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USE POLYNOMAL INTERPOLATION TO DERIVR A FIRST, SECOND, THIRD AND FOURTH ORDER INTERPOLATION AND SOME APPLICATION IN COMPUTER AID DESIGN

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و أَنزَلَ اللَّهُ عَلَيْكَ الكِتَابَ وَالْحِكْمَةَ وَعَلَّمَكَ مَا لَمْ تَكُن تَعْلَمُ وَكَانَ فَضْلُ اللَّهِ عَلَيْكَ عَظِيماً

صدق الله العلي العظيم

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ABSTRACT

In this project, a new approach of image interpolation has been proposed to apply zooming of small segment in digital image using Lagrange interpolation method. The proposed approach is implemented by software which is designed in MATLAB environment. In this software unit, the same mathematical concepts of Lagrange polynomial are considered except that the computation of interpolation is repeated sequentially for each fixed length segment to overcome the problem of complex calculation that is caused by high order of Lagrange polynomial.

CHAPTER 1

INTRODUCTION

1.1 Background

Nowadays, interpolation technique has a large usage in the field of computer vision, digital photography, multimedia and electronic publishing for generating preview images. In image compression, digital zooming, computed tomography (CT), magnetic resonance imaging (MRI), image reconstruction requires interpolation to approximate the discrete data to get the enhanced image, hence there is a need of good interpolator scheme that can efficiently restore the signal and can help to reduce the cost of systems.

Interpolation is a method of constructing new data points from a discrete set which are being fitted in the continuous curves and then sampling at a higher rate interpolates the given data. In the early years, simple algorithms, such as nearest neighbor or linear interpolation, were used for sampling. After the introduction of sinc function a revolutionary idea was born in the field of interpolation because of its acceptance as a ideal interpolation function. However, this ideal interpolator has an infinite impulse response (IIR) and is not suitable for local interpolation with finite impulse response (FIR) [1].

1.2 Image Interpolation

Interpolation is the process of determining the values of a function at positions lying between its samples. It achieves this process by fitting a continuous function through the discrete input samples. This permits input values to be evaluated at arbitrary positions in the input, not just those defined at the sample points. While sampling generates an infinite bandwidth signal from one that is band limited, interpolation plays an opposite role: it reduces the bandwidth of a signal by applying a low-pass filter to the discrete signal. That is, interpolation reconstructs the signal lost in the sampling process by smoothing the data samples with an interpolation function.

The process of interpolation is one of the fundamental operations in image processing. The image quality highly depends on the used interpolation technique. The interpolation techniques are divided into two categories, deterministic and statistical interpolation techniques. The difference is that deterministic interpolation techniques assume certain variability between the sample points, such as linearity in case of linear interpolation. Statistical interpolation methods approximate the signal by minimizing the estimation error. This approximation process may result in original sample values not being replicated. Since statistical methods are computationally inefficient, in this article only deterministic techniques will be discussed. A comparison between methods will also be made. In this article, we are mainly concerned about image interpolation. However, interpolation in two dimensions for the general case is sometimes difficult to describe. For gridded data, the n-dimensional interpolation function can be described as the product of n one-dimensional interpolation functions. Therefore it is permitted to look at one dimensional interpolation functions to discuss the behavior of the n-dimensional interpolation functions [2].

1.3 Mathematical Techniques for Image Interpolation

A digital image is not an exact snapshot of reality, it is only a discrete approximation. This fact should be apparent to the average web surfer, as images commonly become blocky or jagged after being resized to
fit the browser. The goal of image interpolation is to produce acceptable images at different resolutions from a single low-resolution image. The actual resolution of an image is defined as the number of pixels, but the effective resolution is a much harder quantity to define as it depends on subjective human judgment and perception. The goal of this paper is to explore different mathematical formulations of this essentially aesthetic quantity [3].

The image interpolation problem goes by many names, depending on the application: image resizing, image up sampling/down sampling, digital zooming, image magnification, resolution enhancement, etc. The term super-resolution is sometimes used, although in the literature this generally refers to producing a high-resolution image from multiple images such as a video sequence. If we define interpolation as "filling in the pixels in between," the image interpolation problem can be viewed as a subset of the in painting problem (see Figure 1.1) [4].

The applications of image interpolation range from the commonplace viewing of online images to the more sophisticated magnification of satellite images. With the rise of consumer-based digital photography, users expect to have a greater control over their digital images. Digital zooming has a role in picking up clues and details in surveillance images and video. As high-definition television (HDTV) technology enters the marketplace, engineers are interested in fast interpolation algorithms for viewing traditional lowdefinition programs on HDTV. Astronomical images from rovers and probes are received at an extremely low transmission rate (about 40 bytes per second), making the transmission of high-resolution data infeasible [5].

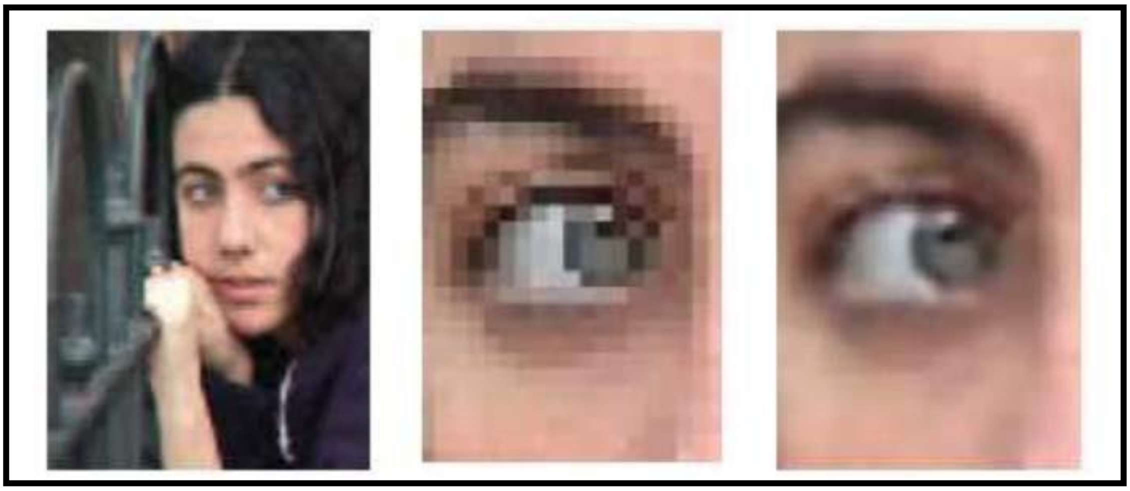


Figure 1.1: Image interpolation using Navier-Stokes in painting. Left: original image. Center: close-up of eye in image. Right: interpolated image.

In medical imaging, neurologists would like to have the ability to zoom in on specific parts of brain tomography images. This is just a short list of applications, but the wide variety cautions us that our desired interpolation result could vary depending on the application and user.

Interpolation methods differ in their mathematical description of a “good” interpolated image. Although, it is difficult to compare image interpolation methods and judge their output. The following eight criteria are visual properties of the interpolated image. Also, the last is a computational property of the interpolation method.

1- **Geometric Invariance**: The interpolation method should preserve the geometry and relative sizes