that d is small compared to the linear dimensions of the plates. The capacitance is _i**S/d**, using a dielectric of permittivity _i. Now let us replace a part of this dielectric by another of permittivity _2, placing the boundary between the two dielectrics parallel to the plates (Fig. 5.i5).

Some of us may immediately suspect that this combination is effectively two capacitors in series, yielding a total capacitance of

C i

c*i* + C*2*

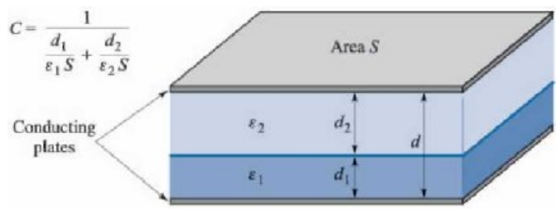


FIGURE 5.15

A parallel-plate capacitor containing two dielectrics with the dielectric interface parallel to the conducting plates.

where C_1 = $\mathbf{e_1 S/d_1}$ and C_2 = $\mathbf{e_2 S/d_2}$. This is the correct result, but we can obtain it using less intuition and a more basic approach.
Since our capacitance definition, C = Q / V, involves a charge and a voltage, we may assume either and then find the other in terms of it. The capacitance is not a function of either, but only of the dielectrics and the geometry. Suppose we assume a potential difference V_0 between the plates. The electric field intensities in the two regions, $\mathbf{E_2}$ and *£1*, are both uniform, and V_0 = $\mathbf{E_1 d_1}$ + $£_2 d_2$. At the dielectric interface, E is normal and $\mathbf{D_{N1}}$ = D^ o r e1 **E**1 = e2**E**2. Eliminating **E**2 in our **V**0 relation, we have

$$£1 \quad \frac{V_0}{d_1 + d_2(e_1/e_2)}$$

and the surface charge density therefore has the magnitude

$$\frac{\mathbf{V_0}}{\frac{\mathbf{d_1}}{e_1} + \frac{\mathbf{d_2}}{e_2}}$$

Since **D**1 = **D**2, the magnitude of the surface charge is the same on each plate. The capacitance is then

$$c = \mathbf{Q} \_ = ps?s\_ = \ \underline{1} \ \_ \ \underline{1}$$

As an alternate (and slightly simpler) solution, we might assume a charge **Q** on one plate, leading to a charge density **Q/S** and a value of **D** that is also **Q/S.** This is true in both regions, as $\mathbf{D_{N1}}$ = $\mathbf{D_{N2}}$ and D is normal. Then $£_1$ = **D**/e1 = **Q**/(e1**S**), E2 = **D**/e2 = **Q**/(e2**S**), and the potential differences across the regions are V_1 = $E_1 d_1$ = $Qd_1/(e_1 S)$, and V_2 = $£_2 d_2$ = $Qd_2/(e_2 S)$. The capacitance is

$$C = {}^{Q} = \quad =_{,,} \ * \qquad (50)$$

How would the method of solution or the answer change if there were a third conducting plane along the interface? We would now expect to find surface charge on each side of this conductor, and the magnitudes of these charges should be equal. In other words, we think of the electric lines not as passing directly fromone outer plate to the other, but as terminating on one side of this interior plane and then continuing on the other side. The capacitance is unchanged, provided, of course, that the added conductor is of negligible thickness. The addition of a thick conducting plate will increase the capacitance if the separation of the outer plates is kept constant, and this is an example of a more general theoremwhich states that the replacement ofany portion ofthe dielectric by a conducting body will cause an increase in the capacitance.

If the dielectric boundary were placed **normal** to the two conducting plates and the dielectrics occupied areas of S_i and S_2, then an assumed potential difference V_0 would produce field strengths $E_i = E_2 = V_0/d$. These are tangential fields at the interface, and they must be equal. Then we may find in succession D_1, D_2, ρ_{S1}, ρ_{S2}, and Q, obtaining a capacitance

$$C = \frac{\epsilon_1 S_1 + \epsilon_2 S_2}{d} = C_1 + C_2 \tag{5i}$$

as we should expect.

At this time we can do very little with a capacitor in which two dielectrics are used in such a way that the interface is not everywhere normal or parallel to the fields. Certainly we know the boundary conditions at each conductor and at the dielectric interface; however, we do not know the fields to which to apply the boundary conditions. Such a problemmust be put aside until our knowledge of field theory has increased and we are willing and able to use more advanced mathematical techniques.

PROBLEMS

5.1 Given the current density $J = -10^4(\sin 2x e^{-2y} a_x + \cos 2x e^{-2y} a_y)$ kA/m^2:
(a) find the total current crossing the plane y = 1 in the a_y direction in the region 0 < x < 1, 0 < z < 2. Find the total current leaving the region 0 < x, y < 1, 2 < z < 3 by: (b) integrating $J \cdot dS$ over the surface of the cube; (c) employing the divergence theorem.

5.2 Let the current density be $J = 2\rho \cos^2 \phi\, a_\rho - \rho \sin 2\phi\, a_\phi$ A/m^2 within the region 2.1 < ρ < 2.5, 0 < φ < 0.1 rad, 6 < z < 6.1. Find the total current I crossing the surface: (a) ρ = 2.2, 0 < φ < 0.1, 6 < z < 6.1 in the a_ρ direction; (b) φ = 0.05, 2.2 < ρ < 2.5, 6 < z < 6.1, in the a_ϕ direction. (c) Evaluate $\nabla \cdot J$ at P(ρ = 2.4, φ = 0.08, z = 6.05).

5.3 Let $J = -\frac{\sin}{r^2 + 4}$ A/m^2. (a) Find the total current flowing through that portion of the spherical surface r = 0.8 bounded by θ = 0.1π, θ = 0.3π, 0 < φ < 2π. (b) Find the average value of J over the defined area.

5.4 The cathode of a planar vacuumtube is at z = 0. Let $E = -4 \times 10^6 a_z$ V/mfor z > 0. An electron (e = 1.602 x 10^{-19} C, m = 9.11 x 10^{-31} kg) is emitted from the cathode with zero initial velocity at t = 0. (a) Find **v(t).** (b) Find z(t), the electron location as a function of time. (c) Determine v(z). (d) Make the assumption that electrons are emitted continuously as a beamwith a 0.25-mmradius and a total current of 60 uA. Find J(z) and ρ_v(z).

5.5 Let $J = -\frac{25}{\rho} a_\rho - \frac{20}{\rho^2 + 0.01} a_z$ A/m^2, and: (a) find the total current crossing the plane z = 0.2 in the a_z direction for p < 0.4. (b) Calculate —. (c)

3t

Find the total outward current crossing the closed surface defined by p = 0.01, p = 0.4, z = 0, and z = 0.2. (J) Show that the divergence the-oremis satisified for *J* and the surface specified.

5.6 Let € = $€_0$ and V = $90z^{4/3}$ in the region z = 0. (a) Obtain expressions for *E, D,* and p„ as functions of z. (b) If the velocity of the charge density is given as $i>_x = 5 \times 10^6 z^{2/3}$ m/s, find $/_z$ at z = 0 and z = 0.1m.

5.7 Assuming that there is no transformation of mass to energy or vice versa, it is possible to write a continuity equation for mass. (a) If we use the continuity equation for charge as our model, what quantities correspond to *J* and p„? (b) Given a cube 1 cmon a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, −9.85, 1.75, −2.00, −4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center.

5.8 The continuity equation for mass equates the divergence of the mass rate of flow (mass per second per square meter) to the negative of the density (mass per cubic meter). After setting up a cartesian coordinate system inside a star, Captain Kirk and his intrepid crew make measurements over the faces of a cube centered at the origin with edges 40 kmlong and parallel to the coordinate axes. They find the mass rate of flow of material outward across the six faces to be −1112, 1183, 201, −196, 1989, and − 1920 kg/km^2 • s. (a) Estimate the divergence of the mass rate of flow at the origin. (b) Estimate the rate of change of the density at the origin. *5.9* (a) Using data tabulated in Appendix C, calculate the required diameter for a 2-mlong nichrome wire that will dissipate an average power of 450 W when 120-V rms at 60 Hz is applied to it. (b) Calculate the rms current density in the wire.

5.9 A steel wire has a radius of 2 mm and a conductivity of 6×10^6 S/m. The steel wire has an aluminum (a = 3.8×10^7 S/m) coating of 2-mm thickness. Let the total current carried by this hybrid conductor be 80 A dc. Find: (a) $/_{st}$; (b) $/_{A1}$; (c) $Z?_{st}$; (J) $Z?_{Al}$; (e) the voltage between the ends of the conductor if it is 1 mi long.

5.10 Two perfectly conducting cylindrical surfaces are located at p = 3 and P = 5 cm. The total current passing radially outward through the med-iumbetween the cylinders is 3 A dc. (a) Find the voltage and resistance between the cylinders, and *E* in the region between the cylinders, if a conducting material having a = 0.05 S/mis present for 3 < p < 5 cm. (b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power.

5.11 The spherical surfaces r = 3 and r = 5cmare perfectly conducting, and the total current passing radially outward through the medium between the surfaces is 3 A dc. (a) Find the voltage and resistance between the spheres, and *E* in the region between them, if a conducting material having a = 0.05 S/mis present for 3 < r < 5cm. (b) Repeat if a = 0. 0005/r for 3 < r < 5cm. (c) Show that integrating the power dissipated per unit volume in part b over the volume gives the total dissipated power.

5.12 A hollow cylindrical tube with a rectangular cross section has external dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that the material is brass for which **a** = 1.5 x 10^7 S/m. A current of 200 A dc is flowing down the tube. (a) What voltage drop is present across a 1 m length of the tube? (b) Find the voltage drop if the interior of the tube is filled with a conducting material for which **a** = 1.5 x 10^5 S/m.

5.13 Find the magnitude of the electric field intensity in a conductor if: (a) the current density is 5MA/m^2, the electron mobility is 3 x $10^{-3}$$m^2$/V • s, and the volume charge density is −2.4 x 10^{10}C/m^3; (b) / = 3MA/m^2 and the resistivity is 3 x 10^{-8} £ • m.

5.14 Let V = 10(p + 1)z^2cos p V in free space. (a) Let the equipotential surface V = 20 V define a conductor surface. Find the equation of the conductor surface. (b) Find P and *E* at that point on the conductor surface where p = 0.2i and z = 1.5. (c) Find p| at that point.

5.15 A potential field in free space is given as V = (80cos 6 sin p)/r^3 V. Point P(r = 2, 6 = i/3, p = i/2) lies on a conducting surface. (a) Write the equation of the conducting surface. (b) Find a unit normal directed outward to the surface, assuming the origin is inside the surface. (c) Find *E* at P.

5.17 Given the potential field $V = -z\dfrac{100xz}{x^2 + 4}$ V in free space: (a) find *D* at the surface z = 0. (b) Show that the z = 0 surface is an equipotential surface. (c) Assume that the z = 0 surface is a conductor and find the total charge on that portion of the conductor defined by 0 < x < 2, −3 < y < 0.

5.18 Let us assume a field $E = 3y^2z^3a_x + 6xyz^3a_y + 9xy^2z^2$ V/min free space, and also assume that point P(2, 1, 0) lies on a conducting surface. (a) Find p„ just adjacent to the surface at P. (b) Find ps at P. (c) Show that V = $-3xy^2z^3$ V. (J) Determine V_{PG}, given g(1, 1, 1).

5.19 Let V = $20x^2yz - 10z^2$ V in free space. (a) Determine the equations of the equipotential surfaces on which V = 0 and 60 V. (b) Assume these are conducting surfaces and find the surface charge density at that point on the V = 60-V surface where x = 2 and z = 1. It is known that 0 < V < 60 V is the field-containing region. (c) Give the unit vector at this point that is normal to the conducting surface and directed toward the V = 0 surface.

5.20 A conducting plane is located at z = 0 in free space, and a 20-nC point charge is present at g(2, 4, 6). (a) If V = 0 at z = 0, find V at P(5, 3, 1). (b) Find *E* at P. (c) Find ps at ,4(5, 3, 0).

5.21 Let the surface y = 0 be a perfect conductor in free space. Two uniform infinite line charges of 30nC/meach are located at x = 0, y = 1, and x = 0, y = 2. (a) Let V = 0 at the plane y = 0, and find V at P(1, 2, 0).
(b) Find *E* at P.

5.22 Let the plane x = 0 be a perfect conductor in free space. Locate a point charge of 4nC at P_1(7, 1, −2), and a point charge of −3nC at P_2(4, 2, 1). (a) Find *E* at 4(5, 0, 0). (b) Find |ps| at 5(3, 0, 0).

5.23 A dipole with $p = 0.1a_z$ uC • mis located at 4(1, 0, 0) in free space, and the x = 0 plane is perfectly conducting. (a) Find V at P(2, 0, 1). (b) Find the equation of the 200-V equipotential surface in cartesian coordinates.

5.24 The mobilities for intrinsic silicon at a certain temperature are $/z_e$ = 0.14m^2/V • s and $/z_h$ = 0.035 m^2/V • s. The concentration of both holes and electrons is 2.2 x 10^{16}m^{-3}. Determine both the conductivity and resistivity of this silicon sample.

5.25 Electron and hole concentrations increase with temperature. For pure silicon suitable expressions are p_h = — p_e = 6200$T^{L5}e^{-7000/r}$C/m^3. The functional dependence of the mobilities on temperature is given by ^{h} = 2.3 x 10^5T^{-27}m^2/V • s and $/z_e$ = 2.1 x 10^5T^{-25}m^2/V • s. Find ct at: (a) 0° C; (b) 40° C; (c) 80° C.

5.26 A little donor impurity, such as arsenic, is added to pure silicon so that the electron concentration is 2 x 10^{17} conduction electrons per cubic meter while the number of holes per cubic meter is only 1.1 x 10^{15}.If $/z_e$ = 0.15m^2/V • s for this sample, and $/z_h$ = 0.045 m^2/V• s, determine the conductivity and resistivity.

5.27 Atomic hydrogen contains 5.5 x 10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/mis applied, each dipole formed by the electron and the positive nucleus has an effective length of 7.1 x 10^{-19} m. Find: (a) P; (b) e_R.

5.28 In a certain region where the relative permittivity is 2.4, $D = 2a_X$ — $4a_Y + 5a_Z$nC/m^2. Find: (a) *E;* (b) *P;* (c) | VV | .

5.29 A coaxial conductor has radii a = 0.8 mm and b = 3 mm and a polystyr-

2 2

ene dielectric for which e_R = 2.56. If *P* = — a_p nC/m^2 in the dielectric,

P

find: (a) *D* and *E* as functions of p; (b) V_{AB} and x«. (c) If there are 4 x 10^{19} molecules per cubic meter in the dielectric, find *p*(p).

5.30 Given the potential field V = 200 — 50x + 20yV in a dielectric material for which <?r = 2.1, find: (a) *E;* (b) *D;* (c) *P;* (J) p„; (e) p_B; (/") p_r.

5.31 The surface x = 0 separates two perfect dielectrics. For x > 0 let e_R = e_{R1} = 3, while e_{R2} = 5 where x < 0. If $E_1 = 80a_x - 60a_y - 30a_z$ V/m, find: (a) 2?^; (b) E_{r1}; (c) E_{r1}; (J) *£1;* (e) the angle 6_1 between E_1 and a normal to the surface; (/*) £V_2; (g) £>$_{I2}$; (h) D_2; (z) P_2 (/) the angle 6_2 between *E2* and a normal to the surface.

5.32 In Fig. 5.18 let $D_1 = 3a_x - 4a_y + 5a_z$ nC/m^2 and find: (a) D_2; (b) $D\#_2$; (c) D_{i2}; (J) the energy density in each region; (e) the angle that D_2 makes with a_Z; (0 ZtyA; (g) $P2/P1$.

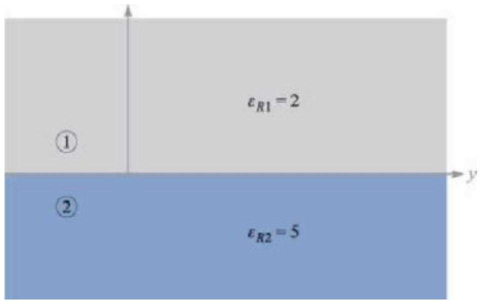
5.33 Two perfect dielectrics have relative permittivities e_{R1} = 2 and e_{R2} = 8. The planar interface between themis the surface x — y + 2z = 5. The origin lies in region 1. If $E_1 = 100a_x + 200a_y - 50a_z$ V/m, find E_2.

5.34 Let the spherical surfaces r = 4 cmand r = 9 cmbe separated by two perfect dielectric shells, e_{R1} = 2 for 4 < r < 6 cm, and e_{R2} = 5 for

6 < r < 9 cm. If E_1 = — — a_r V/m, find: (a) E_2; (b) the total electrostatic

energy stored in each region.

FIGURE 5.18



5.35 Let the cylindrical surfaces p = 4 c m a n d p = 9 cmenclose two wedges
of perfect dielectrics, e^*i* = 2 for 0 <0<7 r / 2 , and €^*₂* = 5 for
,/2 < 0 < 2^.If *E1* = 2000*aP* V/m, find: (a) *E2*; (b) the total electrostatic

energy stored in a 1-mlength of each region.

5.36 Let S = 120cm^2, J = 4mm, and €_r = 12 for a parallel-plate capacitor. (a) Calculate the capacitance. (b) After connecting a 40-V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy. (c) The source is now removed and the dielectric carefully withdrawn frombetween the plates. Again calculate E, D, Q, and the energy. (J) What is the voltage between the plates?

5.37 Capacitors tend to be more expensive as their capacitance and maximum voltage V_{max} increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{bd}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas: (a) air: = 1, E^d = 3 MV/m; (b) bariumtitanate: = 1200, E_{bd} = 3 MV/m; (c) silicon dioxide: €_r = 3.78, E*GD* = 16 MV/m; (J) polyethylene: €_r = 2.26, E_{bd} = 4.7 MV/m.

5.38 A dielectric circular cylinder used between the plates of a capacitor has a thickness of 0.2mm and a radius of 1.4 cm. The dielectric properties are
= 400 and ct = 10^{-5} S/m. (a) Calculate C. (b) Find the quality factor Qq.f(Qqf = &>-RC) of the capacitor at / = 10 kHz. (c) If the maximum field strength permitted in the dielectric is 2 MV/m, what is the maximum permissible voltage across the capacitor? (J) What energy is stored when this voltage is applied?

5.39 A parallel-plate capacitor is filled with a nonuniformdielectric characterized by e_*R* = 2 + 2 x 10^{6}x^2, where x is the distance fromone plate. If S = 0.02m^2 and J = 1 mm, find C.

5.40 (a) The width of the region containing in Fig. 5.19 is 1.2 m. Find if e_*R₂* = 2.5 and the total capacitance is 60 nF. (b) Find the width of each region (containing and e^2) if C_{total} = 80 nF, e^2 = 3e_{R1}, and

C*1* = 2C*2*.

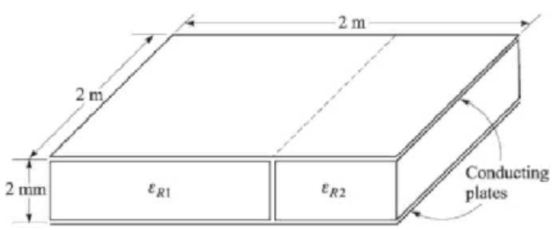


FIGURE 5.19^

5.41 Let $\epsilon_{R1} = 2.5$ for $0 < y < 1$mm, $\epsilon_{R2} = 4$ for $1 < y < 3$mm, and ϵ_{R3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: (a) ϵ_{R3} is air; (b) $<?R3 = <?R1;$ (c) $<?R3 = <?R2;$ (J) $<?R3$ is silver.

5.42 Cylindrical conducting surfaces are located at $\rho = 0.8$cmand 3.6 cm. The region 0.8 cm $< \rho <$ a contains a dielectric for which $\epsilon_R = 4$, while $\epsilon_R = 2$ for $a < \rho <$ 3.6 cm. (a) Find a so that the voltage across each dielectric layer is the same. (b) Find the total capacitance per meter.

5.43 Two coaxial conducting cylinders of radius 2 cmand 4 cmhave a length of 1 m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = J$ with $\epsilon_R = 4$. Find the capacitance if: (a) c = 2 cm, J = 3 cm; (b) J = 4 cm, and the volume of dielectric is the same as in part a.

5.44 Conducting cylinders lie at $\rho = 3$ and 12 mm; both extend from z = 0to z = 1m. Perfect dielectrics occupy the interior region: $\epsilon_R = 1$ for $3 < \rho <$ 6 mm, $\epsilon_R = 4$ for $6 < \rho <$ 9 mm, and $\epsilon_R = 8$ for $9 < \rho <$ 12 mm. (a) Calculate C. (b) If the voltage between the cylinders is 100 V, plot $|Z?_\rho|$ versus ρ.

5.45 Two conducting spherical shells have radii a = 3 cmand b = 6 cm. The interior is a perfect dielectric for which $\epsilon_R = 8$. (a) Find C. (b) A portion of the dielectric is now removed so that $\epsilon_R = 1$, $0 < \rho < 7$ r/2, and $\epsilon_R = 8$, 7 r/2 $< \rho <$ 27 t. Again find C.

5.46 Conducting cylinders lie at $\rho = 3$ and 12 mm; both extend from z = 0to z = 1m. Perfect dielectrics occupy the interior region: $\epsilon_R = 1$ for $3 < \rho <$ 6 mm, $\epsilon_R = 4$ for $6 < \rho <$ 9 mm, and $\epsilon_R = 8$ for $9 < \rho <$ 12 mm. (a) Calculate C. (b) If the voltage between the cylinders is 100 V, plot $|Z?_\rho|$ versus ρ.

5.47 With reference to Fig. 5.17, let b = 6 m, h = 15 m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for A_{11}, ρ^, a, and C.

5.48 A potential function in free space is given by $V = -20 + 10 \ln \dfrac{(5+y)^2 + x^2}{(5-y)^2 + x^2}$ V. Describe: (a) the 0-V equipotential surface; (b) the 10-V equipotential surface.

5.49 A 2-cm-diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the: (a) cylinder at a point nearest the plane; (b) plane at a point nearest the cylinder.

Chapter Six

POISSON'S AND LAPLACE'S EQUATIONS

6.1POISSON'SANDLAPLACE'SEQUATIONS

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$VD \quad Pv \qquad (1)$$

the definition of D,

$$D = eE \qquad (2)$$

and the gradient relationship,

$$E = - VV \qquad (3)$$

by substitution we have

$$V \bullet D = V \bullet (eE) = -V \bullet (eVV) = pv$$

or

$$V \quad \frac{Pv}{e} \qquad (4)$$

for ahomogeneous region in which e is constant.

Equation (4) is Poisson's equation, but the "double V" operation must be interpreted and expanded, at least in cartesian coordinates, before the equation can be useful. In cartesian coordinates,

V-A dAx dAy

d_A±

V Bx dy Bz

BV BV BV

Bx By Bz

and therefore

V d_ (BV\ <5_ (BV_ B_(BV_ Bz\

Bx\ Bx) By\ By Bz

B2 V B2 V B2 V Bx2 (5)

By2 Bz2

Usually the operation V • V is abbreviated V2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in (5), and we have

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}} \qquad (6)$$

in cartesian coordinates.

If $\rho_v = 0$, indicating zero volume charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\nabla^2 V = 0 \qquad (7)$$

which is Laplace's equation. The ∇^2 operation is called the Laplacian of V. In cartesian coordinates Laplace's equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \quad \text{(cartesian)} \qquad (8)$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \quad \text{(cylindrical)}} \qquad (9)$$

and in spherical coordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{(spherical)} \qquad (10)$$

These equations may be expanded by taking the indicated partial derivatives, but it is usually more helpful to have them in the forms given above; furthermore, it is much easier to expand them later if necessary than it is to put the broken pieces back together again.

Laplace's equation is all-embracing, for, applying as it does wherever volume charge density is zero, it states that every conceivable configuration of electrodes or conductors produces afield for which $\nabla^2 V = 0$. All these fields are different, with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. Since every field (if $\rho_v = 0$) satisfies Laplace's equation, how can we expect to reverse the procedure and use Laplace's equation to find one specific field in which we happen to have an interest? Obviously, more information is required, and we shall find that we must solve Laplace's equation subject to certain boundary conditions.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values, perhaps $V_0, V_1, \ldots$, or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved in this chapter. In other types of problems, the boundary conditions take the form of specified values of E on an enclosing surface, or a mixture of known values of V and E.

Before using Laplace's equation or Poisson's equation in several examples, we must pause to show that if our answer satisfies Laplace's equation and also satisfies the boundary conditions, then it is the only possible answer. It would be very distressing to work a problem by solving Laplace's equation with two different approved methods and then to obtain two different answers. We shall show that the two answers must be identical.

Example 6.1

boundary conditions. Laplac Let us assume that V is afunction only of x and worry later about which physical problem we are solving when we have a need for e's equation reduces to

$$d^2v \quad :0$$

and the partial derivative may be replaced by an ordinary derivative, since V is not a function of y or z,

$d\,^2V$

dx^2

We integrate twice, obtaining

$$dV\ dx \quad : A$$

and

$$V = Ax + B \qquad (12)$$

where A and B are constants of integration. Equation (12) contains two such constants, as we should expect for a second-order differential equation. These constants can be determined only from the boundary conditions.

What boundary conditions should we supply? They are our choice, since no physical problem has yet been specified, with the exception of the original hypothesis that the potential varied only with x. We should now attempt to visualize such a field. Most of us probably already have the answer, but it may be obtained by exact methods.

Since the field varies only with x and is not a function of y and z, then V is a constant if x is a constant or, in other words, the equipotential surfaces are described by setting x constant. These surfaces are parallel planes normal to the x axis. The field is thus that of a parallel-plate capacitor, and as soon as we specify the potential on any two planes, we may evaluate our constants of integration.

To be very general, let $V = V_1$ at $x = x_1$ and $V = V_2$ at $x = x_2$. These values are then substituted into (12), giving

$$V_1 = Ax_1 + B \qquad V_2 = Ax_2 + B$$

A $\quad x_1 - x_2 \quad$ B $\quad V_2x_1$

x_1

V_1x_2

x_2

and

V Vj(x — x2)— V2(x —
xQ
x1 — x2

A simpler answer would have been obtained by choosing simpler boundary conditions. If we had fixed V = 0at x = 0and V = V0 at x = d, then

A=V B = 0

$$V = \frac{V_0 x}{d}$$

and

Suppose our primary aim is to find the capacitance of a parallel-plate capacitor. We have solved Laplace's equation, obtaining (12) with the two constants A and B. Should they be evaluated or left alone? Presumably we are not interested in the potential field itself, but only in the capacitance, and we may continue successfully with A and B or we may simplify the algebra by a little foresight. Capacitance is given by the ratio of charge to potential difference, so we may choose now the potential difference as F0, which is equivalent to one boundary condition, and then choose whatever second boundary condition seems to help the form of the equation the most. This is the essence of the second set of boundary conditions which produced (13). The potential difference was fixed as Fo by choosing the potential of one plate zero and the other Fo; the location of these plates was made as simple as possible by letting F = Oat x = 0.

Using (13), then, we still need the total charge on either plate before the capacitance can be found. We should remember that when we first solved this capacitor problem in Chap. 5, the sheet of charge provided our starting point. We did not have to work very hard to find the charge, for all the fields were expressed in terms of it. The work then was spent in finding potential difference. Now the problem is reversed (and simplified).

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given F, use E = -VF to find E.
2. Use D = eE to find D.
3. Evaluate D at either capacitor plate, D = Ds = DNaN.
4. Recognize that ps = .
5. Find Q by a surface integration over the capacitor plate, Q = Js psdS.

Here we have

F x
d

E Fo
~d 2

D ~~ d

Vo

T>s = D = — €~T ax d

x=0

= ax

d

f — VoS

Q = -- $ = — €-.

Js d d

and the capacitance is

eS

c = iQi (14)

" Vo d

We shall use this procedure several times in the examples to follow.

Example 6.2

Since no new problems are solved by choosing fields which vary only with y or with z in cartesian coordinates, we pass on to cylindrical coordinates for our next example. Variations with respect to z are again nothing new, and we next assume variation with respect to p only. Laplace's equation becomes

1 3 / 3V P 3p

Y 3p 0

or

1 d 0

p dp dV

i

— dp

Noting the p in the denominator, we exclude p = 0 from our solution and then multiply by p and integrate,

dV _ A P dp

rearrange, and integrate again,

V = A ln p + B (15)

The equipotential surfaces are given by p = constant and are cylinders, and the problem is that of the coaxial capacitor or coaxial transmission line. We choose a potential difference of V0 by letting V = V0 at p = a, V = Oat p = 6, b > a, and obtain

V V ln(b/p) 0

ln(b/a)

from

which Fo 1

Q

DN(p=a)

(16)

p \n(b/a) eFo2naL a
ln(b/a)
. eFo a
ln(b/a)

C 2neL 'Mb/a (17)

which agrees with our results in Chap. 5.

Example 6.3

Now let us assume that F is afunction only of 0 in cylindrical coordinates. We might look at the physical problem first for a change and see that equipotential surfaces are given by 0 = constant. These are radial planes. Boundary conditions might be F = oat 0 = oand F = Fo at 0 = a, leading to the physical problem detailed in Fig. 7.1. Laplace's equation is now

1 tFL o

We exclude p = o
and have d 2F_
d02 = o

The

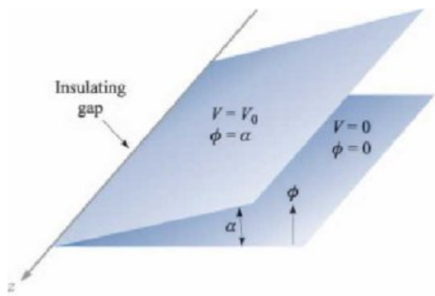


FIGURE 7.1
Two infinite radial planes with an interior angle a. An infinitesimal insulating gap exists at p = o. The potential field may be found by applying Laplace's equation in cylindrical coordinates.

solution is F = A0 + B

The boundary conditions determine A and _8,and

V = V (18

Taking the gradient of (18) produces the electric field intensity,

E V0a^ ap

(19)

and it is interesting to note that E is afunction of p and not of 0. This does not contradict our original assumptions, which were restrictions only on the potential field. Note, however, that the vector field E is afunction of 0.

A problem involving the capacitance of these two radial planes is included at the end of the chapter.

Example 6.4

We now turn to spherical coordinates, dispose immediately of variations with respect to 0 only as having just been solved, and treat first V = V(r). The details are left for a problem later, but the final potential field is given by

$$\boxed{V = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}} \qquad (20)$$

where the boundary conditions are evidently V = 0at r = b and V = V0 at r = a, b > a. The problem is that of concentric spheres. The capacitance was found previously in Sec. 5.10 (by a somewhat different method) and is

4jre 1

1

a

C: (21)

Example 6.5

In spherical coordinates we now restrict the potential function to V = V(0), obtaining

1 d / dV\
------------ sin 6 -- = 0
r2sin 6 d6\ d6)

We exclude r = 0and 6 = 0or it and have

sin 0 -- $\frac{dF}{d\,0}$ ----- = A

The second integral is then

Ad0

sin 0

which is not as obvious as the previous ones. From integral tables (or a good memory) we have

F = A l^tan^ + B

The equipotential surfaces are cones. Fig. 7.2 illustrates the case where F = oat 0 = 7r/2and F = Fo at 0 = a, a<n/2. We obtain

$$\boxed{V = V_0 \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\tan\frac{\alpha}{2}\right)}} \qquad (22)$$

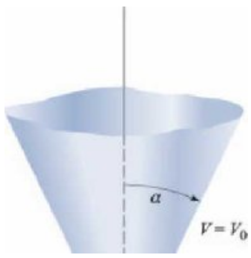
In order to find the capacitance between a conducting cone with its vertex separated from a conducting plane by an infinitesimal insulating gap and its axis normal to the plane, let us first find the field strength:

)

E = -VF = - — ao = ------------------------- -ov ae

r 30 rsin0ln(

The surface charge density on the cone is then



if

FIGURE 7.2

tLs'

For the cone 9 = a at Vq and the plane

! at F = 0, the potential field is

given by V =

-an6» /2)1 /[ln(tana /2)1

PS -€Vq
r sin a ln (tan
a)

producing a total charge Q,

Q t'2n r sin a d0 dr
sin a ln(tan i) r

-2it€o Vo dr
ln
(tan f)

This leads to an infinite value of charge and capacitance, and it becomes necessary to consider a cone of finite size. Our answer will now be only an approximation, because the theoretical equipotential surface is 9 = a, a conical surface extending from r = Oto r = 00, whereas our physical conical surface extends only from r = O to, say, r = ri. The approximate capacitance is

$$C \doteq \frac{2\pi\epsilon r_1}{\ln\left(\cot\frac{\alpha}{2}\right)} \qquad (23)$$

If we desire a more accurate answer, we may make an estimate of the capacitance of the base of the cone to the zero-potential plane and add this amount to our answer above. Fringing, or nonuniform, fields in this region have been neglected and introduce an additional source of error.

%/ D7.3. Find |E| at P(3, 1, 2) for the field of: (a) two coaxial conducting cylinders,
V = 5OV at p = 2 m, and V = 2OV at p = 3 m; (b) two radial conducting planes,
V = 50 V at 0 = 1O°,and V = 20 V at 0 = 30°.

Ans. 23.4 V/m; 27.2 V/m

6.2 EXAMPLE OF THE SOLUTION OF POISSON'S EQUATION

To select a reasonably simple problem which might illustrate the application of Poisson's equation, we must assume that the volume charge density is specified. This is not usually the case, however; in fact, it is often the quantity about which we are seeking further information. The type of problem which we might encounter later would begin with a knowledge only of the boundary values of the potential, the electric field intensity, and the current density. From these we would have to apply Poisson's equation, the continuity equation, and some relationship expressing the forces on the charged particles, such as the Lorentz force equation or the diffusion equation, and solve the whole system of equations simultaneously. Such an ordeal is beyond the scope of this text, and we shall therefore assume a reasonably large amount of information.

As an example, let us select a pn junction between two halves of a semiconductor bar extending in the x direction. We shall assume that the region for x < 0 is doped p type and that the region for x > Ois n type. The degree of doping is identical on each side of the junction. To review qualitatively some of the facts about the semiconductor junction, we note that initially there are excess holes to the left of the junction and excess electrons to the right. Each diffuses across the junction until an electric field is built up in such a direction that the diffusion current drops to zero. Thus, to prevent more holes from moving to the right, the electric field in the neighborhood of the junction must be directed to the left; Ex is negative there. This field must be produced by a net positive charge to the right of the junction and a net negative charge to the left. Note that the layer of positive charge consists of two parts—the holes which have crossed the junction and the positive donor ions from which the electrons have departed. The negative layer of charge is constituted in the opposite manner by electrons and negative acceptor ions.

The type of charge distribution which results is shown in Fig. 7.3a, and the negative field which it produces is shown in Fig. 7.3b. After looking at these two figures, one might profitably read the previous paragraph again.

A charge distribution of this form may be approximated by many different expressions. One of the simpler expressions is

$$\rho_v = 2\rho_{v0}\,\mathrm{sech}\frac{x}{a}\tanh\frac{x}{a} \qquad (24)$$

which has a maximum charge density $\rho_{v\,max} = \rho_{v0}$ that occurs at $x = 0.881a$. The maximum charge density ρ_{v0} is related to the acceptor and donor concentrations N_a and N_d by noting that all the donor and acceptor ions in this region (the depletion layer) have been stripped of an electron or a hole, and thus

$$\rho_{v0} = eN_a = eN_d$$

Let us now solve Poisson's equation,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

subject to the charge distribution assumed above,

$$\frac{d^2F}{dx^2} = -\frac{2\rho_{v0}}{\epsilon}\,\mathrm{sech}\frac{x}{a}\tanh\frac{x}{a}$$

in this one-dimensional problem in which variations with y and z are not present. We integrate once,

$$\frac{dF}{dx} = \frac{2\rho_{v0}a}{\epsilon}\,\mathrm{sech}\frac{x}{a} + Q$$

and obtain the electric field intensity,

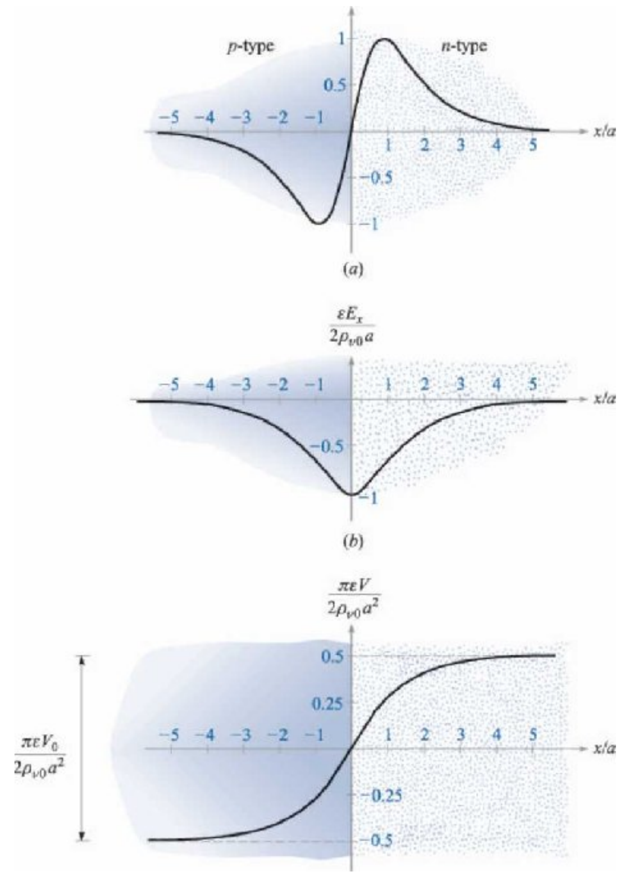


FIGURE 7.3

(a) The charge density, (b) the electric field intensity, and (c) the potential are plotted for a pn junction as functions of distance from the center of the junction. The /?-type material is on the left, and the n-type is on the right.

2pv0a x
€a
Ex =------------------------ sech -- C1

To evaluate the constant of integration C1, we note that no net charge density and no fields can exist far from the junction. Thus, as x —>±oo, Ex must approach zero. Therefore C1 = 0, and

€a
Ex = -2pv0a sech x (25)

Integrating again,

F = 4pv0a2tan-1 ex/a + C2
€

Let us arbitrarily select our zero reference of potential at the center of the junction, x = 0,

4pv0a2 n
0 = —^ 4 + C2

and finally,

F = 4p€°a2 (tan-1 ex/a - n) (26)

Fig. 7.3 shows the charge distribution (a), electric field intensity (b), and the potential (c), as given by (24), (25), and (26), respectively.

The potential is constant once we are a distance of about 4a or 5a from the junction. The total potential difference F0 across the junction is obtained from

(26),

F0 = Fx_- Fx_o= 2npv0a2 (27)

€

This expression suggests the possibility of determining the total charge on one side of the junction and then using (27) to find a junction capacitance. The total positive charge is

Q = S r0° x , x
2pv0sech — tanh — dx = 2pv0aS
0 a a

where S is the area of the junction cross section. If we make use of (27) to eliminate the distance parameter a, the charge becomes

q = s^2^ (28)

Since the total charge is a function of the potential difference, we have to be careful in defining a capacitance. Thinking in "circuit" terms for a moment,

/ = dQ = CrfF0

dt dt and thus

c = ddQQ-

dVo

By differentiating (28) we therefore have the capacitance,

C ^2irVoS 2ita (9)

The first form of (29) shows that the capacitance varies inversely as the square root of the voltage. That is, a higher voltage causes a greater separation of the charge layers and a smaller capacitance. The second form is interesting in that it indicates that we may think of the junction as a parallel-plate capacitor with a "plate" separation of 27ra. In view of the dimensions of the region in which the charge is concentrated, this is a logical result.

Poisson's equation enters into any problem involving volume charge density. Besides semiconductor diode and transistor models, we find that vacuum tubes, magnetohydrodynamic energy conversion, and ion propulsion require its use in constructing satisfactory theories.

PROBLEMS

7.1 Let V = 2xy2z3 and e = eQ. Given point P(1, 2, —1), find: (a) V at P; (b) E at P; (c) pv at P; (d) the equation of the equipotential surface passing through P; (e) the equation of the streamline passing through P. (/) Does V satisfy Laplace's equation?

7.9 The functions V1(p, 0, z) and V2(p, 0, z) both satisfy Laplace's equation in the region a <p< b,0 < 0< 2n, — L < z < L; each is zero on the surfaces p = b for —L < z < L; z =—L for a <p< b;and z = L for a <p< b; and each is 100 V on the surface p = a for —L < z < L. (a) In the region specified above, is Laplace's equation satisfied by the functions V1 + V2, V1 — V2, V1 + 3, and V1V2? (b) On the boundary surfaces specified, are the potential values given above obtained from the functions V1 + V2, V1 — V2, V1 + 3, and V1V2? (c) Are the functions V2, V1 + V2, V1 + 3, and V1V2 identical with V1?

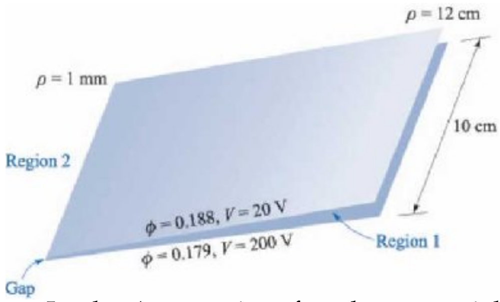
7.10 Conducting planes at z = 2 cm and z = 8 cm are held at potentials of —3 V and 9 V, respectively. The region between the plates is filled with a perfect dielectric with e = 5e0. Find and sketch: (a) V(z); (b) £z(z); (c) £z(z).

7.11 The conducting planes 2x + 3y = 12 and 2x + 3y = 18 are at potentials of 100 V and 0, respectively. Let e = e0 and find: (a) V at P(5, 2, 6); (b) E at P.

7.12 Conducting cylinders at p = 2 cm and p = 8 cm in free space are held at potentials of 60 mV and — 30 mV, respectively. (a) Find V(p). (b) Find £p(p). (c) Find the surface on which V = 30 mV.

7.13 Coaxial conducting cylinders are located at p = 0.5 cm and p = 1.2 cm. The region between the cylinders is filled with ahomogeneous perfect dielectric. If the inner cylinder is at 100 V and the outer at 0 V, find: (a) the location of the 20-V equipotential surface; (b) £pmax; (c) if the charge per meter length on the inner cylinder is 20 nC/m.

7.14 Two semi-infinite planes are located at 0 =— a and 0 = a, where a< n/2. A narrow insulating strip separates them along the z axis. The potential at 0 = —a is V0, while V = Oat 0 = a. (a) Find V(0) in terms of a and V0. (b) Find £"0 at 0 = 20°, p = 2cm, if V0 = 100 V and a 30°.



7.15 (a) Solve Laplace's equation for the potential field in the homogeneous region between two concentric conducting spheres with radii a and b, b > a,if V = Qat r = b,and V = Vq at r = a. (b) Find the capacitance between them.

7.16 Concentric conducting spheres are located at r = 5mm and r = 2Q mm. The region between the spheres is filled with aperfect dielectric. If the inner sphere is at 1QQ V and the outer at Q V: (a) find the location of the 2Q-V equipotential surface; (b) find Ermax; (c) find if the surface charge density on the inner sphere is 1QQ uC/m2.

Chapter Seven

THE STEADY MAGNETIC FIELD

7.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We shall largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present relationships will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current I flowing in a different vector length of the filament dL. The law of Biot-Savart19 then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P. The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P. Of the two possible normals, that one is to be chosen which is in the direction of progress of a right-handed screw turned from dL through the smaller angle to the line from the filament to P. Using rationalized mks units, the constant of proportionality is 1 /47r.

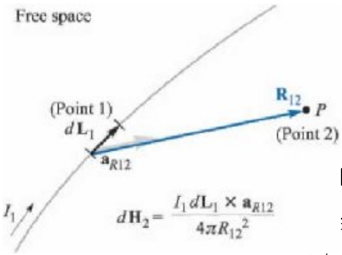
[19] Biot and Savart were colleagues of Ampere, and all three were professors of physics at the College de

The Biot-Savart law, described above in some 150 words, may be written concisely using vector notation as

$$dH = \frac{IdL \times a_R}{4\pi R^2} = \frac{IdL \times R}{4\pi R^3} \qquad (1)$$

The units of the magnetic field intensity H are evidently amperes per meter (A/m). The geometry is illustrated in Fig. 8.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

France at one time or another. The Biot-Savart law was proposed in 1820.



URE 8.1
The law of Biot-
ırt expresses the magnetic field inten-
dH2 produced by a differential current element I1dL1. The direction of dH2 is into the page.

l1dL1 x a^12
d H2 = --------------- = --
(2)

The law of Biot-Savart is sometimes called Ampere's law for the current element, but we shall retain the former name because of possible confusion with Ampere's circuital law, to be discussed later.
In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

2 4Tt€oR212

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.
It is impossible to check experimentally the law of Biot-Savart as expressed by (1) or (2) because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time. The continuity equation in Sec. 5.2, Eq. (5),

VJ = -£

therefore shows that

V • J = 0

or upon applying the divergence theorem,

o J • d S = 0

J*

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit which must be our experimental source, not the differential element.
It follows that only the integral form of the Biot-Savart law can be verified experimentally,

H x a_s IdL

4

ttR2

Equation (1) or (2), of course leads directly to the integral form (3), but other differential expressions also yield the same integral formulation. Any term may be added to (1) whose integral around a closed path is zero. That is, any conservative field could be added to (1). The gradient of any scalar field always yields a conservative field, and we could therefore add a term VG to (1), where G is a general scalar field, without changing (3) in the slightest. This qualification on (1) or (2) is mentioned to show that if we later ask some foolish questions, not subject to any experimental check, concerning the force exerted by one differential current element on another, we should expect foolish answers.

The Biot-Savart law may also be expressed in terms of distributed sources, such as current density J and surface current density K. Surface current flows in a sheet of vanishingly small thickness, and the current density J, measured in amperes per square meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by K. If the surface current density is uniform, the total current I in any width b is

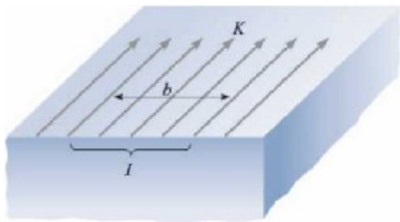
I = Kb

where we have assumed that the width b is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Fig. 8.2. For a nonuniform surface current density, integration is necessary:

I

KdN (4)

where dN is a differential element of the path across which the current is flowing. Thus the differential current element IdL, where dL is in the direction of the



ꞦE 8.2
The total current
n a transverse width b,in which
s a uniform surface current
ᵹ K is Kb

current, may be expressed in terms of surface current density K or current density J,

IdL = K dS = J dv (5)

and alternate forms of the Biot-Savart law obtained,

« = I (6)

and

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2} \qquad (7)$$

We may illustrate the application of the Biot-Savart law by considering an infinitely long straight filament. We shall apply (2) first and then integrate. This, of course, is the same as using the integral form (3) in the first place.2

Referring to Fig. 8.3, we should recognize the symmetry of this field. No variation with z or with <p can exist. Point 2, at which we shall determine the field, is

therefore chosen in the z = 0 plane. The field point r is therefore r = pap. The source point r' is given by r' = z 'az, and therefore

R12 = r - r' = pap - z

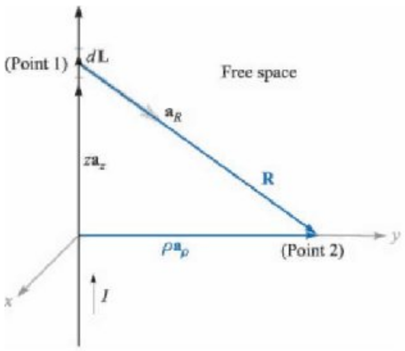


FIGURE 8.3
An infinitely long straight filament carrying a direct current I. The field at point 2 is $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$.

The closed path for the current may be considered to include a return filament parallel to the first filament and infinitely far removed. An outer coaxial conductor of infinite radius is another theoretical possibility. Practically, the problem is an impossible one, but we should realize that our answer will be quite accurate near a very long straight wire having a distant return path for the current.

so that

pap - z'az VP2

+ z'2

We take JL — Jz 'az and (2) becomes

; x(pap — z 'az)

J H2

4tt(p2 + z'2)372

Since the current is directed toward increasing values of z', the limits are —oc and oo on the integral, and we have

/° IJz 'az x(pap — z 'az) 2 I— 00 4tt(p2 + z'2)372 I j00

pJz 'a<

4W —o(p2 + z'2)372

At this point the unit vector a<, under the integral sign should be investigated, for it is not always a constant, as are the unit vectors of the cartesian coordinate system. A vector is constant when its magnitude and direction are both constant. The unit vector certainly has constant magnitude, but its direction may change. Here a< changes with the coordinate < but not with p or z. Fortunately, the integration here is with respect to zand a< is a constant and may be removed from under the integral sign,

Ipa<, f00 Jz'

H2

4tt J—o(p2 + z'2)372 I pa<, z' "°

4tt py p2 + z'2

and

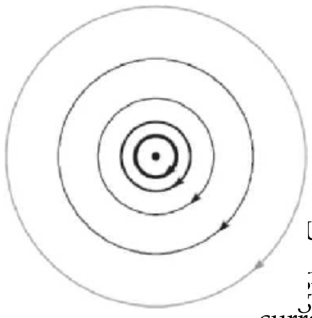
H2 I

27rp (8)

The magnitude of the field is not a function of <p or z and it varies inversely as the distance from the filament. The direction of the magnetic-field-intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Fig. 8.4.

The separation of the streamlines is proportional to the radius, or inversely proportional to the magnitude of H. To be specific, the streamlines have been drawn with curvilinear squares in mind. As yet we have no name for the family of lines[3] which are perpendicular to these circular streamlines, but the spacing of

If you can't wait, see Sec. 8.6



URE 8.4
The streamlines of the
;netic field intensity about an infinitely
; straight filament carrying a direct
current /. The direction of / is into the page.

the streamlines has been adjusted so that the addition of this second set of lines will produce an array of curvilinear squares.
A comparison of Fig. 8.4 with the map of the electric field about an infinite line charge shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and the unnamed (and undrawn) perpendicular family of lines in the magnetic field corresponds to the streamlines of the electric field. This correspondence is not an accident, but there are several other concepts which must be mastered before the analogy between electric and magnetic fields can be explored more thoroughly.
Using the Biot-Savart law to find H is in many respects similar to the use of Coulomb's law to find E. Each requires the determination of a moderately complicated integrand containing vector quantities, followed by an integration. When we were concerned with Coulomb's law we solved a number of examples, including the fields of the point charge, line charge, and sheet of charge. The law of Biot-Savart can be used to solve analogous problems in magnetic fields, and some of these problems now appear as exercises at the end of the chapter rather than as examples here.
One useful result is the field of the finite-length current element, shown in Fig. 8.5. It turns out (see Prob. 8 at the end of the chapter) that H is most easily expressed in terms of the angles a1 and a2, as identified in the

iH a2 — sin a1)a^

figure. The result is

If one or both ends are below point 2, then a1, or both a1 and a2, are negative.

Equation (9) may be used to find the magnetic field intensity caused by current filaments arranged as a sequence of straight line segments.

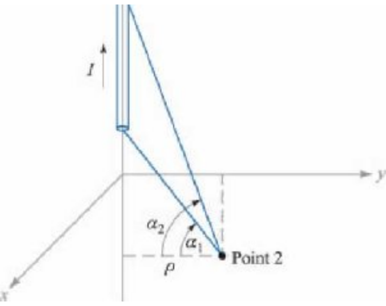
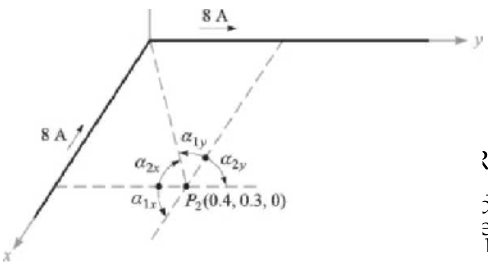


FIGURE 8.5
The magnetic field intensity caused by a finite-length current filament on the z axis is $(I/4\pi\rho)(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$.

NII^Example 8.1

As a numerical example illustrating the use of (9), let us determine H at P2(0.4, 0.3, 0) in the field of an 8-A filamentary current directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in Figure 8.6.

Solution. We first consider the semi-infinite current on the x axis, identifying the two angles, a1x — —90° and a2x — tan—1 (0.470.3) — 53.1°. The radial distance p is measured from the x axis, and we have px — 0.3. Thus, this contribution to H2 is



RE 8.6
The
dual fields of two semi-
e current segments are
by (9) and added to

$$H2(x) = 4\^fcj(\sin 53'10 + 1)a* = oi (1.8)a* =12 a*$$

The unit vector a^, must also be referred to the x axis. We see that it becomes — az. Therefore,

$$H2(x) = -\frac{12}{—} az \quad A/m$$

For the current on the y axis, we have a1y = — tan-1(0.3/0.4) = —36.9°, a2y = 90°, and py = 0.4. It follows that

$$H2(y) = \quad (1 + \sin 36.9°)(-az) = - J21\ az \quad A/m$$

Adding these results, we have

$$H2 = H2(x) + H2(y) = -\frac{20}{\sim} az = -6.37az \quad A/m$$

7.2 ampere's circuital law

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields. Here, the law that helps us solve problems more easily is known as Ampere's circuital[4] law, sometimes called Ampere's work law. This law may be derived from the Biot-Savart law, and the derivation is accomplished in Sec. 8.7. For the present we might agree to accept Ampere's circuital law temporarily as another law capable of experimental proof. As is the case with Gauss's law, its use will also require careful consideration of the symmetry of the problem to determine which variables and components are present.

Ampere's circuital law states that the line integral of h about any closed path is exactly equal to the direct current enclosed by that path,

The preferred pronunciation puts the accent on "circ-."

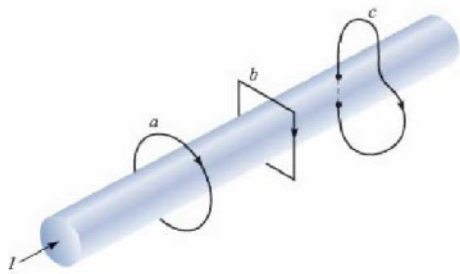
H JL I (10)

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Referring to Fig. 8.7, which shows a circular wire carrying a direct current I, the line integral of H about the closed paths lettered a and b results in an answer of I; the integral about the closed path c which passes through the conductor gives an answer less than I and is exactly that portion of the total current which is enclosed by the path c. Although paths a and b give the same answer, the integrands are, of course, different. The line integral directs us to multiply the component of H in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process, continuing until the path is completely traversed. Since H will generally vary from point to point, and since paths a and b are not alike, the contributions to the integral made by, say, each micrometer of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression "current enclosed by the path." Suppose we solder a circuit together after passing the conductor once through a rubber band, which we shall use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now let us replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.

A ?E 8.7
ctor has a total current /.
ıe integral of H about the
paths a and b is equal to /,
e integral around path c is
an / since the entire



In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

We shall find that the nature of the closed path is usually extremely simple and can be drawn on a plane. The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss's law involves finding the total charge enclosed by a closed surface; the application of Ampere's circuital law involves finding the total current enclosed by a closed path.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current I. The filament lies on the z axis in free space (as in Fig. 8.3), and the current flows in the direction given by az. Symmetry inspection comes first, showing that there is no variation with z or p. Next we determine which components of H are present by using the Biot-Savart law. Without specifically using the cross product, we may say that the direction of dH is perpendicular to the plane conaining dL and R and therefore is in the direction of ap. Hence the only component of H is Hp, and it is a function only of p.

We therefore choose a path to any section of which H is either perpendicular or tangential and along which H is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampere's circuital law with the product of the scalar magnitudes, except along that portion of the path where H is normal to the path and the dot product is zero; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which H is parallel.

In our example the path must be a circle of radius p and Ampere's circuital law becomes

H ■ dL = I Hppdp = Hpp I dp = Hp2np = I
Jo Jo

or

H I
p 27rp

as before.

As a second example of the application of Ampere's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and −I in the outer conductor. The line is shown in Fig. 8.8a. Symmetry shows that H is not a function of p or z. In order to determine the components present, we may use the results of the previous

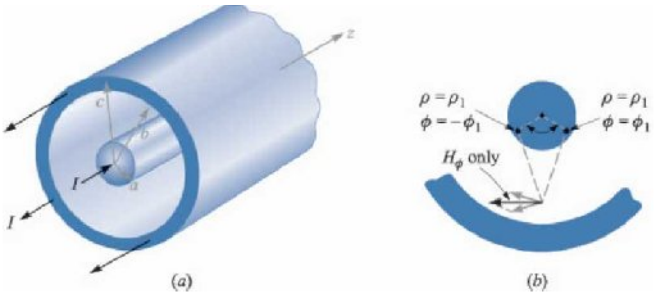


FIGURE 8.8
(a) Cross section of a coaxial cable carrying a uniformly distributed current / in the inner conductor and —/ in the outer conductor. The magnetic field at any point is most easily determined by applying Ampere's circuital law about a circular path. (b) Current filaments at p — p\, p = ±pi, produces Hp components which cancel. For the total field, H = Hpap.

example by considering the solid conductors as being composed of a large number of filaments. No filament has a z component of H. Furthermore, the Hp component at p — 0°, produced by one filament located at p = p1; p — pi,is canceled by the Hp component produced by a symmetrically located filament at p = pl, p =—p>i. This symmetry is illustrated by Fig. 8.8b. Again we find only an Hp component which varies with p.

A circular path of radius p, where p is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

Hp <p2ib) 2itp (a

If we choose p smaller than the radius of the inner conductor, the current enclosed is

^encl — I a2

Hp — I p2 2np

2 Hp a) 2ita Ip (p<

If the radius p is larger than the outer radius of the outer conductor, no current is enclosed and

Hp — 0 (p> c)

Finally, if the path lies within the outer conductor, we have

(p2 — b2\

$- I \left(\frac{}{c^2 - b^2}\right)$

$I \quad c^2 \quad \rho^2$,

The magnetic-field-strength variation with radius is shown in Fig. 8.9 for a coaxial cable in which b = 3a, c = 4a. It should be noted that the magnetic field intensity H is continuous at all the conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the enclosure of a tremendously different current. The value of Hp shows no sudden jumps.
The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude

I/27rp, but complete cancellation occurs. This is another example of "shielding"; such a coaxial cable carrying large currents would not produce any noticeable effect in an adjacent circuit.

As a final example, let us consider a sheet of current flowing in the positive y direction and located in the z = 0 plane. We may think of the return current as equally divided between two distant sheets on either side of the sheet we are considering. A sheet of uniform surface current density K = Kyay is shown in Fig. 8.10. H cannot vary with x or y. If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an Hy component. Moreover, the Biot-Savart law shows that the contributions to Hz produced by a symmetrically located pair of filaments cancel. Thus, Hz is zero also; only an Hx component is present. We therefore choose the path 1-1'-2'-2-1 composed of straight-line segments which are either parallel or perpendicular to Hx. Ampere's circuital law gives

$$H_{x1}L + H_{x2}(-L) = K_yL$$

or

$$H_{x1} - H_{x2} = K_y$$

RE 8.9 The
tic field intensity as a
on of radius in an infinitely
oaxial transmission line
e dimensions shown

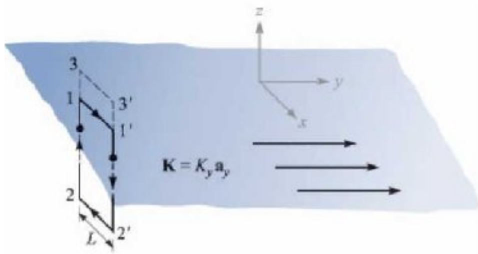
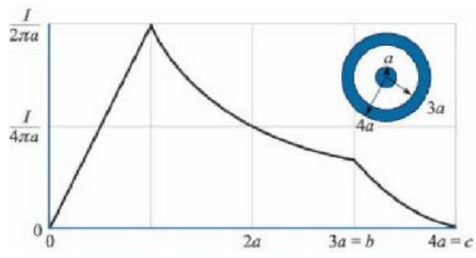


FIGURE 8.10
A uniform sheet of surface current K — K_y**a**$_y$ in the z — 0 plane. H may be found by applying Ampere's circuital law about the paths l-l'-2'-2-l and 3-3'-2'-2-3.



If the path 3-3'-2'-2-3 is now chosen, the same current is enclosed, and

$$H_{X3} - H_{X2} - K_y$$

and therefore

Hx3 — Hxl

It follows that Hx is the same for all positive z. Similarly, Hx is the same for all negative z. Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. Above the sheet,

Hx — 1 Ky (z > 0)

while below it

Hx — — 2 Ky (z < 0)

Letting be a unit vector normal (outward) to the current sheet, the result may be written in a form correct for all z as

$$H - 2 K x \tag{11}$$

If a second sheet of current flowing in the opposite direction, K — —Kyay,is placed at z — h, (11) shows that the field in the region between the current sheets is

$$H - K x a_N \quad (0 < z < h) \tag{12}$$

and is zero elsewhere,

$$H - 0 \quad (z < 0, z > h) \tag{13}$$

The most difficult part of the application of Ampere's circuital law is the determination of the components of the field which are present. The surest method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form.

Problem 13 at the end of this chapter outlines the steps involved in applying Ampere's circuital law to an infinitely long solenoid of radius a and uniform current density Kaa,, as shown in Fig. 8.11a. For reference, the result is

H = Kaaz (p< a)(14a)

H = 0 (p> a)(14b)

If the solenoid has a finite length d and consists of N closely wound turns of a filament that carries a current I (Fig. 8.11b), then the field at points well within the solenoid is given closely by

NI

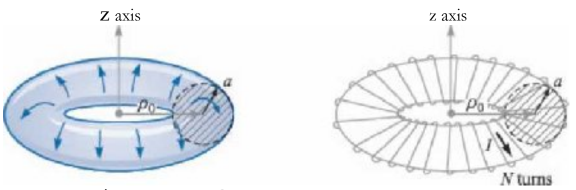
H = — az (well within the solenoid) (15)

The approximation is useful it if is not applied closer than two radii to the open ends, nor closer to the solenoid surface than twice the separation between turns.

For the toroids shown in Fig. 8.12, it can be shown that the magnetic field intensity for the ideal case, Fig. 8.12a,is

H = Ka "Lll a, (inside toroid) (16a)

H = 0 (outside)(16b)



K = Kaa1a\p=p0-a,z = G

R = Ka {inside toroid)

M
H = 2^ (well inside toroid)

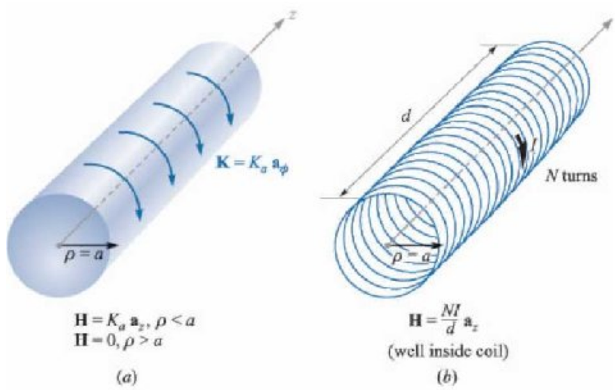


FIGURE 8.11
(*a*) An ideal solenoid of infinite length with a circular current sheet $\mathbf{K} = K_a \mathbf{a}_\phi$. (*b*) An N-turn solenoid of finite length d.

11 "0 (OLlIHKk)

FIGURE 8.12
(a) An ideal toroid carrying a surface current K in the direction shown. (b) An W-turn toroid carrying a filamentary current / .

For the TV-turn toroid of Figure 8.12b, we have the good approximations,

$$H - \frac{NI}{-} a_p \quad \text{(inside toroid)} \tag{17a}$$

$$H - 0 \, \text{(outside)} \tag{17b}$$

as long as we consider points removed from the toroidal surface by several times the separation between turns.

Toroids having rectangular cross sections are also treated quite readily, as you can see for yourself by trying Prob. 14.

Accurate formulas for solenoids, toroids, and coils of other shapes are available in Sec. 2 of the "Standard Handbook for Electrical Engineers'' (see Suggested References for Chap. 5).

7.3 CURL

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampere's circuital law to the perimeter of a differential surface element and discuss the

third and last of the special derivatives of vector analysis, the curl. Our immediate objective is to obtain the point form of Ampere's circuital law.
Again we shall choose cartesian coordinates, and an incremental closed path of sides Ax and Ay is selected (Fig. 8.13). We assume that some current, as yet unspecified, produces a reference value for H at the center of this small rectangle,

Ho = Hxo&x + ^yo^y + HZQaz

The closed line integral of H about this path is then approximately the sum of the four values of H • AL on each side. We choose the direction of traverse as 1-2-34-1, which corresponds to a current in the az direction, and the first contribution is therefore

(H • AL)i_2 = #y,i-2Ay

The value of Hy on this section of the path may be given in terms of the reference value Hy0 at the center of the rectangle, the rate of change of Hy with x, and the distance Ax/2 from the center to the midpoint of side 1-2:

Hy,i-2 = Hyo + d-H (I Ax)

Thus

(H • AL)i_2 Hy0 + 2^ 1 dHy)Ay

Along the next section of the path we have

(H • AL)2_3 = Hx,2-3("Ax) = - +2iyf Ay) Ax

Continuing for the remaining two segments and adding the results,

-y
• d L = (--H - --Hx)AxAy
\ -x

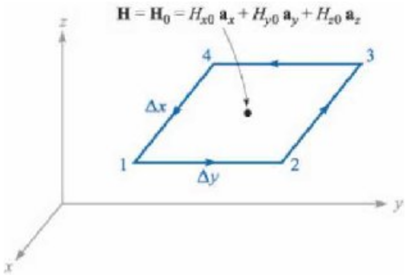


FIGURE 8.13
An incremental closed path in cartesian coordinates is selected for the application of Ampere's circuital law to determine the spatial rate of change of H.

By Ampere's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density J, the enclosed current is then $\Delta I = J_z \Delta x \Delta y$, and

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

or

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the above expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \qquad (18)$$

After beginning with Ampere's circuital law equating the closed line integral of H to the current enclosed, we have now arrived at a relationship involving the closed line integral of H per unit area enclosed and the current per unit area enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface per unit volume enclosed and charge per unit volume enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths which are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the y and z components of the current density,

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \qquad (19)$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \qquad (20)$$

Comparing (18), (19), and (20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of H about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of curl. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that the above definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} \qquad (21)$$

where ASn is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is normal to the surface enclosed by the closed path. It may represent any component in any coordinate system.
In cartesian coordinates the definition (21) shows that the x, y, and z components of the curl H are given by (18), (19), and (20), and therefore

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

This result may be written in the form of a determinant,

c

ax

x y Z

x z (23)

and may also be written in terms of the vector operator,

c = Vx H
url H (24)

Equation (22) is the result of applying the definition (21) to the cartesian coordinate system. We obtained the z component of this expression by evaluating Ampere's circuital law about an incremental path of sides Ax and Ay, and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the cartesian coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.
The expressions for curl H in cylindrical and spherical coordinates are derived in Appendix A by applying the definition (21). Although they may be written in determinant form, as explained there, the determinants do not have one row of unit vectors on top and one row of components on the bottom, and they are not easily memorized. For this reason, the curl expansions in cylindrical

and spherical coordinates which appear below and inside the back cover are usually referred to whenever necessary.

VxH 1 dH dz J" ' dHp v dz dH 1 d(pHp) 1 dHp P a P dp p dp (cylindrical) (2 5)

VxH 1 d(Hp r sin sin ((dHe\ 1/ 1 dHr d(rHp) r \sin e dp dr dHr de (2 6)
+ 9\ 1/ d(rHe) r dr — I ap (spherical)

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that E • dL = 0. Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of H, H • dL = I. Either of these closed line integrals is also known by the name of "circulation," a term obviously borrowed from the field of fluid dynamics.

The circulation of H,or H • dL, is obtained by multiplying the component of H parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampere's circuital law tells us that if H does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of E is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may now describe curl as circulation per unit area. The closed path is vanishingly small, and curl is defined at a point. The curl of E must be zero, for the circulation is zero. The curl of H is not zero, however; the circulation of H per unit area is the current density by Ampere's circuital law [or (18), (19), and (20)].

Skilling5 suggests the use of a very small paddle wheel as a "curl meter." Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the

direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Fig. 8.14a shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction of an inward normal to the surface of the page. If the velocity of water does not change as we go up- or downstream and also shows no variation as we go across the river (or even if it decreases in the same fashion toward either bank), then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the

page.

In Fig. 8.14ft the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since H = (//2np)a^, we may substitute into (25) obtaining

curl H = -dH±ap + -^az = 0

az p dp

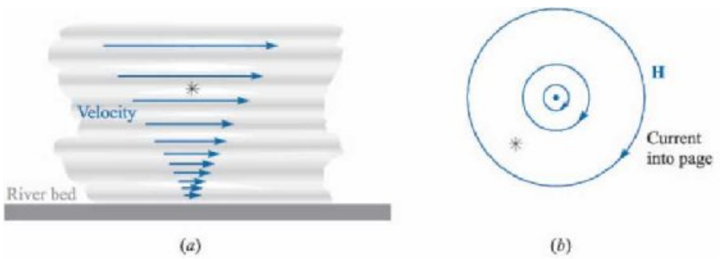


FIGURE 8.14
(*a*) The curl meter shows a component of the curl of the water velocity into the page. (*b*) The curl of the magnetic field intensity about an infinitely long filament is shown.

Example 8.2

As an example of the evaluation of curl H from the definition and of the evaluation of another line integral, let us suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Fig. 8.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > 2d$.

Solution. We evaluate the line integral of H along the four segments, beginning at the top:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0.2(z_1 + \tfrac{1}{2}d)^2 d + 0 - 0.2(z_1 - \tfrac{1}{2}d)^2 d + 0 = 0.4z_1d^2$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d \to 0} \frac{0.4z_1d^2}{d^2} = 0.4z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4z_1\mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the result above when $z = z_1$.

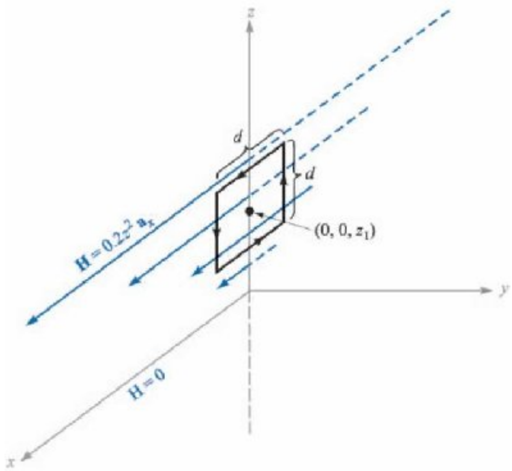


FIGURE 8.15
A square path of side d with its center on the z axis at $z = z_1$ is used to evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ and find curl H.

Returning now to complete our original examination of the application of Ampere's circuital law to a differential-sized path, we may combine (18), (19), (20), (22), and (24),

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J} \qquad (27)$$

and write the point form of Ampere's circuital law,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{28}$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of $\oint \mathbf{E} \cdot d\mathbf{L} = 0$, or

$$\nabla \times \mathbf{E} = 0 \tag{29}$$

The fourth equation appears in Sec. 8.5.

7.4 STOKES' THEOREM

Although the last section was devoted primarily to a discussion of the curl opeation, the contribution to the subject of magnetic fields should not be overlooked. From Ampere's circuital law we derived one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$. This latter equation should be considered the point form of Ampere's circuital law and applies on a "per-unit-area" basis. In this section we shall again devote a major share of the material to the mathematical theorem known as Stokes' theorem, but in the process we shall show that we may obtain Ampere's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. In other words, we are then prepared to obtain the integral form from the point form or to obtain the point form from the integral form.

Consider the surface S of Fig. 8.16 which is broken up into incremental surfaces of area AS. If we apply the definition of the curl to one of these incremental surfaces, then

(V x H)n

where the N subscript again indicates the right-hand normal to the surface. The subscript on dLAS indicates that the closed path is the perimeter of an incremental area AS. This result may also be written

(V x H)- an

or

H ■ JLas = (V x H ■ an AS = (Vx H)- AS

where aN is a unit vector in the direction of the right-hand normal to AS.

Now let us determine this circulation for every AS comprising S and sum the results. As we evaluate the closed line integral for each AS, some cancellation will occur because every interior wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore we have

(Vx H)-dSH ■ dL = (3

S

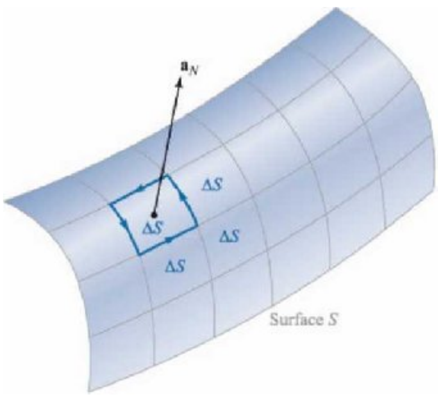


FIGURE 8.16

The sum of the closed line integrals about the perimeter of every ΔS is the same as the closed line integral about the perimeter of S because of cancellation on every interior path.

0)

where dL is taken only on the perimeter of S.

Equation (30) is an identity, holding for any vector field, and is known as Stokes' theorem.

Example 7.3

A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Fig. 8.17. The surface is specified by r = 4, 0 < 0 < 0.1jr,0 < 0 < 0.3jr, and the closed path forming its perimeter is composed of three circular arcs. We are given the field H = 6r sin 0ar + 18r sin 0cos 0a0 and are asked to evaluate each side of Stokes' theorem.

Solution. The first path segment is described in spherical coordinates by r = 4, 0 < 0 < 0.1ir, 0 = 0; the second one by r = 4, 0 = 0.1ir,0 < 0 < 0.3jr; and the third by r = 4, 0 < 0 < 0.1jr, 0 = 0.3jr. The differential path element dL is the vector sum of the three differential lengths of the spherical coordinate system first discussed in Sec.
1.9,

d L = dr ar + rd0 a0 + r sin 0 d 0 a0

The first term is zero on all three segments of the path since r = 4 and dr = 0, the second is zero on segment 2 since 0 is constant, and the third term is zero on both segments 1 and 3. Thus

i^>H • dL = jH0rd0 + jH0rsin0d0 + jH0rd0

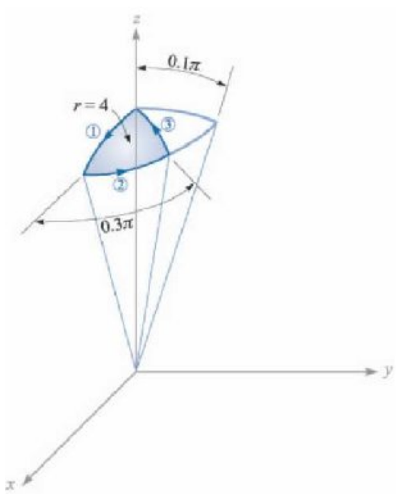


FIGURE 8.17
A portion of a spherical cap is used as a surface and a closed path to illustrate Stokes' theorem.

Since H6 = 0, we have only the second integral to evaluate,

H • dL = J [18(4)sinO.ljrcos0]4sin0.lird0

= 288 sin2 0. lit sin 0.3i = 22.2A We next attack the surface integral. First, we use (26) to find

VxH = —1— (36rsin6cos6cos0)ar + -\ } 6rcos0 — 36rsin6cos0)a6
r sin 6 r \sm 6

Since dS = r2 sin 6 d6 d0 ar, the integral is

p /*0.3ir pO.lit
(Vx H)-dS = / / (36cos6cos0)16sin6d6d0
is J0
00

cos 0d0

,2 a\ I

/ 576(2 sin2

02

= 288 sin2 0. lit sin 0.3it = 22.2A

Thus, the results check Stokes' theorem, and we note in passing that a current of 22.2 A is flowing upward through this section of a spherical cap.

Next, let us see how easy it is to obtain Ampere's circuital law from VxH = J. We merely have to dot each side by dS, integrate each side over the same (open) surface S, and apply Stokes' theorem:

f(Vx H)- dS = I J • dS =<£> H • dL

The integral of the current density over the surface S is the total current / passing through the surface, and therefore

• dL = /

This short derivation shows clearly that the current /, described as being "enclosed by the closed path," is also the current passing through any of the infinite number of surfaces which have the closed path as a perimeter.

Stokes' theorem relates a surface integral to a closed line integral. It should be recalled that the divergence theorem relates a volume integral to a closed surface integral. Both theorems find their greatest use in general vector proofs. As an example, let us find another expression for V-Vx A, where A represents any vector field. The result must be a scalar (why?), and we may let this scalar be T ,or

V-Vx A = T

Multiplying by dv and integrating throughout any volume v,

/ (V-Vx A)dv = Tdv

J vol J vol

we first apply the divergence theorem to the left side, obtaining

<t(V x A) - dS = f Tdv

JS Jvol

The left side is the surface integral of the curl of A over the closed surface surrounding the volume v. Stokes' theorem relates the surface integral of the curl of A over the open surface enclosed by a given closed path. If we think of the path as the opening of a laundry bag and the open surface as the surface of the bag itself, we see that as we gradually approach a closed surface by pulling on the drawstrings, the closed path becomes smaller and smaller and finally disappears as the surface becomes closed. Hence the application of Stokes' theorem to a closed surface produces a zero result, and we have

j Tdv = 0

Since this is true for any volume, it is true for the differential volume dv,

Tdv = 0

and therefore

T=0

or

V-Vx (3
A = i 1)

Equation (31) is a useful identity of vector calculus.6 Of course, it may also be proven easily by direct expansion in cartesian coordinates.
Let us apply the identity to the non-time-varying magnetic field for which

VxH = J

This shows quickly that

V - J = 0

which is the same result we obtained earlier in the chapter by using the continuity equation.
Before introducing several new magnetic field quantities in the following section, we may review our accomplishments at this point. We initially accepted the Biot-Savart law as an experimental result,

J 4nR2

This and other vector identities are tabulated in Appendix A.3.

and tentatively accepted Ampere's circuital law, subject to later proof,

• dL = /

From Ampere's circuital law the definition of curl led to the point form of this same law,

VxH = J

We now see that Stokes' theorem enables us to obtain the integral form of Ampere's circuital law from the point of form.

\/ D8.6. Evaluate both sides of Stokes' theorem for the field H = 6xyax — 3y2ay A/m and the rectangular path around the region, 2 < x < 5, —1 < y < l, z = 0. Let the positive direction of dS be az.

Ans. —126 A; —126 A

7.5 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the magnetic flux density B as

B = /z0H | (free space only) (32)

where B is measured in webers per square meter (Wb/m2) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1

Wb/m2 is the same as 10000 G. The constant /z0 is not dimensionless and has the defined value for free space, in henrys per meter (H/m), of

^0 = 4i x 10 (3
22 H/m 3)

An alternate analogy is presented in Sec. 10.2.[22]

The name given to /z0 is the permeability of free space.

We should note that since H is measured in amperes per meter, the weber is dimensionally equal to the product of henrys and amperes. Considering the henry as a new unit, the weber is merely a convenient abbreviation for the product of henrys and amperes. When time-varying fields are introduced, it will be shown that a weber is also equivalent to the product of volts and seconds.

The magnetic-flux-density vector B, as the name weber per square meter implies, is a member of the flux-density family of vector fields. One of the possible analogies between electric and magnetic fields[23] compares the laws of Biot-Savart and Coulomb, thus establishing an analogy between H and E. The relations B = /z0H and D = e0E then lead to an analogy between B and D.If B is measured in teslas or webers per square meter, then magnetic flux should be measured in webers. Let us represent magnetic flux by $ and define $ as the flux passing through any designated area,

[23] An alternate analogy is presented in Sec. 10.2.

(37)

) s B dS Wb (34)

Our analogy should now remind us of the electric flux measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

1 s D • d S = Q

The charge Q is the source of the lines of electric flux and these lines begin and terminate on positive and negative charge, respectively.

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current /, the H field formed concentric circles about the filament. Since B = /zoH, the B field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge." For this reason Gauss's law for the magnetic field is

1 s B d S = 0 (35)

and application of the divergence theorem shows us that

V-B = i (36)

We have not proved (35) or (36) but have only suggested the truth of these statements by considering the single field of the infinite filament. It is possible to show that (35) or (36) follows from the Biot-Savart law and the definition of B, B = /x0H, but this is another proof which we shall postpone to Sec. 8.7.

Equation (36) is the last of Maxwell's four equations as they apply to static electric fields and steady magnetic fields. Collecting these equations, we then have for static electric fields and steady magnetic fields

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}}$$

To these equations we may add the two expressions relating D to E and B to H in free space,

D = eoE (38)

(37)

B (3
= /z0H 9)

We have also found it helpful to define an electrostatic potential,

V
E = (4
-Vl 0)

and we shall discuss a potential for the steady magnetic field in the following section. In addition, we have extended our coverage of electric fields to include conducting materials and dielectrics, and we have introduced the polarization P. A similar treatment will be applied to magnetic fields in the next chapter.
Returning to (37), it may be noted that these four equations specify the divergence and curl of an electric and a magnetic field. The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is

0 D d S = Q pvdv /
=
Js J vol
j e d L = 0
d L = I = Si • dS
J Is
S B d S = 0
J
s

(4
1)

Our study of electric and magnetic fields would have been much simpler if we could have begun with either set of equations, (37) or (41). With a good knowledge of vector analysis, such as we should now have, either set may be readily obtained from the other by applying the divergence theorem of Stokes' theorem. The various experimental laws can be obtained easily from these equations.
As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Fig. 8.8a. The magnetic field intensity was found to be

H I
p 2
itp a <p< b) (

and therefore

n ii
b = /zo±i = -— a,*
ZJtp

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from p = a to p = b and from, say, z = 0 to z = d

<J> = B • dS = -— a,* • dp dz a,*
Js Jo Ja 2np

(37)

or

$$<J> = \wedge \ln \frac{b}{a} \quad 2n \qquad (42)$$

This expression will be used later to obtain the inductance of the coaxial transmission line.

7.6 THE SCALAR AND VECTOR MAGNETIC POTENTIALS

The solution of electrostatic field problems is greatly simplified by the use of the scalar electrostatic potential V. Although this potential possesses a very real physical significance for us, it is mathematically no more than a stepping-stone which allows us to solve a problem by several smaller steps. Given a charge configuration, we may first find the potential and then from it the electric field intensity.

We should question whether or not such assistance is available in magnetic fields. Can we define a potential function which may be found from the current distribution and from which the magnetic fields may be easily determined? Can a scalar magnetic potential be defined, similar to the scalar electrostatic potential? We shall show in the next few pages that the answer to the first question is "yes," but the second must be answered "sometimes." Let us attack the last question first by assuming the existence of a scalar magnetic potential, which we designate V_m, whose negative gradient gives the magnetic field intensity

$$H = -VV_m$$

The selection of the negative gradient will provide us with a closer analogy to the electric potential and to problems which we have already solved.

This definition must not conflict with our previous results for the magnetic field, and therefore

$$V \times H = J = V \times (-VV_m)$$

However, the curl of the gradient of any scalar is identically zero, a vector identity the proof of which is left for a leisure moment. Therefore we see that if H is to be defined as the gradient of a scalar magnetic potential, then current density must be zero throughout the region in which the scalar magnetic potential is so defined. We then have

$$H = -VV_m \quad (J = 0) \qquad (43)$$

Since many magnetic problems involve geometries in which the current-carrying conductors occupy a relatively small fraction of the total region of interest, it is evident that a scalar magnetic potential can be useful. The scalar magnetic potential is also applicable in the case of permanent magnets. The dimensions of V_m are obviously amperes.

This scalar potential also satisfies Laplace's equation. In free space,

$$V \cdot B = /z0 V \cdot H = 0$$

and hence

$$\wedge V \cdot (-VV_m) = 0$$

or

$$V2\ V_m = 0 \qquad (J = 0) \qquad (44)$$

We shall see later that V_m continues to satisfy Laplace's equation in homogeneous magnetic materials; it is not defined in any region in which current density is present.

Although we shall consider the scalar magnetic potential to a much greater extent in the next chapter, when we introduce magnetic materials and discuss the magnetic circuit, one difference between V and V_m should be pointed out now: V_m is not a single-valued function of position. The electric potential V is single-valued; once a zero reference is assigned, there is only one value of V associated with each point in space. Such is not the case with V_m. Consider the cross section of the coaxial line shown in Fig. 8.18. In the region $a < p < b$, $J = 0$, and we may establish a scalar magnetic potential. The value of H is

$$H = 2\wedge\ a_p$$

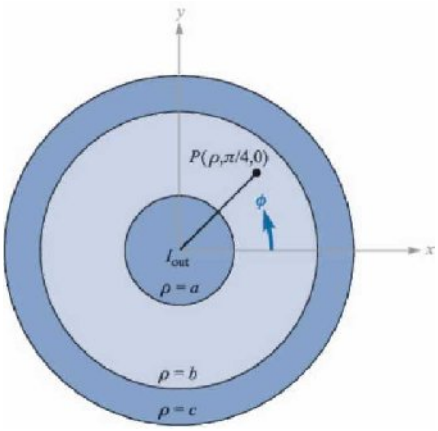


FIGURE 8.18
The scalar magnetic potential V_m is a multivalued function of p in the region $a < p < b$. The electrostatic potential is always single-valued.

where I is the total current flowing in the az direction in the inner conductor. Let us find Vm by integrating the appropriate component of the gradient. Applying (43),

I = -wm 1 dvVn
27 p p dp
rp

or

Vm d I
p 2ttd

Thus

m Vi p
2
n

where the constant of integration has been set equal to zero. What value of potential do we associate with point P, where p = 7r/4? If we let Vm be zero at p = 0 and proceed counterclockwise around the circle, the magnetic potential goes negative linearly. When we have made one circuit, the potential is -i,but that was the point at which we said the potential was zero a moment ago. At P, then, p = 7t/4, 9tt/4, 17tt/4, ...,or -7tt/4, -15tt/4, -23tt/4, ...,or

Vm
P (2n - jr)jr (n = 0, ±1, ±2, ...)

or

VmP = I(n - i)(n = 0, ±1, ±2, ...)

The reason for this multivaluedness may be shown by a comparison with the electrostatic case. There, we know that

VxE = 0 (j)E

■ dL = 0

and therefore the line integral

Vab = — / E ■ dL

Jb

is independent of the path. In the magnetostatic case, however,

VxH = 0 (wherever J = 0)

but

■ dL = /

even if J is zero along the path of integration. Every time we make another complete lap around the current, the result of the integration increases by I .If no current I is enclosed by the path, then a single-valued potential function may be defined. In general, however,

a

Vm>ab = — I H ■ dL (4 5)

(specified path)

b

where a specific path or type of path must be selected. We should remember that the electrostatic potential V is a conservative field; the magnetic scalar potential Vm is not a conservative field. In our coaxial problem let us erect a barrier24 at p = n; we agree not to select a path which crosses this plane. Therefore we cannot encircle I, and a single-valued potential is possible. The result is seen to be

Vm = —1- <j> (—lt<p<lt)

2n

and

vmP = —8 (* = 4)

The scalar magnetic potential is evidently the quantity whose equipotential surfaces will form curvilinear squares with the streamlines of H in Fig. 8.4. This is one more facet of the analogy between electric and magnetic fields about which we will have more to say in the next chapter.

Let us temporarily leave the scalar magnetic potential now and investigate a vector magnetic potential. This vector field is one which is extremely useful in studying radiation from antennas, from apertures, and radiation leakage from transmission lines, waveguides, and microwave ovens. The vector magnetic potential may be used in regions where the current density is zero or nonzero, and we shall also be able to extend it to the time-varying case later.

Our choice of a vector magnetic potential is indicated by noting that

V-B = 0

Next, a vector identity which we proved in Sec. 8.4 shows that the divergence of the curl of any vector field is zero. Therefore we select

This corresponds to the more precise mathematical term "branch cut."

$$B = V \times A \tag{46}$$

where A signifies a vector magnetic potential, and we automatically satisfy the condition that the magnetic flux density shall have zero divergence. The H field is

$$H = — V \times A$$

and

$$V \times H = J = — V \times V \times A$$

The curl of the curl of a vector field is not zero and is given by a fairly complicated expression,[25] which we need not know now in general form. In specific cases for which the form of A is known, the curl operation may be applied twice to determine the current density.

Equation (46) serves as a useful definition of the vector magnetic potential A. Since the curl operation implies differentiation with respect to a length, the units of A are webers per meter.

As yet we have seen only that the definition for A does not conflict with any previous results. It still remains to show that this particular definition can help us to determine magnetic fields more easily. We certainly cannot identify A with any easily measured quantity or history-making experiment.

We shall show in the following section that, given the Biot-Savart law, the definition of B, and the definition of A, then A may be determined from the differential current elements by

[25] $V \times V \times A = V(V \bullet A) - V^2A$. In cartesian coordinates, it may be shown that $V^2A = V^2A_x\mathbf{a}_x + V^2A_y\mathbf{a}_y + V^2A_z\mathbf{a}_z$. In other coordinate systems, V^2A may be found by evaluating the second-order partial derivatives in $V^2A = V(V \bullet A) - V \times V \times A$.

A i iJLQld L, 4nR (47)

The significance of the terms in (47) is the same as in the Biot-Savart law; a direct current I flows along a filamentary conductor of which any differential length dL is distant R from the point at which A is to be found. Since we have defined A only through specification of its curl, it is possible to add the gradient of any scalar field to (47) without changing B or H, for the curl of the gradient is identically zero. In steady magnetic fields, it is customary to set this possible added term equal to zero.

The fact that A is a vector magnetic potential is more apparent when (47) is compared with the similar expression for the electrostatic potential,

v

" PhdL
4ne0R

Each expression is the integral along a line source, in one case line charge and in the other case line current; each integrand is inversely proportional to the distance from the source to the point of interest; and each involves a characteristic of the medium (here free space), the permeability or the permittivity. Equation (47) may be written in differential form,

dA = (48)

if we again agree not to attribute any physical significance to any magnetic fields we obtain from (48) until the entire closed path in which the current flows is considered.

With this reservation, let us go right ahead and consider the vector magnetic potential field about a differential filament. We locate the filament at the origin in free space, as shown in Fig. 8.19, and allow it to extend in the positive z direction so that dL = dz az. We use cylindrical coordinates to find dA at the point (p, p, z):

Joldz az
dA

or

= J°/dz : dAp = 0 dAp = 0 (49)
4^yp_2Tz2

We note that the direction of dA is the same as that of IdL. Each small section of a current-carrying conductor produces a contribution to the total vector magnetic potential which is in the same direction as the current flow in the conductor. The magnitude ofthe vector magnetic potential varies inversely as the distance to the current element, being strongest in the neighborhood of the

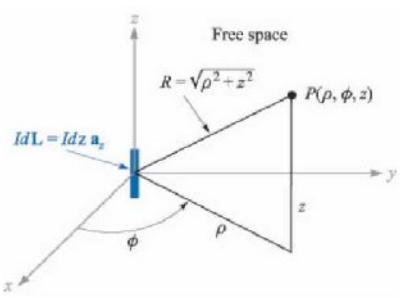


FIGURE 8.19
The differential current element $I\,dz\mathbf{a}_z$ at the origin establishes the differential vector magnetic potential field, $d\mathbf{A} = \dfrac{\mu_0 I\,dz\mathbf{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$ at $P(\rho, \phi, z)$.

current and gradually falling off to zero at distant points. Skilling26 describes the vector magnetic potential field as "like the current distribution but fuzzy around the edges, or like a picture of the current out of focus."
In order to find the magnetic field intensity, we must take the curl of (49) in cylindrical coordinates, leading to

dH = — V x d A =

/Z0 (J,0\ dp

or

... Idz p

4tT (p2 + Z2)3/2 P

which is easily shown to be the same as the value given by the Biot-Savart law.

Expressions for the vector magnetic potential A can also be obtained for a current source which is distributed. For a current sheet K, the differential current element becomes

Id L = K dS

In the case of current flow throughout a volume with a density J, we have

IdL = J dv

In each of these two expressions the vector character is given to the current. For the filamentary element it is customary, although not necessary, to use IdL instead of I dL. Since the magnitude of the filamentary element is constant, we have chosen the form which allows us to remove one quantity from the integral. The alternative expressions for A are then

See the Suggested References at the end of the chapter.

$$\mathbf{A} = \oint_S \frac{\mu_0 \mathbf{K}\, dS}{4\pi R} \tag{50}$$

and

$$\mathbf{A} = \int_{vol} \frac{\mu_0 \mathbf{J}\, dv}{4\pi R} \tag{51}$$

Equations (47), (50), and (51) express the vector magnetic potential as an integration over all of its sources. From a comparison of the form of these integrals with those which yield the electrostatic potential, it is evident that once again the zero reference for $\mathbf{A}$ is at infinity, for no finite current element can produce any contribution as $R \to \infty$. We should remember that we very seldom used the similar expressions for V; too often our theoretical problems included charge distributions which extended to infinity and the result would be an infinite potential everywhere. Actually, we calculated very few potential fields until the differential form of the potential equation was obtained, $\nabla^2 V = -\rho_v/\epsilon$, or better yet, $\nabla^2 V = 0$. We were then at liberty to select our own zero reference.

The analogous expressions for $\mathbf{A}$ will be derived in the next section, and an example of the calculation of a vector magnetic potential field will be completed.

PROBLEMS

8.1 (a) Find $\mathbf{H}$ in cartesian components at $P(2, 3, 4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction. (b) Repeat if the filament is located at $x = -1$, $y = 2$. (c) Find $\mathbf{H}$ if both filaments are present.

8.2 A current filament of $3\mathbf{a}_x$ A lies along the x axis. Find $\mathbf{H}$ in cartesian components at $P(-1, 3, 2)$.

8.3 Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < a$ and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction. (a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$. (b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be half the value obtained for an infinite filament?

8.4 (a) A filament is formed into a circle of radius a, centered at the origin in the plane $z = 0$. It carries a current I in the $\mathbf{a}_\phi$ direction. Find $\mathbf{H}$ at the origin. (b) A filament of the same length is shaped into a square in the $z = 0$ plane. The sides are parallel to the coordinate axes and a current I flows in the general $\mathbf{a}_\phi$ direction. Again find $\mathbf{H}$ at the origin.

8.5 The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$.

8.6 (a) A current filament I is formed into a circle, $\rho = a$, in the $z = z'$ plane. Find H_z at $P(0, 0, z)$ if I flows in the $\mathbf{a}_\phi$ direction. (b) Find H_z at P caused by a uniform surface current density $\mathbf{K} = K_0\mathbf{a}_\phi$, flowing on the cylindrical surface, $\rho = a$, $0 < z < h$. The results of part (a) should help.

8.7 Given points $C(5, -2, 3)$ and $P(4, -1, 2)$, a current element $I\,d\mathbf{L} = 10^{-4}(4, -3, 1)$ A $\cdot$ m at C produces a field $d\mathbf{H}$ at P. (a) Specify the direction of $d\mathbf{H}$ by a unit vector $\mathbf{a}_H$. (b) Find $|d\mathbf{H}|$. (c) What direction $\mathbf{a}_L$ should $I\,d\mathbf{L}$ have at C so that $d\mathbf{H} = 0$?

FIGURE 8.21

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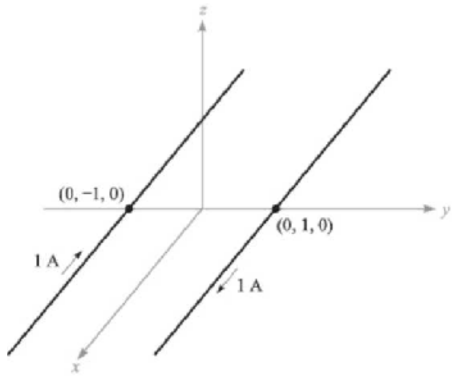
8.8 For the finite-length current element on the z axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1.



8.9 A current sheet K = 8ax A/m flows in the region −2 < y < 2m in the plane z = 0. Calculate H at P(0, 0, 3).

8.10 Let a filamentary current of 5 mA be directed from infinity to the origin on the positive z axis and then back out to infinity on the positive x axis.
Find H at P(0, 1 , 0).

8.11 An infinite filament on the z axis carries 20^mA in the az direction. Three uniform cylindrical current sheets are also present: 400mA/m at p = 1cm, −250mA/mat p = 2 cm, and − 300mA/mat p = 3 cm. Calculate Hp at p = 0.5, 1.5, 2.5, and 3.5 cm.

8.12 In Fig. 8.22, let the regions 0 < z < 0.3m and 0.7 < z < 1.0m be conducting slabs carrying uniform current densities of 10 A/m2 in opposite directions as shown. Find H at z =: (a) −0.2; (b) 0.2 (c) 0.4; (d) 0.75; (e)1.2 m.

8.13 A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of X"aap.(a) Show that H is not a function of p or z.(b) Show that Hp and Hp are everywhere zero. (c) Show that Hz = 0 for p> a.(d) Show that Hz = for p< a.(e)A second shell, p = b, carries a current iQ,ap. Find H everywhere.

8.14 A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders p = 2 cm and p = 3 cm, and the planes z = 1 cm and z = 2.5 cm. The toroid carries a surface current density of −50az A/m on the surface p = 3 cm. Find H at the point P(p, p, z):(a) PA(1.5cm, 0, 2cm);(b) PB(2.1cm, 0, 2cm);(c) Pc(2.7 cm, tt/2, 2cm);(d) 3.5 cm, 7r/2, 2cm).

8.15 Assume that there is a region with cylindrical symmetry in which the conductivity is given by a = 1.5e−150pkS/m. An electric field of 30az V/m is present. (a) Find J.(b) Find the total current crossing the surface p<p0, z = 0, all p.(c) Make use of Ampere's circuital law to find H.

FIGURE 8.22

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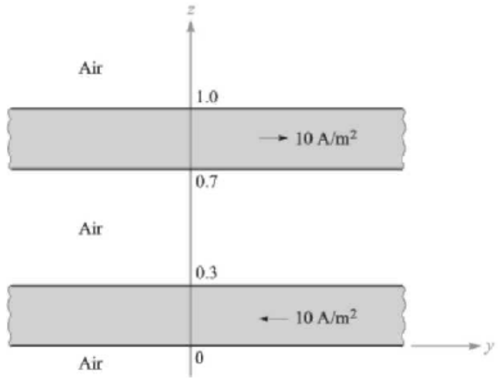

8.16 The cylindrical shell, 2 mm <p< 3 mm, carries a uniformly distributed total current of 8 A in the —az direction, and a filament on the z axis carries 8 A in the az direction. Find H everywhere.

8.17 A current filament on the z axis carries a current of 7 mA in the az direction, and current sheets of 0.5az A/m and —0.2az A/m are located at p = 1 cm and p = 0.5 cm, respectively. Calculate H at p =: (a) 0.5 cm;
(b)1.5 cm; (c) 4cm; (d) What current sheet should be located at p = 4cm
so that H = 0 for all p> 4 cm?

8.18 Current density is distributed as follows: J = 0 for |y| > 2 m, J = 8yaz A/m2 for |y| < 1m, J = 8(2 — y)az A/m2 for 1 < y < 2m, J = —8(2 + y)az A/m2 for —2 < y < — 1 m. Use symmetry and Ampere's law to find H everywhere.

8.19 Calculate V x [V(V • G)] if G = 2x2yzax — 20yay + (x2 — z2)az.

8.20 The magnetic field intensity is given in the square region x = 0, 0.5 < y < 1, 1 < z < 1.5by H = z2ax + x3ay + y4az A/m. (a) Evaluate
H • dL about the perimeter of the square region. (b) Find Vx H.
(c) Calculate (V x H)x at the center of the region. (d) Does (V x H)x = [H • dL]/Area enclosed?

8.21 Points A, B, C, D, E, and F are each 2mm from the origin on the
coordinate axis indicated in Fig. 8.23. The value of H at each point is
given. Calculate an approximate value for Vx H at the origin.

2p
8.22 In the cylindrical region p < 0.6 mm, Hp =- + - A/m, while Hp =
3 p 2
-A/m for p> 0.6mm. (a) Determine J for p< 0.6mm. (b) Deter-
p
mine J for p> 0.6mm. (c) Is there a filamentary current at p = 0? If so, what is its value? (d) What is J at p = 0? 8.23 Given the field H = 20p2ap A/m: (a) determine the current density J;(b) integrate J over the circular surface p = 1,0 <p< 27r, z = 0, to determine the total current passing through that surface in the az direction; (c) find the total current once more, this time by a line integral around the circular path p = 1,0 <p < 27r, z = 0.

Point	H (A/m)		
A	$11.34\mathbf{a}_x$	$-13.78\mathbf{a}_y$	$+14.21\mathbf{a}_z$
B	$10.68\mathbf{a}_x$	$-12.19\mathbf{a}_y$	$+15.82\mathbf{a}_z$
C	$10.49\mathbf{a}_x$	$-12.19\mathbf{a}_y$	$+15.69\mathbf{a}_z$
D	$11.49\mathbf{a}_x$	$-13.78\mathbf{a}_y$	$+14.35\mathbf{a}_z$
E	$11.11\mathbf{a}_x$	$-13.88\mathbf{a}_y$	$+15.10\mathbf{a}_z$
F	$10.88\mathbf{a}_x$	$-13.10\mathbf{a}_y$	$+14.90\mathbf{a}_z$

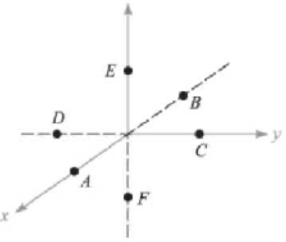

FIGURE 8.23

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8.24 Evaluate both sides of Stokes' theorem for the field G = 10 sin 9ap and the surface r = 3, 0 < 9 < 90°,0 < p < 90°. Let the surface have the a, direction. 1 p p

8.25 Given the field H = 2 cos 2 ap — sin 2 ap A/m, evaluate both sides of

Stokes' theorem for the path formed by the intersection of the cylinder p = 3 and the plane z = 2, and for the surface defined by p = 3, 0 < z < 2, and z = 0, 0 < p < 3.

8.26 Let G = 15rap.(a) Determine §G • dL for the circular path r = 5, 9 = 25°,0 < p < 27r. Evaluate Js(V x G)^ dS over the spherical cap r = 5, 0 < 9 < 25°,0 < p < 2tt.

8.27 The magnetic field intensity is given in a certain region of space as

H = x + 2y ay + 2 az A/m. (a) Find VxH.(b) Find J.(c) Use J to

z2 z

find the total current passing through the surface z = 4, 1 < x < 2, 3 < z < 5, in the az direction. (d) Show that the same result is obtained using the other side of Stokes' theorem.

8.28 Given H = (3r2/ sin 9)a9 + 54r cos 9ap A/m in free space: (a) find the total current in the A9 direction through the conical surface 9 = 20°, 0 < p < 27r,0 < r < 5, by whichever side of Stokes' theorem you like the best. (b) Check the result by using the other side of Stokes' theorem.

8.29 A long straight nonmagnetic conductor of 0.2-mm radius carries a uniformly distributed current of 2 A dc. (a) Find J within the conductor. (b) Use Ampere's circuital law to find H and B within the conductor. (c) Show that Vx H = J within the conductor. (d) Find H and B within the conductor. (e) Show that VxH = J outside the conductor.

8.30 A solid nonmagnetic conductor of circular cross section has a radius of 2mm. The conductor is inhomogeneous, with a = 106(1 + 106p2) S/m. If the conductor is 1 m in length and has a voltage of 1 mV between its ends, find: (a) H;(b) the total magnetic flux inside the conductor.

8.31 The cylindrical shell defined by 1 cm <p< 1.4 cm consists of a nonmagnetic conducting material and carries a total current of 50 A in the az direction. Find the total magnetic flux crossing the plane p = 0, 0 < z < 1: (a)0 <p< 1.2cm; (b) 1.4cm <p< 1.4cm; (c) 1.4cm <p< 20 cm.

8.32 The free-space region defined by 1 < z < 4cm and 2 <p<3cm is a toroid of rectangular cross section. Let the surface at p = 3 cm carry a surface current K = 2az kA/m. (a) Specify the currents on the surfaces at p = 2 cm, z = 1 cm, and z = 4 cm. (b) Find H everywhere. (c) Calculate the total flux within the toroid.

8.33 Use an expansion in cartesian coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero.

8.34 A filamentary conductor on the z axis carries a current of 16 A in the az direction, a conducting shell at p = 6 carries a total current of12A in the —az direction, and another shell at p = 10 carries a total current of 4 A in the — az direction. (a) Find H for 0 <p < 12. (b) Plot Hp versus p.(c) Find the total flux <J> crossing the surface 1 <p< 7, 0 < z < 1.

8.35 A current sheet, K = 20az A/m, is located at p = 2, and a second sheet, K =—10az A/m, is located at p = 4. (a) Let Vm = 0at P(p = 3, p = 0, z = 5) and place a barrier at p = 7r. Find Vm(p, p, z) for —7r< p<7T.(b) Let A = 0at P and find A(p, p, z) for 2 <p< 4.

8.36 Let A = (3y — z)ax + 2xzay Wb/m in a certain region of free space. (a) Show that V- A = 0. (b)At P(2, —1, 3), find A, B, H, and J.

8.37 Let N = 1000, / = 0.8A, p0 = 2 cm, and a = 0.8 cm for the toroid shown in Fig. 8.12b. Find Vm in the interior of the toroid if Vm = 0at p = 2.5 cm, p = 0.37T. Keep p within the range 0 <p< 27r.

8.38 The solenoid shown in Fig. 8.11b contains 400 turns, carries a current / = 5 A, has a length of 8 cm, and a radius a = 1.2 cm. (a) Find H within the solenoid. (b)If Vm = 0 at the origin, specify Vm(p, p, z) inside the solenoid. (c) Let A = 0 at the origin, and specify A(p, p, z) inside the solenoid if the medium is free space.

8.39 Planar current sheets of K = 30az A/m and —30az A/m are located in free space at x = 0.2 and x = —0.2, respectively. For the region — 0.2 < x < 0.2: (a) find H;(b) obtain an expression for Vm if Vm = 0 at P(0.1, 0.2, 0.3);(c) find B;(d) obtain an expression for A if A = 0at P.

8.40 Let A = (3y2 — 2z)ax — 2x2zay + (x + 2y)az Wb/m in free space. Find VxVx A at P(—2, 3, —1).

8.41 Assume that A = 50p2az Wb/m in a certain region of free space. (a) Find H and B.(b) Find J.(c) Use J to find the total current crossing the surface 0 < p < 1,0 < p< 27r, z = 0. (d) Use the value of Hp at p = lto calculate §H • dL for p = 1, z = 0.

8.42 Show that V2(1/Ri2) = —Vi(1/Ri2) = R21/RL.

8.43 Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is 7a. Select the proper zero reference and sketch the results on the figure.

8.44 By expanding Eq. (58), Sec. 8.7, in cartesian coordinates, show that (59) is correct.

Chapter Eight

MAGNETIC FORCES, MATERIALS, AND INDUCTANCE

8.1 FORCE ON A MOVING CHARGE

In an electric field the definition of the electric field intensity shows us that the force on a charged particle is

$$F = QE \tag{1}$$

The force is in the same direction as the electric field intensity (for a positive charge) and is directly proportional to both E and Q. If the charge is in motion, the force at any point in its trajectory is then given by (1).

A charged particle in motion in a magnetic field of flux density B is found experimentally to experience a force whose magnitude is proportional to the product of the magnitudes of the charge Q, its velocity v, and the flux density B, and to the sine of the angle between the vectors v and B. The direction of the force is perpendicular to both v and B and is given by a unit vector in the direction of v x B. The force may therefore be expressed as

$$= Q\mathbf{v} \times \qquad F \tag{2}$$

A fundamental difference in the effect of the electric and magnetic fields on charged particles is now apparent, for a force which is always applied in a direction at right angles to the direction in which the particle is proceeding can never change the magnitude of the particle velocity. In other words, the acceleration vector is always normal to the velocity vector. The kinetic energy of the particle remains unchanged, and it follows that the steady magnetic field is incapable of transferring energy to the moving charge. The electric field, on the other hand, exerts a force on the particle which is independent of the direction in which the particle is progressing and therefore effects an energy transfer between field and particle in general.

The first two problems at the end of this chapter illustrate the different effects of electric and magnetic fields on the kinetic energy of a charged particle moving in free space.
The force on a moving particle due to combined electric and magnetic fields is obtained easily by superposition,

$$= Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

This equation is known as the Lorentz force equation, and its solution is required in determining electron orbits in the magnetron, proton paths in the cyclotron, plasma characteristics in a magnetohydrodynamic (MHD) generator, or, in general, charged-particle motion in combined electric and magnetic fields.

\/ D9.1. The point charge Q = 18 nC has a velocity of 5 x 106m/s in the direction a„ = 0.04ax — 0.05ay + 0.2az. Calculate the magnitude of the force exerted on the charge by the field: (a) B = —3ax + 4ay + 6az mT; (b) E = —3ax + 4ay + 6az kV/m; (c) B and E acting together.

Ans. 124.6 /N; 140.6 /N; 187.8 /xN

8.2 FORCE ON A DIFFERENTIAL CURRENT ELEMENT

The force on a charged particle moving through a steady magnetic field may be written as the differential force exerted on a differential element of charge,

$$dF = dQ\, v \times B \qquad (4)$$

Physically, the differential element of charge consists of a large number of very small discrete charges occupying a volume which, although small, is much larger than the average separation between the charges. The differential force expressed by (4) is thus merely the sum of the forces on the individual charges. This sum, or resultant force, is not a force applied to a single object. In an analogous way, we might consider the differential gravitational force experienced by a small volume taken in a shower of falling sand. The small volume contains a large number of sand grains, and the differential force is the sum of the forces on the individual grains within the small volume.
If our charges are electrons in motion in a conductor, however, we can show that the force is transferred to the conductor and that the sum of this extremely large number of extremely small forces is of practical importance. Within the conductor, electrons are in motion throughout a region of immobile positive ions which form a crystalline array giving the conductor its solid properties. A magnetic field which exerts forces on the electrons tends to cause them to shift position slightly and produces a small displacement between the centers of "gravity" of the positive and negative charges. The Coulomb forces between electrons and positive ions, however, tend to resist such a displacement. Any attempt to move the electrons, therefore, results in an attractive force between electrons and the positive ions of the crystalline lattice. The magnetic force is thus transferred to the crystalline lattice, or to the conductor itself. The

Coulomb forces are so much greater than the magnetic forces in good conductors that the actual displacement of the electrons is almost immeasurable. The charge separation that does result, however, is disclosed by the presence of a slight potential difference across the conductor sample in a direction perpendicular to both the magnetic field and the velocity of the charges. The voltage is known as the Hall voltage, and the effect itself is called the Hall effect.

Fig. 9.1 illustrates the direction of the Hall voltage for both positive and negative charges in motion. In Fig. 9.1a, v is in the − ax direction, v x B is in the ay direction, and Q is positive, causing Fq to be in the ay direction; thus, the positive charges move to the right. In Figure 9.1ft, v is now in the +ax direction, B is still in the az direction, v x B is in the −ay direction, and Q is negative; thus Fq is again in the ay direction. Hence, the negative charges end up at the right edge. Equal currents provided by holes and electrons in semiconductors can therefore be differentiated by their Hall voltages. This is one method of determining whether a given semiconductor is «-type or />-type.

Devices employ the Hall effect to measure the magnetic flux density and, in some applications where the current through the device can be made proportional to the magnetic field across it, to serve as electronic wattmeters, squaring elements, and so forth.

Returning to (4), we may therefore say that if we are considering an element of moving charge in an electron beam, the force is merely the sum of the forces on the individual electrons in that small volume element, but if we are considering an element of moving charge within a conductor, the total force is applied to the solid conductor itself. We shall now limit our attention to the forces on current-carrying conductors.

In Chap. 5 we defined convection current density in terms of the velocity of the volume charge density,

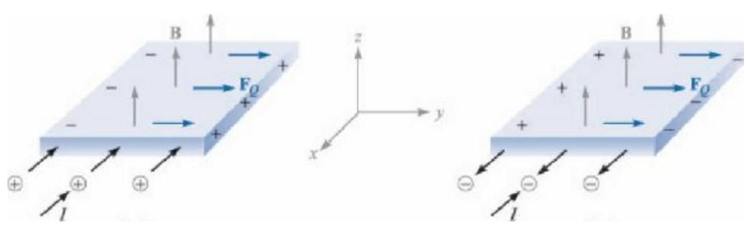


FIGURE 9.1
EQUAL CURRENTS DIRECTED INTO THE MATERIAL ARE PROVIDED BY POSITIVE CHARGES MOVING INWARD IN (A) AND NEGATIVE CHARGES MOVING OUTWARD IN (B). THE TWO CASES CAN BE DISTINGUISHED BY OPPOSITELY DIRECTED HALL VOLTAGES, AS SHOWN.

J = pvv

The differential element of charge in (4) may also be expressed in terms of volume charge density,1
dQ = pvdv dF = pvdv v x B

Thus

or

dF = J x B dv (5)

We saw in the previous chapter that J dv may be interpreted as a differential current element; that is,

J dv = K dS = Id L

and thus the Lorentz force equation may be applied to surface current density,

dF = K x B dS (6)

or to a differential current filament,

dF = IdL x B (7)

Integrating (5), (6), or (7) over a volume, a surface which may be either open or closed (why?), or a closed path, respectively, leads to the integral formulations

(8) (9)

F = / J x B dv
J vol

F K B dS (10)
S

and

F = (bid LxB = -I(f> B x d L

One simple result is obtained by applying (7) or (10) to a straight conductor in a uniform magnetic field, Remember that dv is a differential volume element and not a differential increase in velocity.

$$F = / L x B \qquad (11)$$

The magnitude of the force is given by the familiar equation

$$F= BIL \sin 8 \qquad (12)$$

where 8 is the angle between the vectors representing the direction of the current flow and the direction of the magnetic flux density. Equation (11) or (12) applies only to a portion of the closed circuit, and the remainder of the circuit must be considered in any practical problem.

Example 8.1

As a numerical example of these equations, consider Fig. 9.2. We have a square loop of wire in the z = 0 plane carrying 2 mA in the field of an infinite filament on the y axis, as shown. We desire the total force on the loop.

Solution. The field produced in the plane of the loop by the straight filament is

$$H = \frac{-J}{2irx} az = \frac{---}{2irx} az \quad A/m$$

Therefore,

$$B = /x0H = 4n \times 10-7H = \frac{3 \times 10-6}{x} az \quad T$$

We use the integral form (10),

$$F = -lj> B \times dL$$

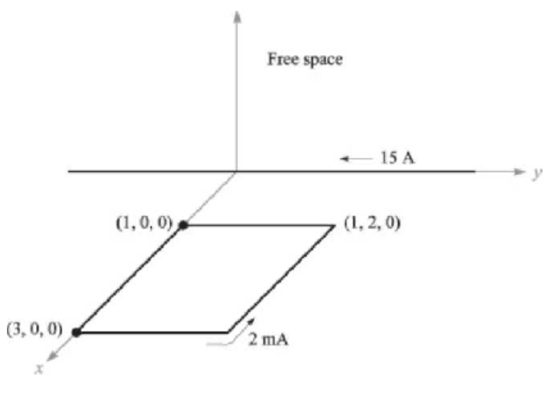


FIGURE 9.2

Let us assume a rigid loop so that the total force is the sum of the forces on the four sides. Beginning with the left side:

F = -2 x 10-3 x 3 x 10-6 " 'x= 1 x dx ax + 2 Jy=0 az 3 x dy ay

+I – x dx a_x + iv=3 x Jv=2 + 3 1 y 2 N X 1 y + 0

= -6 x 10-9 (LN3)ay - 3ax + (LN1) a y + 2a x

x

= -8ax pN

Thus, the net force on the loop is in the -ax direction.

8.3 FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS

The concept of the magnetic field was introduced to break into two parts the problem of finding the interaction of one current distribution on a second current distribution. It is possible to express the force on one current element directly in terms of a second current element without finding the magnetic field. Since we claimed that the magnetic-field concept simplifies our work, it then behooves us to show that avoidance of this intermediate step leads to more complicated expressions.

The magnetic field at point 2 due to a current element at point 1 was found to be

d H2 =- r ---

Now, the differential force on a differential current element is

dF = IdL x B

and we apply this to our problem by letting B be dB2 (the differential flux density at point 2 caused by current element 1), by identifying IdL as I2dL2, and by symbolizing the differential amount of our differential force on element 2 as d (dF 2):

d(dF2) = I2dL2 x dB2

Since dB2 = /z0dH2, we obtain the force between two differential current elements,

I1I2

d(dF2) = — dL2 x(dL1 x ai?12)(13)

Example 8.2

As an example that illustrates the use (and misuse) of these results, consider the two differential current elements shown in Fig. 9.3. We seek the differential force on dL2.

Solution. We have I1dL1 = −3ay A-mat P1(5,2,1), and I2dL2 = −4az A-mat P2(1, 8, 5). Thus, R12 = −4ax + 6ay + 4az, and we may substitute these data into (13),

dfdF) = 4^10−7 (−

4ax) x [(−3ay)x(−4ax + 6ay + 4az)]

(4ir (16 + 36 +16)1.5

= 8.56ay nN

Many chapters ago when we discussed the force exerted by one point charge on another point charge, we found that the force on the first charge was the negative of that on the second. That is, the total force on the system

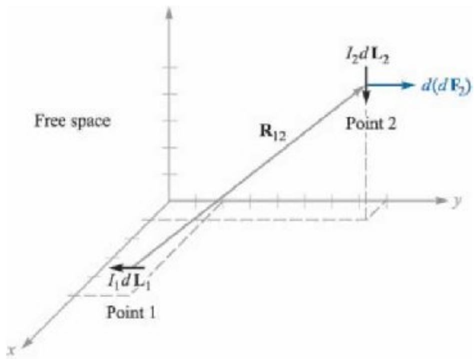


FIGURE 9.3
Given P1(5, 2, 1), P2(1, 8, 5), h dL1 = −3ay A-m, and /2 <iL2 = −4az A-m, the force on /2<iL2 is 8.56nN in the ay direction

was zero. This is not the case with the differential current elements, and d(dF1) = -12.84az nN in the example above. The reason for this different behavior lies with the nonphysical nature of the current element. Whereas point charges may be approximated quite well by small charges, the continuity of current demands that a complete circuit be considered. This we shall now do.

The total force between two filamentary circuits is obtained by integrating twice:

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\mathbf{L}_2 \times \oint \frac{d\mathbf{L}_1 \times \mathbf{a}_{R12}}{R_{12}^2} \right] = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} \right] \times d\mathbf{L}_2 \qquad (14)$$

Equation (14) is quite formidable, but the familiarity gained in the last chapter with the magnetic field should enable us to recognize the inner integral as the integral necessary to find the magnetic field at point 2 due to the current element at point 1. Although we shall only give the result, it is not very difficult to make use of (14) to find the force of repulsion between two infinitely long, straight, parallel, filamentary conductors with separation d, and carrying equal but opposite currents /, as shown in Fig. 9.4. The integrations are simple, and most errors are made in determining suitable expressions for a^12, dL1, and dL2. However, since the magnetic field intensity at either wire caused by the other is already known to be //(2nd), it is readily apparent that the answer is a force of /u.0/2/(2nd) newtons per meter length.

8.4 FORCE AND TORQUE ON A CLOSED CIRCUIT

We have already obtained general expressions for the forces exerted on current systems. One special case is easily disposed of, for if we take our relationship for the force on a filamentary closed circuit, as given by Eq. (10), Sec. 9.2,

F = —IJ> B x dL

and assume a uniform magnetic flux density, then B may be removed from the integral:

F = —IB x ^> dL

However, we discovered during our investigation of closed line integrals in an electrostatic potential field that dL = 0, and therefore the force on a closed filamentary circuit in a uniform magnetic field is zero.

If the field is not uniform, the total force need not be zero.

This result for uniform fields does not have to be restricted to filamentary circuits only. The circuit may contain surface currents or volume current density as well. If the total current is divided into filaments, the force on each one is zero, as we showed above, and the total force is again zero. Therefore any real closed circuit carrying direct currents experiences a total vector force of zero in a uniform magnetic field.

Although the force is zero, the torque is generally not equal to zero.

In defining the torque,or moment, of a force, it is necessary to consider both an origin at or about which the torque is to be calculated, as well as the point at which the force is applied. In Fig.

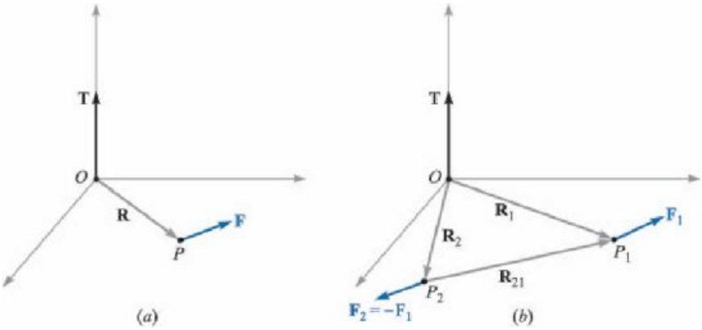


FIGURE 9.5
(A) GIVEN A LEVER ARM R *EXTENDING FROM AN ORIGIN* O *TO A POINT* P *WHERE FORCE* F *IS APPLIED, THE TORQUE ABOUT* O *IS* T = R *X* F. *(FT) IF* F_2 = —F_1, *THEN THE TORQUE* T = R_{21} *X* F_1 *IS INDEPENDENT OF THE CHOICE OF ORIGIN FOR* R_1 *AND* R2.

9.5a, we apply a force F at point P, and we establish an origin at O with a rigid lever arm R extending from O to P. The torque about point O is a vector whose magnitude is the product of the magnitudes of R,of F, and of the sine of the angle between these two vectors. The direction of the vector torque T is normal to both the force F and lever arm R and is in the direction of progress of a right-handed screw as the lever arm is rotated into the force vector through the smaller angle. The torque is expressible as a cross product,

T = R x F

Now let us assume that two forces, F1 at P1 and F2 at P2, having lever arms R1 and R2 extending from a common origin O, as shown in Fig. 9.56, are applied to an object of fixed shape and that the object does not undergo any translation. Then the torque about the origin is

$$T = R1 \times F1 + R2 \times F2$$

where

$$F1 + F2 = 0$$

and therefore

$$T = (R1 - R2) \times F1 = R21 \times F1$$

The vector R21 = R1 - R2 joins the point of application of F2 to that of F1 and is independent of the choice of origin for the two vectors R1 and R2. Therefore, the torque is also independent of the choice of origin, provided that the total force is zero. This may be extended to any number of forces.

Consider the application of a vertically upward force at the end of a horizontal crank handle on an elderly automobile. This cannot be the only applied force, for if it were, the entire handle would be accelerated in an upward direction. A second force, equal in magnitude to that exerted at the end of the handle, is applied in a downward direction by the bearing surface at the axis of rotation. For a 40-N force on a crank handle 0.3m in length, the torque is 12Nm. This figure is obtained regardless of whether the origin is considered to be on the axis of rotation (leading to 12 Nm plus 0 Nm), at the midpoint of the handle (leading to 6Nm plus 6Nm), or at some point not even on the handle or an extension of the handle.

We may therefore choose the most convenient origin, and this is usually on the axis of rotation and in the plane containing the applied forces if the several forces are coplanar.

With this introduction to the concept of torque, let us now consider the torque on a differential current loop in a magnetic field B. The loop lies in the xy plane (Fig. 9.6); the sides of the loop are parallel to the x and y axes and are of length dx and dy. The value of the magnetic field at the center of the loop is taken as B0. Since the loop is of differential size, the value of B at all points on the loop may be taken as B0. (Why was this not possible in the discussion of curl and

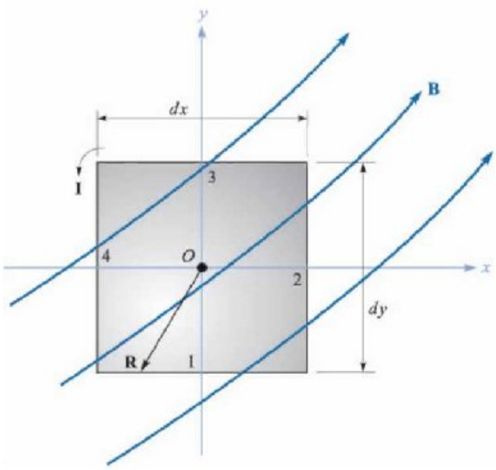


FIGURE 9.6
A differential current loop in a magnetic field **B**. The torque on the loop is $d\mathbf{T} = I(dx\,dy\,\mathbf{a}_z) \times \mathbf{B}_0 = I\,d\mathbf{S} \times \mathbf{B}$.

divergence?) The total force on the loop is therefore zero, and we are free to choose the origin for the torque at the center of the loop. The vector force on side 1 is

dF1 = Idx ax x B0

or

dF1 = Idx(^0yaz — ^a^)

For this side of the loop the lever arm R extends from the origin to the midpoint of the side, R1 = —1 dj ay, and the contribution to the total torque is

dT1 = R1 x dF1
= — idy aj x
Idx(^0yaz — i?0zay)
= —1dxdjIi?0yax

The torque contribution on side 3 is found to be the same,

dT3 = R3 x dF3 = 2djay x(—
Idxax x B0) = —2
dxdjI^0yax = dT1

and

dT1 + dT3 = — dxdyZB0Jax

Evaluating the torque on sides 2 and 4, we find d T2 + d T4 = dx dj I50xay

and the total torque is then

dT = Idx dy(BQxay - BQyax) The quantity within the parentheses may be represented by a cross product,

dT = Idxdj(az x Bo)

or

dT = IdS x B (15)

where dS is the vector area of the differential current loop and the subscript on Bo has been dropped.

We now define the product of the loop current and the vector area of the loop as the differential magnetic dipole moment dm, with units of A-m2. Thus

d m = Id S (16)

and

$$d\mathbf{T} = d\mathbf{m} \times \mathbf{B} \tag{17}$$

If we extend the results we obtained in Sect. 4.7 for the differential electric dipole by determining the torque produced on it by an electric field, we see a similar result,

$$d\mathbf{T} = d\mathbf{p} \times \mathbf{E}$$

Equations (15) and (17) are general results which hold for differential loops of any shape, not just rectangular ones. The torque on a circular or triangular loop is also given in terms of the vector surface or the moment by (15) or (17).

Since we selected a differential current loop so that we might assume B was constant throughout it, it follows that the torque on a planar loop of any size or shape in a uniform magnetic field is given by the same expression,

$$\mathbf{T} = I\mathbf{S} \times \mathbf{B} = \mathbf{m} \times \mathbf{B} \tag{18}$$

We should note that the torque on the current loop always tends to turn the loop so as to align the magnetic field produced by the loop with the applied magnetic field that is causing the torque. This is perhaps the easiest way to determine the direction of the torque.

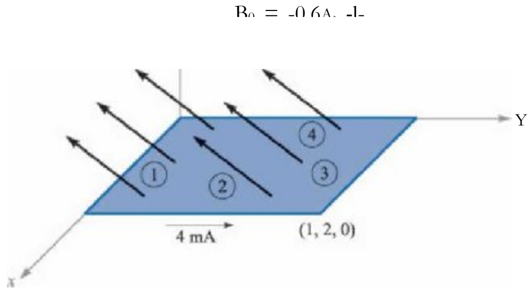


FIGURE 9.7

1 ^Example 9.3

To illustrate some force and torque calculations, consider the rectangular loop shown in Fig. 9.7. Calculate the torque by using $T = IS \times B$.

Solution. The loop has dimensions of 1 m by 2 m and lies in the uniform field $B_0 = -0.6a_y + 0.8a_z$ T. The loop current is 4mA, a value that is sufficiently small to avoid causing any magnetic field that might affect B_0. We have

$$T = 4 \times 10^{-3}[(1)(2)a_z] \times (-0.6a_y + 0.8a_z) = 4.8a_x \quad \text{mN - m}$$

Thus, the loop tends to rotate about an axis parallel to the positive x axis. The small magnetic field produced by the 4-mA loop current tends to line up with B_0.

NIIL^Example 9.4

Now let us find the torque once more, this time by calculating the total force and torque contribution for each side.

Solution. On side 1 we have

$$F_1 = IL_1 \times B_0 = 4 \times 10^{-3}(1a_x) \times (-0.6a_y + 0.8a_z) = -3.2a_y - 2.4a_z \quad \text{mN}$$

On side 3 we obtain the negative of this result,

$$F_3 = 3.2a_y + 2.4a_z \quad \text{mN}$$

Next we attack side 2:

$$F_2 = IL_2 \times B_o = 4 \times 10^{-3}(2a_y) \times (-0.6a_y + 0.8a_z) = 6.4a_x \text{ mN}$$

with side 4 again providing the negative of this result,

$$F_4 = -6.4a_x \text{ mN}$$

Since these forces are distributed uniformly along each of the sides, we treat each force as if it were applied at the center of the side. The origin for the torque may be established anywhere since the sum of the forces is zero, and we choose the center of the loop. Thus,

$$T = T_1 + T_2 + T_3 + T_4 = R_1 \times F_1 + R_2 \times F_2 + R_3 \times F_3 + R_4 \times F_4 = (-1a_y) \times (-3.2a_y - 2.4a_z) + (0.5a_x) \times (6.4a_x) + (1a_y) \times (3.2a_y + 2.4a_z) + (-0.5a_x) \times (-6.4a_x) = 2.4a_x + 2.4a_x = 4.8a_x \text{ mN - m}$$

Crossing the loop moment with the magnetic flux density is certainly easier.

9.5 THE NATURE OF MAGNETIC MATERIALS

We are now in a position to combine our knowledge of the action of a magnetic field on a current loop with a simple model of an atom and obtain some appreciation of the difference in behavior of various types of materials in magnetic fields.

Although accurate quantitative results can only be predicted through the use of quantum theory, the simple atomic model which assumes that there is a central positive nucleus surrounded by electrons in various circular orbits yields reasonable quantitative results and provides a satisfactory qualitative theory. An electron in an orbit is analogous to a small current loop (in which the current is directed oppositely to the direction of electron travel) and as such experiences a torque in an external magnetic field, the torque tending to align the magnetic field produced by the orbiting electron with the external magnetic field. If there were no other magnetic moments to consider, we would then conclude that all the orbiting electrons in the material would shift in such a way as to add their magnetic fields to the applied field, and thus that the resultant magnetic field at any point in the material would be greater than it would be at that point if the material were not present.

A second moment, however, is attributed to electron spin. Although it is tempting to model this phenomenon by considering the electron as spinning about its own axis and thus generating a magnetic dipole moment, satisfactory quantitative results are not obtained from such a theory. Instead, it is necessary to digest the mathematics of relativistic quantum theory to show that an electron may have a spin magnetic moment of about $\pm 9 \times 10^{-24}$ A - m^2; the plus and minus signs indicate that alignment aiding or opposing an external magnetic field is possible. In an atom with many electrons present, only the spins of those electrons in shells which are not completely filled will contribute to a magnetic moment for the atom.

A third contribution to the moment of an atom is caused by nuclear spin. Although this factor provides a negligible effect on the overall magnetic properties of materials, it is the basis of the nuclear magnetic resonance imaging (MRI) procedure now provided by many of the larger hospitals.

Thus each atom contains many different component moments, and their combination determines the magnetic characteristics ofthe material and provides its general magnetic classification. We shall describe briefly six different types of material: diamagnetic, paramagnetic, ferromagnetic, antiferromagnetic, ferri-magnetic, and superparamagnetic.

Let us first consider those atoms in which the small magnetic fields produced by the motion of the electrons in their orbits and those produced by the electron spin combine to produce a net field of zero. Note that we are considering here the fields produced by the electron motion itself in the absence of any external magnetic field; we might also describe this material as one in which the permanent magnetic moment m0 of each atom is zero. Such a material is termed diamagnetic. It would seem, therefore, that an external magnetic field would produce no torque on the atom, no realignment of the dipole fields, and consequently an internal magnetic field that is the same as the applied field. With an error that only amounts to about one part in a hundred thousand, this is correct.

Let us select an orbiting electron whose moment m is in the same direction as the applied field B0 (Fig. 9.8). The magnetic field produces an outward force on the orbiting electron. Since the orbital radius is quantized and cannot change, the inward Coulomb force of attraction is also unchanged. The force unbalance created by the outward magnetic force must therefore be compensated for by a reduced orbital velocity. Hence, the orbital moment decreases, and a smaller internal field results.

If we had selected an atom for which m and B0 were opposed, the magnetic force would be inward, the velocity would increase, the orbital moment would increase, and greater cancellation of B0 would occur. Again a smaller internal field would result.

Metallic bismuth shows a greater diamagnetic effect than most other dia-magnetic materials, among which are hydrogen, helium, the other "inert" gases,

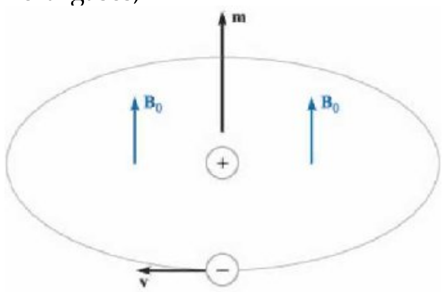


FIGURE 9.8
AN ORBITING ELECTRON IS SHOWN HAVING A MAGNETIC MOMENT M IN THE SAME DIRECTION AS AN APPLIED FIELD B_0

sodium chloride, copper, gold, silicon, germanium, graphite, and sulfur. We should also realize that the diamagnetic effect is present in all materials, because it arises from an interaction of the external magnetic field with every orbiting electron; however, it is overshadowed by other effects in the materials we shall consider next.

Now let us discuss an atom in which the effects of the electron spin and orbital motion do not quite cancel. The atom as a whole has a small magnetic moment, but the random orientation of the atoms in a larger sample produces an average magnetic moment of zero. The material shows no magnetic effects in the absence of an external field. When an external field is applied, however, there is a small torque on each atomic moment, and these moments tend to become aligned with the external field. This alignment acts to increase the value of B within the material over the external value. However, the diamagnetic effect is still operating on the orbiting electrons and may counteract the above increase. If the net result is a decrease in B, the material is still called diamagnetic. However, if there is an increase in B, the material is termed paramagnetic.

Potassium, oxygen, tungsten, and the rare earth elements and many of their salts, such as erbium chloride, neodymium oxide, and yttrium oxide, one of the materials used in masers, are examples of paramagnetic substances.

The remaining four classes of material, ferromagnetic, antiferromagnetic, ferrimagnetic, and superparamagnetic, all have strong atomic moments. Moreover, the interaction of adjacent atoms causes an alignment of the magnetic moments of the atoms in either an aiding or exactly opposing manner.

In ferromagnetic materials each atom has a relatively large dipole moment, caused primarily by uncompensated electron spin moments. Interatomic forces cause these moments to line up in a parallel fashion over regions containing a large number of atoms. These regions are called domains, and they may have a variety of shapes and sizes ranging from one micrometer to several centimeters, depending on the size, shape, material, and magnetic history of the sample. Virgin ferromagnetic materials will have domains which each have a strong magnetic moment; the domain moments, however, vary in direction from domain to domain. The overall effect is therefore one of cancellation, and the material as a whole has no magnetic moment. Upon application of an external magnetic field, however, those domains which have moments in the direction of the applied field increase their size at the expense of their neighbors, and the internal magnetic field increases greatly over that of the external field alone. When the external field is removed, a completely random domain alignment is not usually attained, and a residual, or remnant, dipole field remains in the macroscopic structure. The fact that the magnetic moment of the material is different after the field has been removed, or that the magnetic state of the material is a function of its magnetic history, is called hysteresis, a subject which will be discussed again when magnetic circuits are studied a few pages from now.

Ferromagnetic materials are not isotropic in single crystals, and we shall therefore limit our discussion to polycrystalline materials, except for mentioning that one of the characteristics of anisotropic magnetic materials is magnetostriction, or the change in dimensions of the crystal when a magnetic field is impressed on it.

The only elements that are ferromagnetic at room temperature are iron, nickel, and cobalt, and they lose all their ferromagnetic characteristics above a temperature called the Curie temperature, which is 1043 K (770°C) for iron. Some alloys of these metals with each other and with other metals are also ferromagnetic, as for example alnico, an aluminum-nickel-cobalt alloy with a small amount of copper. At lower temperatures some of the rare earth elements, such as gadolinium and dysprosium, are ferromagnetic. It is also interesting that some alloys of nonferromagnetic metals are ferromagnetic, such as bismuth-manganese and copper-manganese-tin.

In antiferromagnetic materials, the forces between adjacent atoms cause the atomic moments to line up in an antiparallel fashion. The net magnetic moment is zero, and antiferromagnetic materials are affected only slightly by the presence of an external magnetic field. This effect was first discovered in manganese oxide, but several hundred antiferromagnetic materials have been identified since then. Many oxides, sulfides, and chlorides are included, such as nickel oxide (NiO), ferrous sulfide (FeS), and cobalt chloride (CoCl2). Antiferromagnetism is only present at relatively low temperatures, often well below room temperature. The effect is not of engineering importance at present.

The ferrimagnetic substances also show an antiparallel alignment of adjacent atomic moments, but the moments are not equal. A large response to an external magnetic field therefore occurs, although not as large as that in ferromagnetic materials. The most important group of ferrimagnetic materials are the ferrites, in which the conductivity is low, several orders of magnitude less than that of semiconductors. The fact that these substances have greater resistance than the ferromagnetic materials results in much smaller induced currents in the material when alternating fields are applied, as for example in transformer cores which operate at the higher frequencies. The reduced currents (eddy currents) lead to lower ohmic losses in the transformer core. The iron oxide magnetite (Fe_3O_4), a nickel-zinc ferrite ($Ni_{1/2}Zn_{1/2}Fe_2O_4$), and a nickel ferrite ($NiFe_2O_4$)

TABLE 9.1

Characteristics of magnetic materials

Classification	Magnetic moments	B values	Comments
Diamagnetic	$m_{orb} + m_{spin}$ 0	$B_{int} < B_{appl}$	
Paramagnetic	$m_{orb} + m_{spin}$ small	$B_{int} > B_{appl}$	
Ferromagnetic	$\lvert m_{spin}\rvert \gg \lvert m_{orb}\rvert$	$B_{int} \gg B_{appl}$	Domains
Antiferromagnetic	$\lvert m_{spin}\rvert \gg \lvert m_{orb}\rvert$	$B_{int} = B_{appl}$	Adjacent moments oppose
Ferrimagnetic	$\lvert m_{spin}\rvert > \lvert m_{orb}\rvert$	$B_{int} > B_{appl}$	Unequal adjacent moments oppose; low σ
Superparamagnetic	$\lvert m_{spin}\rvert \gg \lvert m_{orb}\rvert$	$B_{int} > B_{appl}$	Nonmagnetic matrix; recording tapes

are examples of this class of materials. Ferrimagnetism also disappears above the Curie temperature.

Superparamagnetic materials are composed of an assemblage of ferromagnetic particles in a nonferromagnetic matrix. Although domains exist within the individual particles, the domain walls cannot penetrate the intervening matrix material to the adjacent particle. An important example is the magnetic tape used in audiotape or videotape recorders.

Table 9.1 summarizes the characteristics of the six types of magnetic materials discussed above.

9.6 MAGNETIZATION AND PERMEABILITY

To place our description of magnetic materials on a more quantitative basis, we shall now devote a page or so to showing how the magnetic dipoles act as a distributed source for the magnetic field. Our result will be an equation that looks very much like Ampere's circuital law, <^ H • dL — I. The current, however, will be the movement of bound charges (orbital electrons, electron spin, and nuclear spin), and the field, which has the dimensions of H, will be called the magnetization M. The current produced by the bound charges is called a bound current or Amperian current.

Let us begin by defining the magnetization M in terms of the magnetic dipole moment m. The bound current I, circulates about a path enclosing a differential area dS, establishing a dipole moment (Am2),

m — Ii,d S

If there are n magnetic dipoles per unit volume and we consider a volume Au, then the total magnetic dipole moment is found by the vector sum

nAu

mtotal — ^ mi (19)

i— 1

Each of the mi may be different. Next, we define the magnetization M as the magnetic dipole moment per unit volume,

M = lim —V m,
Av^0 Av^—i
,=1

and see that its units must be the same as for H, amperes per meter.

Now let us consider the effect of some alignment of the magnetic dipoles as the result of the application of a magnetic field. We shall investigate this alignment along a closed path, a short portion of which is shown in Fig. 9.9. The figure shows several magnetic moments m that make an angle 9 with the element of path dL; each moment consists of a bound current /ft circulating about an area dS. We are therefore considering a small volume, dS cos 9dL,or dS • dL, within which there are «dS • dL magnetic dipoles. In changing from a random orientation to this partial alignment, the bound current crossing the surface enclosed by the path (to our left as we travel in the direction in Fig. 9.9) has increased by /ft for each of the «dS • dL dipoles. Thus

d/ft = «/ftdS • dL = M • dL (20)

and within an entire closed contour,

Ib = j> M • dL (21)

Equation (21) merely says that if we go around a closed path and find dipole moments going our way more often than not, there will be a corresponding current composed of, for example, orbiting electrons crossing the interior surface.
This last expression has some resemblance to Ampere's circuital law, and we may now generalize the relationship between B and H so that it applies to media other than free space. Our present discussion is based on the forces and torques on differential current loops in a B field, and we therefore take B as our fundamental quantity and seek an improved definition of H. We thus write Ampere's circuital law in

$$\oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T \qquad (22)$$

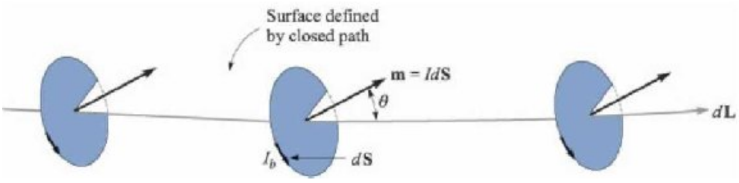


FIGURE 9.9
A SECTION DL OF A CLOSED PATH ALONG WHICH MAGNETIC DIPOLES HAVE BEEN PARTIALLY ALIGNED BY SOME EXTERNAL MAGNETIC FIELD. THE ALIGNMENT HAS CAUSED THE BOUND CURRENT CROSSING THE SURFACE DEFINED BY THE CLOSED PATH TO

terms of the total current, bound plus free, where

It = lb + /

and I is the total free current enclosed by the closed path. Note that the free current appears without subscript since it is the most important type of current and will be the only current appearing in Maxwell's equations.

Combining these last three equations, we obtain an expression for the free current enclosed,

I = It - Ib = & (— - M We may now define H in terms of B and M,

(23)

H =------ M (24)
Mo

and we see that B = /zoH in free space where the magnetization is zero. This relationship is usually written in a form that avoids fractions and minus signs:

B = Mo(H + M) (25)

We may now use our newly defined H field in (23),

I = j> H • d L

obtaining Ampere's circuital law in terms of the free currents. Utilizing the several current densities, we have

) Jb • dS

(2
6)

s

It = <*> Jt • dS
is

I J dS
s

With the help of Stokes' theorem, we may therefore transform (21), (26), and (22) into the equivalent curl relationships:

V J
xM b
¥1
B
V
x —
Mo

Vx (2
H = 7)

We shall emphasize only (26) and (27), the two expressions involving the free charge, in the work that follows.

The relationship between B, H, and M expressed by (25) may be simplified for linear isotropic media where a magnetic susceptibility xm can be defined:

M = (2
XmH 8)

Thus we have B = /Z0(H + XmH) = Me)Me?H

where

lR = 1 + Xm (29)

is defined as the relative permeability /aR. We next define the permeability /a:

l = a0lR (30)

and this enables us to write the simple relationship between B and H,

B = /aH (31)

I IIIL^Example 9.5

Given a ferrite material which we shall specify to be operating in a linear mode with B = 0.05 T, let us assume aR = 50, and calculate values for xm, M, and H.

Solution. Since aR = 1 + xm, we have

Xm = lR - 1 = 49

Also,

B = MrM0h

and

H = ---- 005--- ? = 796 A/m

The magnetization is xmH, or 39 000 A/m. The alternate ways of relating B and H are, first,

B = |0(H + A)

or

0.05 = 4?r x 10-7(796 + 39 000)

showing that Amperian currents produce 49 times the magnetic field intensity that the free charges do; and second,

B = HrHoH

or

0.05 = 50 x 4it x 10-7 x 796

where we utilize a relative permeability of 50 and let this quantity account completely for the notion of the bound charges. We shall emphasize the latter interpretation in the chapters that follow.

The first two laws that we investigated for magnetic fields were the Biot-Savart law and Ampere's circuital law. Both were restricted to free space in their application. We may now extend their use to any homogeneous, linear, isotropic magnetic material that may be described in terms of a relative permeability iir.

Just as we found for anisotropic dielectric materials, the permeability of an anisotropic magnetic material must be given as a 3 x 3 matrix, while B and H are both 3 x 1 matrices. We have

Bx = n.xxHx + iixyHy + ^XZHZ

By = H-yxHx + H-yyHy

+ /Ly2Hz Bz = MzxHx

+ M-zyHy + MzzHz

For anisotropic materials, then, B = //H is a matrix equation; however B = /x0(H + M) remains valid, although B, H, and M are no longer parallel in general. The most common anisotropic magnetic material is a single ferromagnetic crystal, although thin magnetic films also exhibit anisotropy. Most applications of ferromagnetic materials, however, involve polycrystalline arrays that are much easier to make.

Our definitions of susceptibility and permeability also depend on the assumption of linearity. Unfortunately, this is true only in the less interesting paramagnetic and diamagnetic materials for which the relative permeability rarely differs from unity by more than one part in a thousand. Some typical values of the susceptibility for diamagnetic materials are hydrogen, —2 x 10-5; copper, —0.9 x 10—5; germanium, —0.8 x 10—5; silicon, —0.3 x 10—5; and graphite, —12 x 10—5. Several representative paramagnetic susceptibilities are oxygen, 2 x 10—6; tungsten, 6.8 x 10—5; ferric oxide (Fe2O3),1.4 x 10—3; and yttrium oxide (Y2O3),0.53 x 10—6. If we simply take the ratio of B to /z0H as the relative permeability of a ferromagnetic material, typical values of iir would range from 10 to 100000. Diamagnetic, paramagnetic, and antiferromagnetic materials are commonly said to be nonmagnetic.

9.7 MAGNETIC BOUNDARY CONDITIONS

We should have no difficulty in arriving at the proper boundary conditions to apply to B, H, and M at the interface between two different magnetic materials, for we have solved similar problems for both conducting materials and dielectrics. We need no new techniques.

Fig. 9.10 shows a boundary between two isotropic homogeneous linear materials with permeabilities /i1 and //2. The boundary condition on the normal components is determined by allowing the surface to cut a small cylindrical gaussian surface. Applying Gauss's law for the magnetic field from Sec. 8.5,

B d S = 0

s

we find that

B^AS — 0

B^AS

or

BjV2 = (3

Bjv1 2)

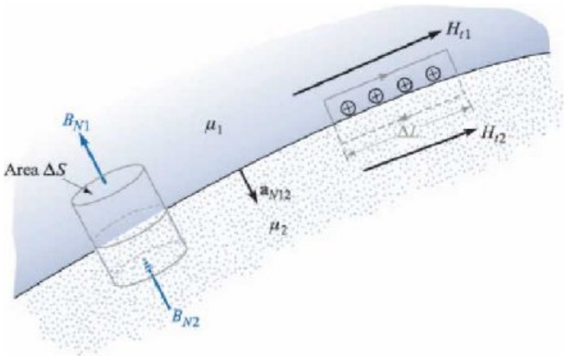


FIGURE 9.10
A GAUSSIAN SURFACE AND A CLOSED PATH ARE CONSTRUCTED AT THE BOUNDARY BETWEEN MEDIA 1 AND 2, HAVING PERMEABILITIES OF /1 AND /I2, RESPECTIVELY. FROM THIS WE DETERMINE THE BOUNDARY CONDITIONS = Bm *AND* H_{T1} — H_{T2} = *K, THE*

Thus

HN2 = — HNi (33)

The normal component of B is continuous, but the normal component of H is discontinuous by the ratio —i/—2.
The relationship between the normal components of M, of course, is fixed once the relationship between the normal components of H is known. For linear magnetic materials, the result is written simply as

MN2 = Xm2 —1 Hvi = ^1 Mm (34)
—2 Xm1—2

Next, Ampere's circuital law

• dL = /

is applied about a small closed path in a plane normal to the boundary surface, as shown to the right in Fig. 9.10. Taking a clockwise trip around the path, we find that

Ht1AL — Ht2AL = KAL

where we assume that the boundary may carry a surface current K whose component normal to the plane of the closed path is K. Thus

H;1 - Ht2 = K (35)

The directions are specified more exactly by using the cross product to identify the tangential components,

(Hi - H2)x aNi2 = K

where aN12 is the unit normal at the boundary directed from region 1 to region 2. An equivalent formulation in terms of the vector tangential components may be more convenient for H:

Hn — H^2 = aN12 x K For tangential B,
we have

Bt1 Ba

= K (36)

The boundary condition on the tangential component of the magnetization for linear materials is therefore

M,2 = — M* — Xm2K (37)
Xm1

The last three boundary conditions on the tangential components are much simpler, of course, if the surface current density is zero. This is a free current density, and it must be zero if neither material is a conductor.

I III^Example 9.6

To illustrate these relationships with an example, let us assume that = /1 = 4 //H/m in region 1 where z > 0, while //i2 = 7 //H/m wherever z < 0. Moreover, let K = 80ax A/m on the surface z = 0. We establish a field, B1 = 2ax — 3ay + az mT, in region 1 and seek the value of B2.

Solution. The normal component of B1 is

Bjv1 =(B1 • aW12)aW12 = [(2ax — 3ay + az)-(—az)](—az) = az mT

Thus,

BW2 = Bjv-1 = az mT We next determine the tangential components:

Ba = B1 — Bw1 = 2ax — 3ay mT

and

„ B,1 (2ax — 3ay)10—3
Hn = — = ------------f— - = 500ax — 750ay A/m

Thus,

H,2 = H — aW12 x K = 500ax — 750ay — (— az)x 80ax = 500ax — 750ay + 80ay = 500ax — 670ay A/m

and

Bfl = ja2Hfl = 7 x 10—6(500ax — 670ay) = 3.5ax — 4.69ay mT Therefore,

B2 = BV2 + Bfl = 3.5ax — 4.69ay + az mT

\/ D9.8. Let the permittivity be 5 /H/m in region A where x < 0, and 20 /H/m in region B where x > 0. If there is a surface current density K = 150ay — 200az A/mat x = 0, and if 77a = 300ax — 400ay + 500az A/m, find: (a) |HM|;(b) |H«aI;(c) |H,B|;(d) |Hjvb|.

Ans. 640 A/m; 300 A/m; 695 A/m; 75 A/m

9.8 THE MAGNETIC CIRCUIT

In this section we shall digress briefly to discuss the fundamental techniques involved in solving a class of magnetic problems known as magnetic circuits. As we shall see shortly, the name arises from the great similarity to the dc-resistive-circuit analysis with which it is assumed we are all facile. The only important difference lies in the nonlinear nature of the ferromagnetic portions of the magnetic circuit; the methods which must be adopted are similar to those required in nonlinear electric circuits which contain diodes, thermistors, incandescent filaments, and other nonlinear elements.

As a convenient starting point, let us identify those field equations upon which resistive circuit analysis is based. At the same time we shall point out or derive the analogous equations for the magnetic circuit. We begin with the electrostatic potential and its relationship to electric field intensity,

$$E = -\nabla V \tag{38a}$$

The scalar magnetic potential has already been defined, and its analogous relation to the magnetic field intensity is

$$H = -\nabla V_m \tag{38b}$$

In dealing with magnetic circuits, it is convenient to call V_m the magnetomotive force, or mmf, and we shall acknowledge the analogy to the electromotive force, or emf, by doing so. The units of the mmf are, of course, amperes, but it is customary to recognize that coils with many turns are often employed by using the term "ampere-turns." Remember that no current may flow in any region in which V_m is defined.

The electric potential difference between points A and B may be written as

$$V_{AB} = \int_A^B E \cdot dL \tag{39a}$$

and the corresponding relationship between the mmf and the magnetic field intensity,

$$V_{mAB} = \int_A^B H \cdot dL \tag{39b}$$

was developed in Chap. 8, where we learned that the path selected must not cross the chosen barrier surface.

Ohm's law for the electric circuit has the point form

$$J = \sigma E \tag{40a}$$

and we see that the magnetic flux density will be the analog of the current density,

B =
uH (4
0b)

To find the total current, we must integrate:

i J (4
• d S 1a)
is

A corresponding operation is necessary to determine the total magnetic flux flowing through the cross section of a magnetic circuit:

<J> = (4
B • dS 16)
s

We then defined resistance as the ratio of potential difference and current,

or

(42a)

and we shall now define reluctance as the ratio of the magnetomotive force to the total flux; thus

Fm (4
= <J>9? 26)

where reluctance is measured in ampere-turns per weber (At/Wb). In resistors which are made of a linear isotropic homogeneous material of conductivity a and have a uniform cross section of area S and length d, the total resistance is

R d_
as (4
3a)

If we are fortunate enough to have such a linear isotropic homogeneous magnetic material of length d and uniform cross section s, then the total reluctance is

91 _
: d_ (4
/ 36)
s

The only such material to which we shall commonly apply this relationship is air.

Finally, let us consider the analog of the source voltage in an electric circuit. We know that the closed line integral of E is zero,

j>E • dL = 0

In other words, Kirchhoffs voltage law states that the rise in potential through the source is exactly equal to the fall in potential through the load. The expression for magnetic phenomena takes on a slightly different form,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I_{total}$$

for the closed line integral is not zero. Since the total current linked by the path is usually obtained by allowing a current I to flow through an TV-turn coil, we may express this result as

H d L = NI (4 4)

In an electric circuit the voltage source is a part of the closed path; in the magnetic circuit the current-carrying coil will surround or link the magnetic circuit. In tracing a magnetic circuit, we shall not be able to identify a pair of terminals at which the magnetomotive force is applied. The analogy is closer here to a pair of coupled circuits in which induced voltages exist (and in which we shall see in Chap. 10 that the closed line integral of E is also not zero).

Let us try out some of these ideas on a simple magnetic circuit. In order to avoid the complications of ferromagnetic materials at this time, we shall assume that we have an air-core toroid with 500 turns, a cross-sectional area of 6 cm2,a mean radius of 15 cm, and a coil current of 4 A. As we already know, the magnetic field is confined to the interior of the toroid, and if we consider the closed path of our magnetic circuit along the mean radius, we link 2000 At,

2000 At

Although the field in the toroid is not quite uniform, we may assume that it is for all practical purposes and calculate the total reluctance of the circuit as

99

y0- d 15^4 = 1.25 x 109 A-t/Wb
4tt10-7 x 6 x 10-4

Thus

Vm 2000 $ = 1.6 x10-6 Wb
9 1.25 x 109

This value of the total flux is in error by less than 4 percent, in comparison with the value obtained when the exact distribution of flux over the cross section is used.

Hence

B = B S 1-6 x 10-6 / 6 x 10-4 = 2.67 x 10-3 T

and finally,

H 2.67 x 10-3 / 4tt10-7 2120 A-t/m

As a check, we may apply Ampere's circuital law directly in this symmetrical problem,

H^2nr = NI

and obtain

TV/ 500 x 4
H = = ...o 015 = 2120 A/m 27rr 6.28 x
0.15

at the mean radius.

Our magnetic circuit in this example does not give us any opportunity to find the mmf across different elements in the circuit, for there is only one type of material. The analogous electric circuit is, of course, a single source and a single resistor. We could make it look just as long as the above analysis, however, if we found the current density, the electric field intensity, the total current, the resistance, and the source voltage.

More interesting and more practical problems arise when ferromagnetic materials are present in the circuit. Let us begin by considering the relationship between B and H in such a material. We may assume that we are establishing a curve of B versus H for a sample of ferromagnetic material which is completely demagnetized; both B and H are zero. As we begin to apply an mmf, the flux density also rises, but not linearly, as the experimental data of Fig. 9.11 show near the origin. After H reaches a value of about 100 At/m, the flux density rises more slowly and begins to saturate when H is several hundred At/m. Having reached partial saturation, let us now turn to Fig. 9.12, where we may continue our experiment at point x by reducing H. As we do so, the effects of hysteresis begin to show, and we do not retrace our original curve. Even after H is zero, B = Br, the remnant flux density. As H is reversed, then brought back to zero, and the complete cycle traced several times, the hysteresis loop of Fig. 9.12 is obtained. The mmf required to

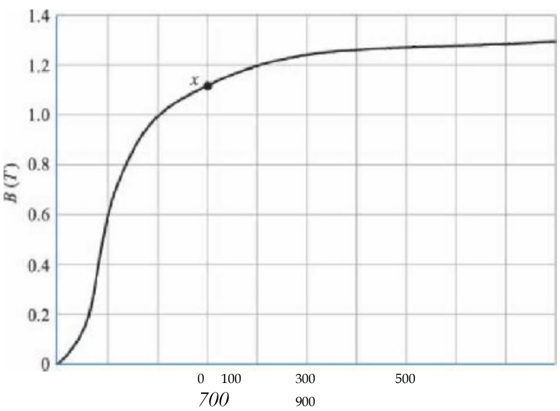


FIGURE 9.11

reduce the flux density to zero is identified as Hc,

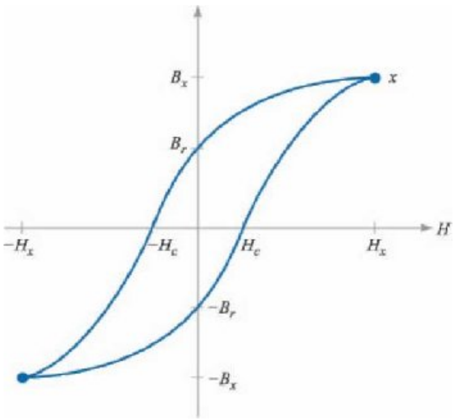


FIGURE 9.12
A HYSTERESIS LOOP FOR SILICON STEEL. THE COERCIVE FORCE H_C *AND REMNANT FLUX DENSITY* B_R *ARE INDICATED.*

the coercive "force." For smaller maximum values of H smaller hysteresis loops are obtained and the locus of the tips is about the same as the virgin magnetization curve of Figure 9.11.

I ^Example 9.7

Let us make use of the magnetization curve for silicon steel to solve a magnetic circuit problem that is slightly different from our previous example. We shall use a steel core in the toroid, except for an air gap of 2 mm. Magnetic circuits with air gaps occur because gaps are deliberately introduced in some devices, such as inductors, which must carry large direct currents, because they are unavoidable in other devices such as rotating machines, or because of unavoidable problems in assembly. There are still 500 turns about the toroid, and we ask what current is required to establish a flux density of 1 T everywhere in the core.

Solution. This magnetic circuit is analogous to an electric circuit containing a voltage source and two resistors, one of which is nonlinear. Since we are given the "current," it is easy to find the "voltage" across each series element, and hence the total "emf." In the air gap,

9air = = 4 1 027X T3 10 4 = 2.65 x 106 A-t/Wb

Knowing the total flux,

$ = BS = 1(6 x 10-4) = 6 x 10-4 Wb which is the same in both steel and air, we may find the mmf required for the gap,

Vm,a!r = (6 x 10-4)(2.65 x 106)= 1590 A-t

Referring to Fig. 9.11, a magnetic field strength of 200 A-t/m is required to produce a flux density of 1 T in the steel. Thus

Hsteel = 200 A-t

Pm.steel =

Hsteeldsteel = 200

X 0.30jT

= 188A-t

The total mmf is therefore 1778 A-t, and a coil current of 3.56 A is required.

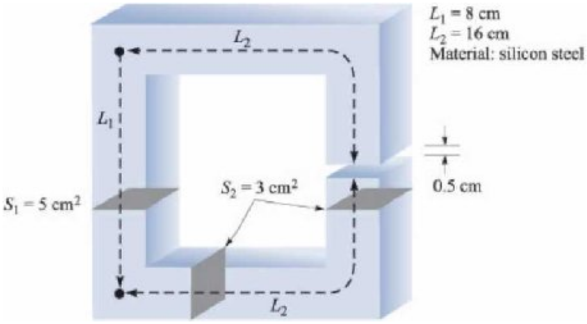
We should realize that we have made several approximations in obtaining this answer. We have already mentioned the lack of a completely uniform cross section, or cylindrical symmetry; the path of every flux line is not of the same length. The choice of a "mean" path length can help compensate for this error in problems in which it may be more important than it is in our example. Fringing flux in the air gap is another source of error, and formulas are available by which we may calculate an effective length and cross-sectional area for the gap which will yield more accurate results. There is also a leakage flux between the turns of wire, and in devices containing coils concentrated on one section of the core, a few flux lines bridge the interior of the toroid. Fringing and leakage are problems which seldom arise in the electric circuit because the ratio of the conductivities of air and the conductive or resistive materials used is so high. In contrast, the magnetization curve for silicon steel shows that the ratio of H to B in the steel is about 200 up to the "knee" of the magnetization curve; this compares with a ratio in air of about 800000. Thus, although flux prefers steel to air by the commanding ratio of 4000 to 1, this is not very close to the ratio of conductivities of, say, 1015 for a good conductor and a fair insulator.

NII^Example 9.8

As a last example, let us consider the reverse problem. Given a coil current of 4 A in the previous magnetic circuit, what will the flux density be?

Solution. First let us try to linearize the magnetization curve by a straight line from the origin to B = 1, H = 200. We then have B = H/200 in steel and B = /x0H in air. The two reluctances are found to be 0.314 x 106 for the steel path and 2.65 x 106 for the air gap, or 2.96 x 106 A-t/Wb total. Since Fm is 2000 A-t, the flux is 6.76 x 10~4Wb, and B = 1.13 T. A more accurate solution may be obtained by assuming several values of B and calculating the necessary mmf. Plotting the results enables us to determine the true value of B by interpolation. With this method we obtain B = 1.10T. The good accuracy of the linear model results from the fact that the reluctance of the air gap in a magnetic circuit is often much greater than the reluctance of the ferromagnetic portion of the circuit. A relatively poor approximation for the iron or steel can thus be tolerated.

\/ D9.9. Given the magnetic circuit of Fig. 9.13, assume B = 0.6 T at the midpoint of the left leg and find: (a) FOTjair;(6) FOT,steel;(c) the current required in a 1300-turn coil linking the left leg.



FIGURE

Ans. 3980A-t; 72A-t; 3.12A

%/ D9.10. The magnetization curve for material X under normal operating conditions may be approximated by the expression B =(H/160)(0.25 + e-H/32°), where H is in A/m and B is in T. If a magnetic circuit contains a 12-cm length of material X, as well as a 0.25mm air gap, assume a uniform cross section of 2.5 cm2 and find the total mmf required to produce a flux of: (a)10 /xWb; (b) 100 /xWb.

Ans. 13.14A-t; 86.7A-t

9.9 POTENTIAL ENERGY AND FORCES ON MAGNETIC MATERIALS

In the electrostatic field we first introduced the point charge and the experimental law of force between point charges. After defining electric field intensity, electric flux density, and electric potential, we were able to find an expression for the energy in an electrostatic field by establishing the work necessary to bring the prerequisite point charges from infinity to their final resting places. The general expression for energy is

WE = 1 / D - E dv (45)
2 ./vol

where a linear relationship between D and E is assumed.

This is not as easily done for the steady magnetic field. It would seem that we might assume two simple sources, perhaps two current sheets, find the force on one due to the other, move the sheet a differential distance against this force, and equate the necessary work to the change in energy. If we did, we would be wrong, because Faraday's law (coming up in the next chapter) shows that there will be a voltage induced in the moving current sheet against which the currentmust be maintained. Whatever source is supplying the current sheet turns out to receive half the energy we are putting into the circuit by moving it.

In other words, energy density in the magnetic field may be determined more easily after time-varying fields are discussed. We shall develop the appropriate expression in discussing Poynting's theorem in Chap. 11.

An alternate approach would be possible at this time, however, for we might define a magnetostatic field based on assumed magnetic poles (or "magnetic charges"). Using the scalar magnetic potential, we could then develop an energy expression by methods similar to those used in obtaining the electrostatic energy relationship. These new magnetostatic quantities we would have to introduce would be too great a price to pay for one simple result, and we shall therefore merely present the result at this time and show that the same expression arises in the Poynting theorem later. The total energy stored in a steady magnetic field in which B is linearly related to H is

$$W_H = \frac{1}{2}\int_{vol} \mathbf{B} \cdot \mathbf{H}\, dv \qquad (46)$$

Letting $\mathbf{B} = \mu\mathbf{H}$, we have the equivalent formulations

$$W_H = \frac{1}{2}\int_{vol} \mu H^2 dv \qquad (47)$$

or

$$W_H = \frac{1}{2}\int_{vol} \frac{B^2}{\mu} dv \qquad (48)$$

It is again convenient to think of this energy as being distributed throughout the volume with an energy density of $\frac{1}{2}\mathbf{B} \cdot \mathbf{H}$ J/m3, although we have no mathematical justification for such a statement.

In spite of the fact that these results are valid only for linear media, we may use them to calculate the forces on nonlinear magnetic materials if we focus our attention on the linear media (usually air) which may surround them. For example, suppose that we have a long solenoid with a silicon-steel core. A coil containing n turns/m with a current I surrounds it. The magnetic field intensity in the core is therefore nI At/m, and the magnetic flux density can be obtained from the magnetization curve for silicon steel. Let us call this value B_{st}. Suppose that the core is composed of two semi-infinite cylinders[2] which are just touching. We now apply a mechanical force to separate these two sections of the core while keeping the flux density constant. We apply a force F over a distance dL, thus doing work FdL. Faraday's law does not apply here, for the fields in the core have not changed, and we can therefore use the principle of virtual work todetermine that the work we have done in moving one core appears as stored energy in the air gap we have created. By (48) above, this increase is

$$dW_H = F\,dL = \frac{1}{2}\frac{B_{st}^2}{\mu_0} S\,dL$$

where S is the core cross-sectional area. Thus

$$F = \frac{B_{st}^2 S}{2\mu_0}$$

[2] A semi-infinite cylinder is a cylinder of infinite length having one end located in finite space.

If, for example, the magnetic field intensity is sufficient to produce saturation in the steel, approximately 1.4T, the force is

$$F = 7.80 \times 105S \quad N$$

or about 113 lb//in2.

\/ D9.11. (a) What force is being exerted on the pole faces of the circuit described in Prob. D9.9 and Figure 9.13? (b) Is the force trying to open or close the air gap?

Ans. 1194 N; as Wilhelm Eduard Weber would put it, "schliessen"

9.10 INDUCTANCE AND MUTUAL INDUCTANCE

Inductance is the last of the three familiar parameters from circuit theory which we are defining in more general terms. Resistance was defined in Chap. 5 as the ratio ofthe potential difference between two equipotential surfaces ofa conducting material to the total current crossing either equipotential surface. The resistance is a function of conductor geometry and conductivity only. Capacitance was defined in the same chapter as the ratio of the total charge on either of two equipotential conducting surfaces to the potential difference between the surfaces. Capacitance is a function only of the geometry of the two conducting surfaces and the permittivity of the dielectric medium between or surrounding them. The interpretation of resistance and capacitance as circuit elements will be inspected more closely in Sec. 13.1.
As a prelude to defining inductance, we first need to introduce the concept of flux linkage. Let us consider a toroid of N turns in which a current / produces a total flux $. We shall assume first that this flux links or encircles each of the N turns, and we also see that each of the N turns links the total flux $. The flux linkage N$ is defined as the product of the number of turns N and the flux $ linking each of them.[28] For a coil having a single turn, the flux linkage is equal to the total flux.

however, and we will continue to write it as N$.

We now define inductance (or self-inductance) as the ratio of the total flux linkages to the current which they link,

[28] THE SYMBOL A IS COMMONLY USED FOR FLUX LINKAGES. WE SHALL ONLY OCCASIONALLY MAKE USE OF THIS CONCEPT,

_ N $
L = ~T (4
9)

The current I flowing in the N-turn coil produces the total flux $ and N$ flux linkages, where we assume for the moment that the flux $ links each turn. This definition is applicable only to magnetic media which are linear, so that the flux is proportional to the current. If ferromagnetic materials are present, there is no single definition of inductance which is useful in all cases, and we shall restrict our attention to linear materials.

The unit of inductance is the henry (H), equivalent to one weber-turn per ampere.

Let us apply (49) in a straightforward way to calculate the inductance per meter length of a coaxial cable of inner radius a and outer radius b. We may take the expression for total flux developed as Eq. (42) in Chap. 8,

Mold, b
$ = --------ln-
2n a

and obtain the inductance rapidly for a length d,

Mod b
L = ----ln- H
2n a

or, on a per-meter basis,

L = ^°lnb H/m (50)
27t a

In this case, N = 1 turn, and all the flux links all the current.

In the problem of a toroidal coil of N turns and a current I, as shown in Fig. 8.12b, we have

MoNI

27rp

o If the dimensions ofthe cross
the toroid n0 then the section are small compared with the mean radius of

2npo

where S is the cross-sectional area. Multiplying the total flux by N, we have the flux linkages, and dividing by I, we have the inductance

L = MoN2S (51)
27rpo

Once again we have assumed that all the flux links all the turns, and this is a good assumption for a toroidal coil of many turns packed closely together. Suppose, however, that our toroid has an appreciable spacing between turns, a short part of which might look like Fig. 9.14. The flux linkages are no longer the product of the flux at the mean radius times the total number of turns. In order to obtain the total flux linkages we must look at the coil on a turn-by-turn basis.

$$\sum_{i=1}^{N} \Phi_i$$

where <J>, is the flux linking the z'th turn. Rather than doing this, we usually rely on experience and empirical quantities called winding factors and pitch factors to adjust the basic formula to apply to the real physical world.

An equivalent definition for inductance may be made using an energy point of view,

where / is the total current flowing in the closed path and Wh is the energy in the magnetic field produced by the current. After using (52) to obtain several other general expressions for inductance, we shall show that it is equivalent to (49). We first express the potential energy Wh in terms of the magnetic fields,

L /vol B ■ H dv (53)

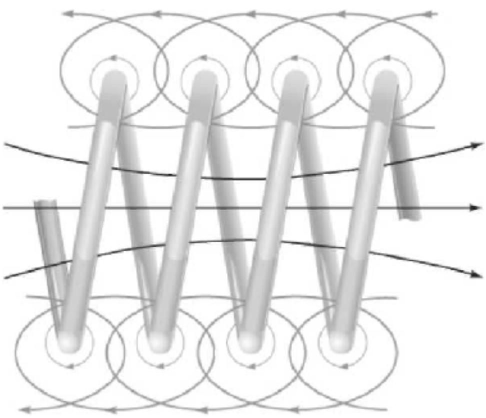


FIGURE
A PORTION OF A COIL SHOWING PARTIAL FLUX LINKAGES. THE TOTAL FLUX LINKAGES ARE OBTAINED BY ADDING THE FLUXES LINKING EACH TURN.

and then replace B by Vx A,

L = \ [H - (Vx A)dv
IZ Jvol

The vector identity

(55)

V-(A x H) = H - (Vx A) - A - (Vx H)(54) may be proved by

expansion in cartesian coordinates. The inductance is then 1

J V-(A x H)dv + I A - (Vx H) dv
L~I 2 J vol J vol

After applying the divergence theorem to the first integral J
and letting Vx H in the second integral, we have

1 c^(A x H)-dS + I A
L = I* - J dv
Js Jvol

The surface integral is zero, since the surface encloses the volume containing all the magnetic energy, and this requires that A and H be zero on the bounding surface. The inductance may therefore be written as

L = 4r I A - J dv (56)
12 ./vol

Equation (56) expresses the inductance in terms of an integral of the values of A and J at every point. Since current density exists only within the conductor, the integrand is zero at all points outside the conductor and the vector magnetic potential need not be determined there. The vector potential is that which arises from the current J, and any other current source contributing a vector potential field in the region of the original current density is to be ignored for the present. Later we shall see that this leads to a mutual inductance.

The vector magnetic potential A due to J is given by Eq. (51), Chap. 8,

A = f ^J dv
Jvol

and the inductance may therefore be expressed more basically as a rather formidable double volume integral,

L = i 1(1 7^ dv) - J dv (57)
12 Jvo\Jvo\4TTR 1 v '

A slightly simpler integral expression is obtained by restricting our attention to current filaments of small cross section for which J dv may be replaced by IdL and the volume integral by a closed line integral along the axis of the filament, Our only present interest in Eqs. (57) and (58) lies in their implication that the inductance is a function of the distribution of the current in space or the geometry of the conductor configuration.

To obtain our original definition of inductance (49) let us hypothesize a uniform current distribution in a filamentary conductor of small cross section so that J dv in (56) becomes IdL,

L = jj> A . dL (59)

For a small cross section, dL may be taken along the center of the filament. We now apply Stokes' theorem and obtain

L = -((V x A). d S
1 Js

or

L = -fB . dS
Is

or

L = y (60)

Retracing the steps by which (60) is obtained, we should see that the flux $ is that portion of the total flux which passes through any and every open surface whose perimeter is the filamentary current path.

If we now let the filament make N identical turns about the total flux, an idealization which may be closely realized in some types of inductors, the closed line integral must consist of N laps about this common path and (60) becomes

N $
L = — (61)

The flux $ is now the flux crossing any surface whose perimeter is the path occupied by any one of the N turns. The inductance of an N-turn coil may still be obtained from (60), however, if we realize that the flux is that which crosses the complicated surface4 whose perimeter consists of all N turns.

Use of any of the inductance expressions for a true filamentary conductor (having zero radius) leads to an infinite value of inductance, regardless of the configuration of the filament. Near the conductor Ampere's circuital law shows

Somewhat like a spiral ramp.

that the magnetic field intensity varies inversely with the distance from the conductor, and a simple integration soon shows that an infinite amount of energy and an infinite amount of flux are contained within any finite cylinder about the filament. This difficulty is eliminated by specifying a small but finite filamentary radius.

The interior of any conductor also contains magnetic flux, and this flux links a variable fraction of the total current, depending on its location. These flux linkages lead to an internal inductance, which must be combined with the external inductance to obtain the total inductance. The internal inductance of a long straight wire of circular cross section, radius a, and uniform current distribution is

^a,int 8tt- H /m (62)

a result requested in Prob. 43 at the end of this chapter.

In Chap. 11 it will be seen that the current distribution in a conductor at high frequencies tends to be concentrated near the surface. The internal flux is reduced, and it is usually sufficient to consider only the external inductance. At lower frequencies, however, internal inductance may become an appreciable part of the total inductance.

We conclude by defining the mutual inductance between circuits 1 and 2, M12, in terms of mutual flux linkages,

M12 h (63)

where <J>12 signifies the flux produced by I1 which links the path of the filamentary current I2, and 7V2 is the number of turns in circuit 2. The mutual inductance, therefore, depends on the magnetic interaction between two currents. With either current alone, the total energy stored in the magnetic field can be found in terms of a single inductance, or self-inductance; with both currents having nonzero values, the total energy is a function of the two self-inductances and the mutual inductance. In terms of a mutual energy, it can be shown that (63) is equivalent to

M12 1 I (B1 • Hi)dc hh J vol (64)

or

M12 (/iH1 • Hi)dc /vol (65)

where Bi is the field resulting from I1 (with I2 = 0) and H2 is the field due to I2 (with Ii = 0). Interchange of the subscripts does not change the right-hand side of (65), and therefore

M12 = (6
M21 6)

Mutual inductance is also measured in henrys, and we rely on the context to allow us to differentiate it from magnetization, also represented by M.

I ^Example 9.9

Calculate the self-inductances of and the mutual inductances between two coaxial solenoids of radius R1 and R2, R2 > R1, carrying currents I1 and I2 with n1 and n2 turns/m, respectively.

Solution. We first attack the mutual inductances. From Eq. (15), Chap. 8, we let n1 = N/d, and obtain

H1 = w1I1az (0
<p< R1) = 0 (p>
R1)

and

H2 = n2I2az (0
<p< R2) = 0 (p>
R2)

Thus, for this

uniform field and

M12 = ^0n1n2TtR21

Similarly,

M21 = ^0n1n2 = M12 If n1 = 50 turns/cm, n2 = 80 turns/cm, R1 = 2 cm, and R2 = 3 cm, then

M12 = M21 = Ait x
10"7(5000)(8000)w(0.022) = 63.2 mH/m The self-inductances are easily found. The flux produced in coil 1 by I1 is

*n = M0n1I1JrR2

and thus

L1 = /x0ni>S1d H The inductance per unit length is therefore

L1 = /x0n2 S1 H/m

or

L1 = 39.5mH/m

Similarly,

L2 = /o«2 & = 22.7mH/m

We see, therefore, that there are many methods available for the calculation of self-inductance and mutual inductance. Unfortunately, even problems possessing a high degree of symmetry present very challenging integrals for evaluation, and only a few problems are available for us to try our skill on.

Inductance will be discussed in circuit terms in Chap. 13.

PROBLEMS

9.1 A point charge, Q = — 0.3 //C and m = 3 x 10—16 kg, is moving through the field E = 30az V/m. Use Eq. (1) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at t = 0: v= 3 x 105ax m/s at the origin. At t = 3 /zs, find: (a) the position P(x, y, z) of the charge; (b) the velocity v;(c) the kinetic energy of the charge.

9.2 A point charge, Q = — 0.3 uC and m = 3 x 10—16 kg, is moving through the field B = 30az mT. Make use of Eq. (2) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial condition at t = 0, v = 3 x 105ax m/s at the origin. Solve these equations (perhaps with the help of an example given in Sec. 7.5) to evaluate at t = 3 us: (a) the position P(x, y, z) of the charge; (b) its velocity; (c) and its kinetic energy.

9.3 A point charge for which Q = 2 x 10—16 C and m = 5 x 10—26 kg is moving in the combined fields E = 100ax — 200ay + 300az V/m and B =—3ax + 2ay — az mT. If the charge velocity at t = 0is v(0) = (2ax — 3ay — 4az)105m/s: (a) give the unit vector showing the direction in which the charge is accelerating at t = 0; (b) find the kinetic energy of the charge at t = 0.

9.4 An electron (qe = —1.60219 x 10—19 C, m = 9.10956 x 10—31 kg) is moving at a constant velocity, v = 4.5 x 107ay m/s, along the negative y axis. At the origin, it encounters the uniform magnetic field B = 2.5az mT, and remains in it up to y = 2.5 cm. If we assume (with good accuracy) that the electron remains on the y axis while it is in the magnetic field, find its x-, y-, and z-coordinate values when y = 50 cm.

9.5 A rectangular loop of wire in free space joins points A(1,0, 1) to B(3, 0, 1) to C(3, 0, 4) to D(1, 0, 4) to A. The wire carries a current of 6 mA, flowing in the az direction from B to C. A filamentary current of 15 A flows along the entire z axis in the az direction. (a) Find F on side BC.(b) Find F on side AB.(c) Find Ftotal on the loop.

9.6 The magnetic flux density in a region of free space is given as B = — 3xax + 5yay — 2zaz T. Find the total force on the rectangular loop shown in Fig. 9.15 if it lies in the plane z = 0 and is bounded by x = 1, x = 3, y = 2, and y = 5, all dimensions in cm.

9.7 Uniform current sheets are located in free space as follows: 8az A/m at y = 0, —4az A/mat y = 1, and —4az A/mat y = —1. Find the

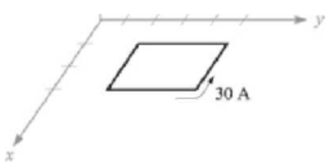


FIGURE 9.15

vector force per meter length exerted on a current filament carrying 7 mA in the aL direction if the filament is located at: (a) x = 0, y = 0.5, and aL = az;(b) y = 0.5, z = 0, and aL = ax;(c) x = 0, y = 1.5, and aL = az.

9.8 Filamentary currents of —25az and 25az A are located in the x = 0 plane in free space at y = —1 and y = 1m, respectively. A third filamentary current of 10—3az A is located at x = k, y = 0. Find the vector force on a 1-m length of the 1-mA filament and plot |F| versus k.

9.9 A current of — 100az A/m flows on the conducting cylinder p = 5 mm and +500az A/m is present on the conducting cylinder p = 1 mm. Find the magnitude of the total force per meter length acting to split the outer cylinder apart along its length.

9.10 Two infinitely long parallel filaments each carry 50 A in the az direction. If the filaments lie in the plane y = 0 and x = 5 mm, find the vector force per meter length on the filament passing through the origin.

9.11 (a) Use Eq. (14), Sec. 9.3, to show that the force of attraction per unit length between two filamentary conductors in free space with currents I1az at x = 0, y = d/2, and I2az at x = 0, y = —d/2, is I1I2/(2itd).(b) Show how a simpler method can be used to check your result.

9.12 A conducting current strip carrying K = 12az A/m lies in the x = 0 plane between y = 0.5 and y = 1.5 m. There is also a current filament of I = 5 A in the az direction on the z axis. Find the force exerted on the: (a) filament by the current strip; (b) strip by the filament.

9.13 A current of 6 A flows from M(2, 0, 5) to 7V(5, 0, 5) in a straight solid conductor in free space. An infinite current filament lies along the z axis and carries 50 A in the az direction. Compute the vector torque on the wire segment using an origin at: (a) (0,0,5); (b) (0,0,0); (c) (3,0,0).

9.14 The rectangular loop of Prob. 6 is now subjected to the B field produced by two current sheets, K1 = 400ay A/mat z = 2, and K2 = 300az A/mat y = 0, in free space. Find the vector torque on the loop, referred to an origin: (a) at (0,0,0); (b) at the center of the loop.

9.15 A solid conducting filament extends from x = —b to x = b along the line y = 2, z = 0. This filament carries a current of 3 A in the ax direction. An infinite filament on the z axis carries 5 A in the az direction. Obtain an expression for the torque exerted on the finite conductor about an origin located at (0,2,0).

9.16 Assume that an electron is describing a circular orbit of radius a about a positively charged nucleus. (a) By selecting an appropriate current and area, show that the equivalent orbital dipole moment is ea2&>/2, where m is the electron's angular velocity. (b) Show that the torque produced by a magnetic field parallel to the plane of the orbit is ea2&>B/2. (c)By equating the Coulomb and centrifugal forces, show that m is (47re0mea3/e2)—1/2, where me is the electron mass. (d) Find values for the angular velocity, torque, and the orbital magnetic moment for a hydrogen atom, where a is about 6 x 10—11 m; let B = 0.5T.

9.17 The hydrogen atom described in Prob. 16 is now subjected to a magnetic field having the same direction as that of the atom. Show that the forces caused by B result in a decrease of the angular velocity by eB/(2me) and a decrease in the orbital moment by e2a2B/(4me). What are these decreases for the hydrogen atom in parts per million for an external magnetic flux density of 0.5 T?

9.18 Calculate the vector torque on the square loop shown in Fig. 9.16 about an origin at A in the field B, given: (a) A(0, 0, 0) and B = 100ay mT; (b) A(0, 0, 0) and B = 200ax + 100ay mT; (c) A(1, 2, 3) and B = 200ax + 100ay — 300az mT; (d) A(1, 2, 3) and B = 200ax + 100ay — 300az mT for x > 2 and B = 0 elsewhere.

9.19 Given a material for which xm = 3.1 and within which B = 0.4yaz T, find: (a) H;(b) o;(c) Or;(d) M;(e) J;(f) Jb;fe) Jr.

9.20 Find H in a material where (a) oR = 4.2, there are 2.7 x 1029 atoms/m3, and each atom has a dipole moment of 2.6 x 10—30ay A •

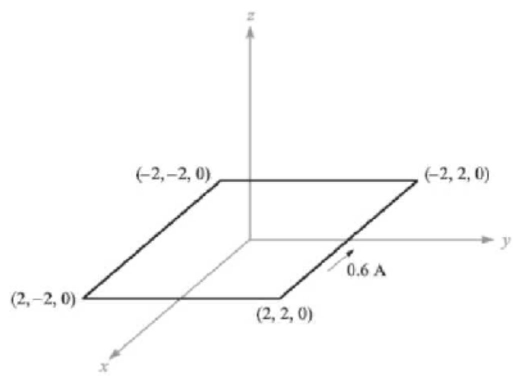
FIGURE 9.16

m2;(b) M = 270az A/m and o = 2 oH/m; (c) xm = 0.7 and B = 2az T. (d) Find M in a material where bound surface current densities of 12az A/m and −9az A/m exist at p = 0.3 m and 0.4 m, respectively.

9.21 Find the magnitude of the magnetization in a material for which: (a) the magnetic flux density is 0.02 Wb/m2;(b) the magnetic field intensity is 1200 A/m and the relative permeability is 1.005; (c) there are 7.2 x 1028 atoms per cubic meter, each having a dipole moment of 4 x 10−30 A-m2 in the same direction, and the magnetic susceptibility is 0.0003.

9.22 Three current sheets are located as follows: 160az A/mat x = 1cm, −40az A/mat x = 5 cm, and 50az A/mat x = 8 cm. Let o = Oo for x < 1 cm and x > 8 cm; for 1 < x < 5 cm, o = 3o0, and for 5 < x < 8 cm, u = 2u0. Find B everywhere.

9.23 Calculate values for B^, and at p = c for a coaxial cable with a = 2.5 mm and b = 6 mm if it carries a current I = 12 A in the center conductor, and o = 3 oH/m for 2.5 mm <p< 3.5 mm, o = 5 uH/m for



3.5mm <p< 4.5mm, and /z = 10 /zH/m for 4.5mm <p< 6mm. Use c =: (a) 3 mm; (b) 4 mm; (c) 5 mm.

9.24 A coaxial transmission line has a = 5 mm and b = 20 mm. Let its center lie on the z axis and let a dc current I flow in the az direction in the center
conductor. The volume between the conductors contains a magnetic material for which /z_r = 2.5, as well as air. Find H, B, and M everywhere
between conductors if YY^ = 6j|0 A/mat p = 10 mm, 4> = ^ and the
magnetic material is located where: (a) a <p < 3a;(b)0 <<^<7r.

9.25 A conducting filament at z = 0 carries 12 A in the az direction. Let //ft = 1 for p< 1 cm, /z_r = 6 for 1 <p < 2 cm, and /z_r = 1 for p> 2 cm. Find: (a) H everywhere; (b) B everywhere.

9.26 Point P(2, 3, 1) lies on the planar boundary separating region 1 from region 2. The unit vector a#12 = 0.6ax + 0.48ay + 0.64az is directed from region 1 to region 2. Let /zffl = 2, /z_r2 = 8, and H1 = 100ax — 300ay + 200az A/m. Find H2.

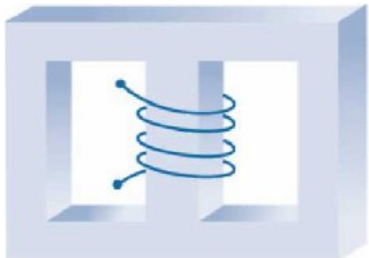
9.27 Let /Zffl = 2 in region 1, defined by 2x + 3y — 4z > 1, while /z_r2 = 5 in region 2 where 2x + 3y — 4z < 1. In region 1, H1 = 50ax — 30ay + 20az A/m. Find: (a) H^b) Ht1;(c) Ht2;(d) H^fc) 01, the angle between H1 and a^21;(/") 02, the angle between H2 and a#21.

9.28 For values of B below the knee on the magnetization curve for silicon steel approximate the curve by a straight line with / = 5mH/m. The core shown in Fig. 9.17 has areas of 1.6 cm2 and lengths of 10 cm in each outer leg, and an area of 2.5 cm2 and a length of 3 cm in the central leg. A coil of 1200 turns carrying 12 mA is placed around the central leg. Find B in the: (a) center leg; (b) center leg, if a 0.3-mm air gap is present in the center leg.

FIGURE
9.17

9.29 In Prob. 28, the linear approximation suggested in the statement of the problem leads to flux density of 0.666 T in the central leg. Using this value of B and the magnetization curve for silicon steel, what current is required in the 1200-turn coil?

9.30 A toroidal core has a circular cross section of 4 cm2 area. The mean radius of the toroid is 6 cm. The core is composed of two semicircular segments, one of silicon steel and the other of a linear material with



UR = 200. There is a 0.4-mm air gap at each of the two joints, and the core is wrapped by a 4000-turn coil carrying a dc current I1.(a) Find I1 if the flux density in the core is 1.2 T. (b) Find the flux density in the core if I1 =—.3A.

9.31 A toroid is constructed of a magnetic material having a cross-sectional area of 2.5 cm2 and an effective length of 8 cm. There is also a short air gap of 0.25-mm length and an effective area of 2.8 cm2. An mmf of 200 A-t is applied to the magnetic circuit. Calculate the total flux in the toroid if the magnetic material: (a) is assumed to have infinite permeability; (b) is assumed to be linear with oR = 1000; (c) is silicon steel.

9.32 Determine the total energy stored in a spherical region 1 cm in radius, centered at the origin in free space, in the uniform field: (a) H1 = —600ay A/m; (b) 600ax + 1200ay A/m; (c) H3 = —600ax + 1200ay A/m; (d) H4 = H2 + H3;(e) 1000ax A/m + 0.001ax T.

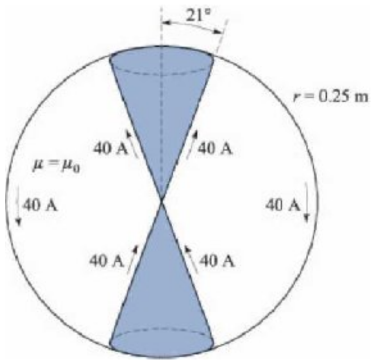
9.33 A toroidal core has a square cross section, 2.5 cm <p< 3.5 cm, —0.5 cm < z < 0.5 cm. The upper half of the toroid, 0 < z < 0.5 cm, is constructed of a linear material for which oR = 10, while the lower half, —0.5 cm < z < 0, has oR = 20. An mmf of 150 A-t establishes a flux in the direction. For z > 0, find: (a) 7/^(p);(b) (p);(c) <J>z>0.(d) Repeat for z < 0. (e) Find <J>total.

9.34 Three planar current sheets are located in free space as follows: —100ax A/mat z =—1, 200ax A/mat z = 0, and —100ax A/mat z = 1. Let w# = 2 B • H J/m3 and find w# for all z.

9.35 The cones 9 = 21° and 9 = 159° are conducting surfaces and carry total currents of 40 A, as shown in Fig. 9.18. The currents return on a spherical conducting surface of 0.25-m radius. (a) Find H in the region 0 < r < 0.25, 21° <9 < 159°,0 <<£< 2^.(b) How much energy is stored in this region?

9.36 A filament carrying a current I in the az direction lies on the z axis, and cylindrical current sheets of 5az A/m and —2az A/m are located at p = 3

FIGURE 9.18


21°
r = 0.25 m
40 A
40 A
$\mu = \mu_0$
40 A
40 A
40 A
40 A

and p = 10, respectively. (a) Find I if H = 0 for p> 10. (b) Using this value of I, calculate H for all p,3 <p< 10. (c) Calculate and plot WY versus p0, where WY is the total energy stored within the volume 0 < z < 1,0 <^< 27t,3 <p< P0.

9.37 Find the inductance of the cone-sphere configuration described in Prob. 35 and Fig. 9.18. The inductance is that offered at the origin between the vertices of the cone.

9.38 A toroidal core has a rectangular cross section defined by the surfaces p = 2 cm, p = 3 cm, z = 4 cm, and z = 4.5 cm. The core material has a relative permeability of 80. If the core is wound with a coil containing 8000 turns of wire, find its inductance.

9.39 Conducting planes in air at z = 0 and z = d carry surface currents of ±X"0ax A/m. (a) Find the energy stored in the magnetic field per unit length (0 < x < 1) in a width w (0 < y < w).(b) Calculate the inductance per unit length of this transmission line from WY = ^ LI2, where I is the total current in a width w in either conductor. (c) Calculate the total flux passing through the rectangle 0 < x < 1, 0 < z < d, in the plane y = 0, and from this result again find the inductance per unit length.

9.40 A coaxial cable has conductor dimensions of 1 and 5 mm. The region between conductors is air for 0 <0< — and — <(/>< —, and a noncon-
ducting material having /z_r = 8 for — <^< — and — <^< 2—. Find the inductance per meter length. 2 2

9.41 A rectangular coil is composed of 150 turns of a filamentary conductor.
Find the mutual inductance in free space between this coil and an infinite
straight filament on the z axis if the four corners of the coil are located
at: (a) (0,1,0), (0,3,0), (0,3,1), and (0,1,1); (b) (1,1,0), (1,3,0), (1,3,1), and (1,1,1).

9.42 Find the mutual inductance of this conductor system in free space: (a)
the solenoid of Fig. 8.11b and a square filamentary loop of side length b
coaxially centered inside the solenoid, if a > b/ 2;(b) a cylindrical con-
ducting shell of a radius a, axis on the z axis, and a filament at x = 0, y = d.

9.43 (a) Use energy relationships to show that the internal inductance of a nonmagnetic cylindrical wire of radius a carrying a uniformly distributed
current I is /z0/(8—) H/m. (b) Find the internal inductance if the portion
of the conductor for which p< c < a is removed.

Chapter Nine

TIME-VARYING FIELDS AND MAXWELL'S EQUATIONS

9.1 FARADAY'S LAW

After Oersted[1] demonstrated in 1820 that an electric current affected a compass needle, Faraday professed his belief that if a current could produce a magnetic field, then a magnetic field should be able to produce a current. The concept of the "field" was not available at that time, and Faraday's goal was to show that a current could be produced by "magnetism."

He worked on this problem intermittently over a period of ten years, until he was finally successful in 1831.[2] He wound two separate windings on an iron toroid and placed a galvanometer in one circuit and a battery in the other. Upon closing the battery circuit, he noted a momentary deflection of the galvanometer; a similar deflection in the opposite direction occurred when the battery was disconnected. This, of course, was the first experiment he made involving a changing magnetic field, and he followed it with a demonstration that either a moving magnetic field or a moving coil could also produce a galvanometer deflection.

In terms of fields, we now say that a time-varying magnetic field produces an electromotive force (emf) which may establish a current in a suitable closed circuit. An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields, and we shall define it below. Faraday's law is customarily stated as

emf
= —
V dt (1)

Equation (1) implies a closed path, although not necessarily a closed conducting path; the closed path, for example, might include a capacitor, or it might be a purely imaginary line in space. The magnetic flux is that

(5)

flux which passes through any and every surface whose perimeter is the closed path, and d<J>/dt is the time rate of change of this flux.

A nonzero value of d<J>/dt may result from any of the following situations:

1. A time-changing flux linking a stationary closed path
2. Relative motion between a steady flux and a closed path
3. A combination of the two

The minus sign is an indication that the emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf. This statement that the induced voltage acts to produce an opposing flux is known as Lenz's law.3

If the closed path is that taken by an TV-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

emf V d $
dt
(2)

where $ is now interpreted as the flux passing through any one of TV coincident paths.

We need to define emf as used in (1) or (2). The emf is obviously a scalar, and (perhaps not so obviously) a dimensional check shows that it is measured in volts. We define the emf as

emf (3)

and note that it is the voltage about a specific closed path. If any part of the path is changed, generally the emf changes. The departure from static results is clearly shown by (3), for an electric field intensity resulting from a static charge distribution must lead to zero potential difference about a closed path. In electrostatics, the line integral leads to a potential difference; with time-varying fields, the result is an emf or a voltage.

Replacing $ in (1) by the surface integral of B, we have

emf /E -dL= -dt/s: B *d* S (4)

where the fingers of our right hand indicate the direction of the closed path, and our thumb indicates the direction of dS. A flux density B in the direction of dS and increasing with time thus produces an average value of E which is opposite to the positive direction about the closed path. The right-handed relationship between the surface integral and the closed line integral in (4) should always be kept in mind during flux integrations and emf determinations.

Let us divide our investigation into two parts by first finding the contribution to the total emf made by a changing field within a

(5)

stationary path (transformer emf), and then we will consider a moving path within a constant (motional, or generator, emf).

We first consider a stationary path. The magnetic flux is the only time-varying quantity on the right side of (4), and a partial derivative may be taken under the integral sign,

emf - dL =~-jdt dS

Before we apply this simple result to an example, let us obtain the point form of this integral equation. Applying Stokes' theorem to the closed line integral, we have

f(Vx E)- dS = — ! — • dS

Js Js ot

where the surface integrals may be taken over identical surfaces. The surfaces are perfectly general and may be chosen as differentials,

(V x E)- dS =------dS

ot

and

E OB (6)

This is one of Maxwell's four equations as written in differential, or point, form, the form in which they are most generally used. Equation (5) is the integral form of this equation and is equivalent to Faraday's law as applied to a fixed path. If B is not a function of time, (5) and (6) evidently reduce to the electrostatic equations,

<j> E • dL = 0 (electrostatics)

and

Vx E = 0 (electrostatics)

As an example of the interpretation of (5) and (6), let us assume a simple magnetic field which increases exponentially with time within the cylindrical region p< b,

B = Boektaz (7)

where B0 = constant. Choosing the circular path p = a, a < b, in the z = 0 plane, along which must be constant by symmetry, we then have from (5)

emf = 2naElp = —kB0ektna2

The emf around this closed path is —kB0ektna2. It is proportional to a2, because the magnetic flux density is uniform and the flux passing through the surface at any instant is proportional to the area.

If we now replace a by p, p< b, the electric field intensity at any point is

E =— 2kB0ektpa^ (8)

Let us now attempt to obtain the same answer from (6), which becomes

(5)

(Vx E)z = -
kB0e- = 1 ^ P
dp

Multiplying by p and integrating from 0 to p (treating t as a constant, since the derivative is a partial derivative),

-\kBoektp2 = pE^

or

E = —kBoekt pa<p

once again.

If B0 is considered positive, a filamentary conductor of resistance R would have a current flowing in the negative direction, and this current would establish a flux within the circular loop in the negative az direction. Since increases exponentially with time, the current and flux do also, and thus tend to reduce the time rate of increase of the applied flux and the resultant emf in accordance with Lenz's law.

Before leaving this example, it is well to point out that the given field B does not satisfy all of Maxwell's equations. Such fields are often assumed (always in ac-circuit problems) and cause no difficulty when they are interpreted properly. They occasionally cause surprise, however. This particular field is discussed further in Prob. 19 at the end of the chapter.

Now let us consider the case of a time-constant flux and a moving closed path. Before we derive any special results from Faraday's law (1), let us use the basic law to analyze the specific problem outlined in Fig. 10.1. The closed circuit consists of two parallel conductors which are connected at one end by a high-resistance voltmeter of negligible dimensions and at the other end by a sliding bar moving at a velocity v. The magnetic flux density B is constant (in space and time) and is normal to the plane containing the closed path.

'(iinifunn)

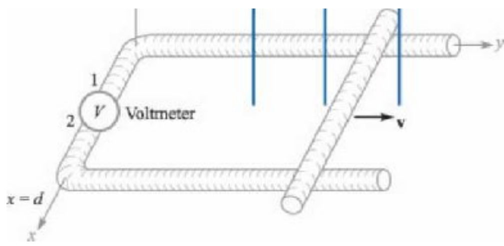


FIGURE 10.1
An example illustrating the application of Faraday's law to the case of a constant magnetic flux density B and a moving path. The shorting bar moves to the right with a velocity v, and the circuit is completed through the two rails and an

Let the position of the shorting bar be given by y; the flux passing through the surface within the closed path at any time t is then

<J> = Byd

From (1), we obtain

emf = -d$/dt = —B dy/dt d =-Bvd (9)

The emf is defined as E • dL and we have a conducting path; so we may actually determine E at every point along the closed path. We found in electrostatics that the tangential component of E is zero at the surface of a conductor, and we shall show in Sec. 10.4 that the tangential component

(5)

is zero at the surface of a perfect conductor (a = 00) for all time-varying conditions. This is equivalent to saying that a perfect conductor is a "short circuit." The entire closed path in Figure 10.1 may be considered as a perfect conductor, with the exception of the voltmeter. The actual computation of (f E • dL then must involve no contribution along the entire moving bar, both rails, and the voltmeter leads. Since we are integrating in a counterclockwise direction (keeping the interior of the positive side of the surface on our left as usual), the contribution E AL across the voltmeter must be —Bvd, showing that the electric field intensity in the instrument is directed from terminal 2 to terminal 1. For an up-scale reading, the positive terminal of the voltmeter should therefore be terminal 2.

The direction of the resultant small current flow may be confirmed by noting that the enclosed flux is reduced by a clockwise current in accordance with Lenz's law. The voltmeter terminal 2 is again seen to be the positive terminal.

Let us now consider this example using the concept of motional emf. The force on a charge Q moving at a velocity v in a magnetic field B is

$$F = Qv \times B$$

or

$$Q = v \times B \qquad (10)$$

The sliding conducting bar is composed of positive and negative charges, and each experiences this force. The force per unit charge, as given by (10), is called the motional electric field intensity Em,

$$Em = v \times B \qquad (11)$$

If the moving conductor were lifted off the rails, this electric field intensity would force electrons to one end of the bar (the far end) until the static field due to these charges just balanced the field induced by the motion of the bar. The resultant tangential electric field intensity would then be zero along the length of the bar. The motional emf produced by the moving conductor is then

$$emf = j> Em \blacksquare dL = j>(v \times B)- dL \qquad (12)$$

where the last integral may have a nonzero value only along that portion of the path which is in motion, or along which v has some nonzero value. Evaluating the right side of (12), we obtain

$$\hat{}(v \times B)- dL = j\ vBdx = -Bvd$$

as before. This is the total emf, since B is not a function of time.

In the case of a conductor moving in a uniform constant magnetic field, we may therefore ascribe a motional electric field intensity Em = v x B to every portion of the moving conductor and evaluate the resultant emf by

$$\oint E \blacksquare dL = \oint E_m \blacksquare dL = j. \qquad emf = (b\ E \blacksquare dL = (b\ Em$$

(5)

■ dL = <p(v x B)-dL (13
)

If the magnetic flux density is also changing with time, then we must include both contributions, the transformer emf (5) and the motional emf (12),

emf = j> E ■ dL = — j ^B ■ dS + ^(v x dL (14)

This expression is equivalent to the simple statement

emf=— ? (1)

and either can be used to determine these induced voltages.

Although (1) appears simple, there are a few contrived examples in which its proper application is quite difficult. These usually involve sliding contacts or switches; they always involve the substitution of one part of a circuit by a new part.4 As an example, consider the simple circuit of Fig. 10.2, containing several perfectly conducting wires, an ideal voltmeter, a uniform constant field B, and a switch. When the switch is opened, there is obviously more flux enclosed in the voltmeter circuit; however, it continues to read zero. The change in flux has not been produced by either a time-changing B [first term of (14)] or a conductor moving through a magnetic field [second part of (14)]. Instead, a new circuit has been substituted for the old. Thus it is necessary to use care in evaluating the change in flux linkages.

The separation of the emf into the two parts indicated by (14), one due to the time rate of change of B and the other to the motion of the circuit, is some-

See Bewley, in Suggested References at the end of the chapter,

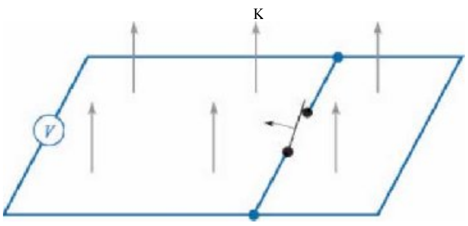


FIGURE 10.2
An apparent increase in flux linkages does not lead to an induced voltage when one part of a circuit is simply substituted for another by opening the switch. No indication will be observed on the voltmeter.

particularly pp. 12-19.

what arbitrary in that it depends on the relative velocity of the observer and the system. A field that is changing with both time and space may look constant to an observer moving with the field. This line of reasoning is developed more fully in applying the special theory of relativity to electromagnetic theory.5

10.2 DISPLACEMENTCURRENT

Faraday's experimental law has been used to obtain one of Maxwell's equations in differential form,

vxE=—§ (15)

which shows us that a time-changing magnetic field produces an electric field. Remembering the definition of curl, we see that this electric field has the special property of circulation; its line integral about a general closed path is not zero. Now let us turn our attention to the time-changing electric field.

We should first look at the point form of Ampere's circuital law as it applies to steady magnetic fields,

This is discussed in several of the references listed in the Suggested References at the end of the chapter. See Panofsky and Phillips, pp. 142-151; Owen, pp. 231-245; and Harman in several places.

Vx H = J (16)

and show its inadequacy for time-varying conditions by taking the divergence of each side,

V-VxH ee 0 = V-J

The divergence of the curl is identically zero, so V- J is also zero. However, the equation of continuity,

V-J =
dt

then shows us that (16) can be true only if dpv/dt = 0. This is an unrealistic limitation, and (16) must be amended before we can accept it for time-varying fields. Suppose we add an unknown term G to (16),

VxH=J+G

Again taking the divergence, we have

0 = V-J + V- G

Thus

Replacing pv by V- D,

V-G = d (V-D) = V-dD
dt ' dt

from which we obtain the simplest solution for G,

G = dD
dt

Ampere's circuital law in point form therefore becomes

VxH = J + dD
(17) d

Equation (17) has not been derived. It is merely a form we have obtained which does not disagree with the continuity equation. It is also consistent with all our other results, and we accept it as we did each experimental law and the equations derived from it. We are building a theory, and we have every right to our equations until they are proved wrong. This has not yet been done.

We now have a second one of Maxwell's equations and shall investigate its significance. The additional term dD/dt has the dimensions of current density, amperes per square meter. Since it results from a time-varying electric flux density (or displacement density), Maxwell termed it a displacement current density. We sometimes denote it by :

Vxh = j + Jd

x ad
jd " IF

This is the third type of current density we have met. Conduction current density,

j = oe

is the motion of charge (usually electrons) in a region of zero net charge density, and convection current density,

j = PvV

is the motion of volume charge density. Both are represented by j in (17). Bound current density is, of course, included in h. In a nonconducting medium in which no volume charge density is present, j = 0, and then

Vxh = ^ (if j = 0)(18)
at

Notice the symmetry between (18) and (15):

vxe=-f (15)

Again the analogy between the intensity vectors e and h and the flux density vectors d and b is apparent. Too much faith cannot be placed in this analogy, however, for it fails when we investigate forces on particles. The force on a charge is related to e and to b, and some good arguments may be presented showing an analogy between e and b and between d and h. We shall omit them, however, and merely say that the concept of displacement current was probably suggested to Maxwell by the symmetry first mentioned above.[6]

The total displacement current crossing any given surface is expressed by the surface integral,

Id = j Jd ■ ds = j add • ds

and we may obtain the time-varying version of Ampere's circuital law by integrating (17) over the surface S,

f(Vx h)- ds =
[j ■ ds + [ad ■ ds at
Js Js Js

The analogy that relates B to D and H to E is strongly advocated by Fano, Chu, and Adler (see Suggested References for Chap. 5) on pp. 159-160 and 179; the case for comparing B to E and D to H is presented in Halliday and Resnick (see Suggested References for this chapter) on pp. 665-668 and 832836.

and applying Stokes' theorem,

H - d L = I + Id = I dD • IT d S
+

(19)

What is the nature of displacement current density? Let us study the simple circuit of Fig. 10.3, containing a filamentary loop and a parallel-plate capacitor. Within the loop a magnetic field varying sinusoidally with time is applied to produce an emf about the closed path (the filament plus the dashed portion between the capacitor plates) which we shall take as

emf = V0 cos mt

Using elementary circuit theory and assuming the loop has negligible resistance and inductance, we may obtain the current in the loop as

I = -fflCV0 sin mt

= — m — V
sin mt d

where the quantities e, S, and d pertain to the capacitor. Let us apply Ampere's circuital law about the smaller closed circular path k and neglect displacement current for the moment:

<jk H - d L = Ik

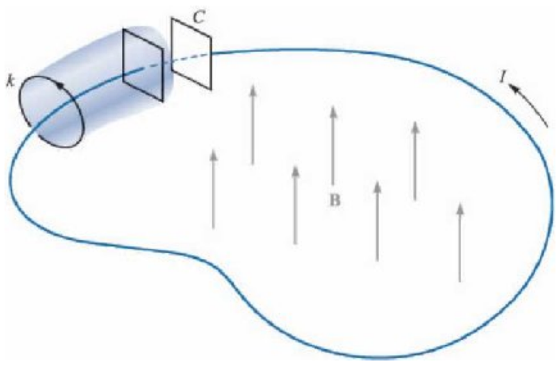


FIGURE 10.3

A filamentary conductor forms a loop connecting the two plates of a parallel-plate capacitor. A time-varying magnetic field inside the closed path produces an emf of V_0 cos mt around the closed path. The conduction current I is equal to the displacement current between the capacitor plates.

The path and the value of H along the path are both definite quantities (although difficult to determine), and <fk H ■ dL is a definite quantity. The current Ik is that current through every surface whose perimeter is the path k. If we choose a simple surface punctured by the filament, such as the plane circular surface defined by the circular path k, the current is evidently the conduction current. Suppose now we consider the closed path k as the mouth of a paper bag whose bottom passes between the capacitor plates. The bag is not pierced by the filament, and the conductor current is zero. Now we need to consider displacement current, for within the capacitor

B eE F e —(V0D
cot J7 cos
d

and therefore

(19)

aD es
Id =— S = -
co— Vo sin
cot
at d

This is the same value as that of the conduction current in the filamentary loop. Therefore the applicaton of Ampere's circuital law including displacement current to the path k leads to a definite value for the line integral ofH. This value must be equal to the total current crossing the chosen surface. For some surfaces the current is almost entirely conduction current, but for those surfaces passing between the capacitor plates, the conduction current is zero, and it is the displacement current which is now equal to the closed line integral of H.

Physically, we should note that a capacitor stores charge and that the electric field between the capacitor plates is much greater than the small leakage fields outside. We therefore introduce little error when we neglect displacement current on all those surfaces which do not pass between the plates.

Displacement current is associated with time-varying electric fields and therefore exists in all imperfect conductors carrying a time-varying conduction current. The last part of the drill problem below indicates the reason why this additional current was never discovered experimentally. This comparison is illustrated further in Sec. 11.3.

10.3 MAXWELL'S EQUATIONS IN POINT FORM

We have already obtained two of Maxwell's equations for time-varying fields,

V x E dB
dt (20)

and

VxH=J+ 3D
~dt
(21)

The remaining two equations are unchanged from their non-time-varying form:

VD =Pv (22)

VB = i (23)

Equation (22) essentially states that charge density is a source (or sink) of electric flux lines. Note that we can no longer say that all electric flux begins and terminates on charge, because the point form of Faraday's law (20) shows that E, and hence D, may have circulation if a changing magnetic field is present. Thus the lines of electric flux may form closed loops. However, the converse is still true, and every coulomb of charge must have one coulomb of electric flux diverging from it.

Equation (23) again acknowledges the fact that "magnetic charges," or poles, are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

These for equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources, charge and current density. The auxiliary equations relating D and E.

$$D = eE \tag{24}$$

relating B and H,

$$B = /zH$$

defining conduction current density,

$$\tag{26}$$

and defining convection current density in terms of the volume charge density pv,

$$J = PvV \tag{27}$$

are also required to define and relate the quantities appearing in Maxwell's equations.

The potentials V and A have not been included above because they are not strictly necessary, although they are extremely useful. They will be discussed at the end of this chapter.

If we do not have "nice" materials to work with, then we should replace (24) and (25) by the relationships involving the polarization and magnetization fields,

$$D = e0E + P \tag{28}$$

$$B \qquad M) \tag{29}$$

For linear materials we may relate P to E

$$P = X^E$$

and M to H

$$M = XmH$$

(25)

(31)

(30)

Finally, because of its fundamental importance we should include the Lorentz force equation, written in point form as the force per unit volume,

= Pv(E + v B) (32)
x

The following chapters are devoted to the application of Maxwell's equations to several simple problems.

|/ D10.4. Let it = 10-5 H/m, e = 4 x 10-9 F/m, o = 0, and pv = 0. Find k (including units) so that each of the following pairs of fields satisfies Maxwell's equations: (a) D = 6ax - 2yay + 2zaz nC/m2, H = kxax + 10yay - 25zaz A/m; (b) E = (20^ -k0ax V/m, H = (j + 2 x 106i)az A/m.

Ans. 15 A/m2; -2.5 x 108 V/(m -s)

10.4 MAXWELL'S EQUATIONS IN INTEGRAL FORM

The integral forms of Maxwell's equations are usually easier to recognize in terms of the experimental laws from which they have been obtained by a generalization process. Experiments must treat physical macroscopic quantities, and their results therefore are expressed in terms of integral relationships. A differential equation always represents a theory. Let us now collect the integral forms of Maxwell's equations of the previous section.

Integrating (20) over a surface and applying Stokes' theorem, we obtain Faraday's law,

<j> E • d L = 3B It ' (33)
- J d S

and the same process applied to (21) yields Ampere's circuital law, dL / +j

iH • dL = / + I — • (34)
dS
Is ot

Gauss's laws for the electric and magnetic fields are obtained by integrating (22) and (23) throughout a volume and using the divergence theorem:

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{vol} \rho_v \, dv \qquad (35)$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad (36)$$

These four integral equations enable us to find the boundary conditions on B, D, H, and E which are necessary to evaluate the constants obtained in solving Maxwell's equations in partial differential form. These boundary conditions are in general unchanged from their forms for static or steady fields, and the same methods may be used to obtain them. Between any two real physical media (where K must be zero on the boundary surface), (33) enables us to relate the tangential E-field components,

Eti = Ea (37)

and from (34),

Ha = Ha (38)

The surface integrals produce the boundary conditions on the normal components,

Avi - DN2 = ps (39)

and

*N1 = bn2 (40)

It is often desirable to idealize a physical problem by assuming a perfect conductor for which a is infinite but J is finite. From Ohm's law, then, in a perfect conductor,

$$E = 0$$

and it follows from the point form of Faraday's law that

$$H = 0$$

for time-varying fields. The point form of Ampere's circuital law then shows that the finite value of J is

J=0

and current must be carried on the conductor surface as a surface current K. Thus, if region 2 is a perfect conductor, (37) to (40) become, respectively,

E,i = 0 (41)

Hti = K (Hri = K x av)(42)

Dni = ps (43)

*vi = 0 (44)

where an is an outward normal at the conductor surface.

Note that surface charge density is considered a physical possibility for either dielectrics, perfect conductors, or imperfect conductors, but that surface current density is assumed only in conjunction with perfect conductors.

The boundary conditions stated above are a very necessary part of Maxwell's equations. All real physical problems have boundaries and require the solution of Maxwell's equations in two or more regions and the matching of these solutions at the boundaries. In the case of perfect conductors, the solution of the equations within the conductor is trivial (all time-varying fields are zero), but the application of the boundary conditions (4i) to (44) may be very difficult.

Certain fundamental properties of wave propagation are evident when Maxwell's equations are solved for an unbounded region. This problem is treated in the following chapter. It represents the simplest application of

Maxwell's equations, becuase it is the only problem which does not require the application of any boundary conditions.

PROBLEMS

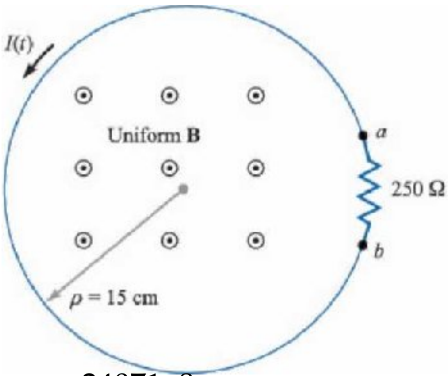
10.1 In Fig. 10.4, let B = 0.2cos120^t T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by /(t) is negligible. Find: (a) Vab(t);(b) /(t).

10.2 Given the time-varying magnetic field B = (0.5ax + 0.6ay-0.3az) cos5000t T and a square filamentary loop with its corners at (2,3,0), (2,-3,0), (-2,3,0), and (-2,-3,0), find the time-varying current flowing in the general direction if the total loop resistance is 400 k^.

10.3 Given H = 300az cos(3 x 1081 - y) A/m in free space, find the emf developed in the general direction about the closed path having corners at: (a) (0,0,0), (1,0,0), (1,1,0), and (0,1,0); (b) (0,0,0) (2tt,0,0), (2jt,27t,0), (0, 2jt,0).

10.4 Conductor surfaces are located at p = 1 cm and p = 2 cm in free space. The volume 1 cm <p< 2 cm contains the fields H =

FIGURE 10.4



2 8 24071 8
pp-cos(6 x 1087rt — 2^z) A/m and Ep = cos(6 x 1087rt — 2^z) V/m.

(a) Show that these two fields satisfy Eq. (6), Sec. 10.1. (b) Evaluate both integrals in Eq. (4) for the planar surface defined by <p = 0, 1 cm <p< 2 cm, 2 = 0.1, and its perimeter, and show that the same results are obtained.

10.5 The location of the sliding bar in Figure 10.5 is given by x = 5t + 2t3, and the separation of the two rails is 20 cm. Let B = 0.8x2a2 T. Find the voltmeter reading at: (a) t = 0.4 s; (b) x = 0.6 m.

10.6 A perfectly conducting filament containing a small 500-S2 resistor is formed into a square, as illustrated by Fig. 10.6. Find I(t) if B =: (a)0.3 cos(1207rt — 30°)a2 T; (b)0.4cos[7r(ct — y)]a2 //T, where c = 3 x 108 m/s.

10.7 The rails in Fig. 10.7 each have a resistance of 2.2 £2/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find I(t),0< t < 1 s, if the bar is at x = 2matt = 0 and: (a)a resistor is present across the left end with the right end open-circuited; (b) a resistor is present across each end.

FIGURE 10.6

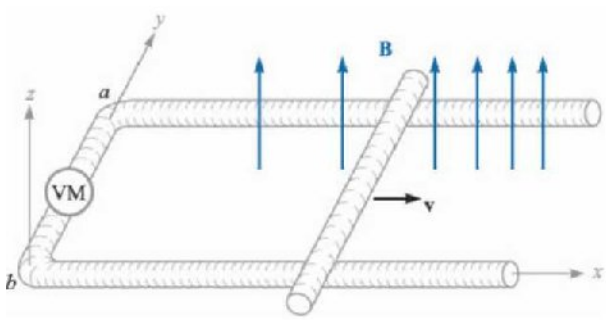
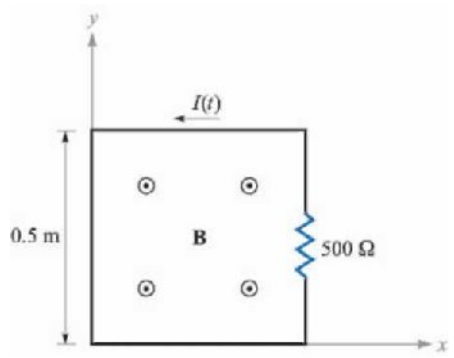


FIGURE 10.5
See Prob. 10.5.



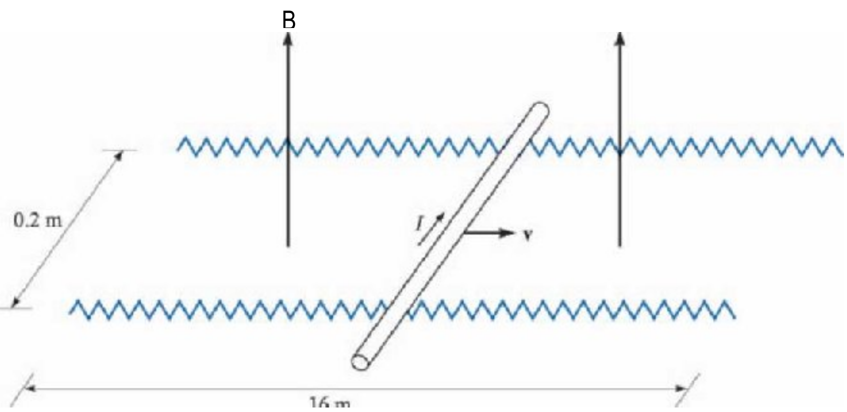


FIGURE 10.7

10.8 Fig. 10.1 is modified to show that the rail separation is larger when y is larger. Specifically, let the separation d = 0.2 + 0.02y. Given a uniform velocity vy = 8 m/s and a uniform magnetic flux density Bz = 1.1 T, find V12 as a function of time if the bar is located at y = 0at t = 0.

10.9 A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 £ per meter length. The loop lies in the z = 0 plane with its corners at (0,0,0), (0.25,0,0), (0.25,0.25,0), and (0,0.25,0) at t = 0. The loop is moving with a velocity vy = 50 m/s in the field Bz = 8cos(1.5 x 108t - 0.5x) //T. Develop a function of time which expresses the ohmic power being delivered to the loop.

10.10 (a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is a/me for the applied field E = Em cos cot. Assume / = /z0 .(b) What is the amplitude ratio if the applied field is E = Eme-t/x, where t is real?

10.11 Let the internal dimensions of a coaxial capacitor be a = 1.2 cm, b = 4 cm, and / = 40 cm. The homogeneous material inside the capacitor has the parameters e = 10-11 F/m, / = 10-5 H/m, and a = 10-5 S/m. If the electric field intensity is E = (106/p) cos 105ap V/m, find: (a) J; (b) the total conduction current Ic through the capacitor; (c) the total displacement current Id through the capacitor; (d) the ratio of the amplitude of Id to that of Ic, the quality factor of the capacitor.

b 2 5

10.12 Given a coaxial transmission line with - = e , /zr = eR = 1, and an

200 a

electric field intensity E = cos(109t - 3.336z)ap V/m, find: (a) Vab,

the voltage between the conductors, if it is known that the electrostatic relationship E =-VV is valid; (b) the displacement current density.

10.13 Consider the region defined by |x|, |y|, and |2| < 1. Let eR = 5, /R = 4, and a = 0. If /d = 20 cos(1.5 x 108t — bx)ay /zA/m2:(a) find D and E;(b) use the point form of Faraday's law and an integration with respect to time to find B and H;(c) use VxH = Jd + J to find Jd .(d) What is the numerical value of b?

10.14 A voltage source V0 sin mt is connected between two concentric conducting spheres, r = a and r = b, b > a, where the region between them is a material for which e = eRe0, / = /z0, and a = 0. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Sec. 5.10) and circuit-analysis methods.

FIGURE
10.8

10.15 Let / = 3 x 10—5 H/m, e = 1.2 x 10—10 F/m, and a = 0 everywhere. If H = 2cos(1010t — /Sx)a2 A/m, use Maxwell's equations to obtain expressions for B, D, E, and /S.

10.16 (a) A certain material has a = 0 and eR = 1. If H = 4sin (106t — 0.012)ay A/m, make use of Maxwell's equations to find /zR.(b) Find E(2, t).

10.17 The electric field intensity in the region 0 < x < 5, 0 < y <7r/12, 0 < 2 < 0.06 m in free space is given by E = C sin 12y sin a2 cos 2 x 1010tax V/m. Beginning with the VxE relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero.

10.18 The parallel-plate transmission line shown in Fig. 10.8 has dimensions b = 4 cm and d = 8 mm, while the medium between the plates is characterized by /zR = 1, eR = 20, and a = 0. Neglect fields outside the dielectric. Given the field H = 5cos(109t — /2)ay A/m, use Maxwell's equations to help find: (a) /S,if 0; (b) the displacement current density at 2 = 0; (c) the total displacement current crossing the surface x = 0.5d,0 < y < b,0 < 2 < 0.1 m in the ax direction.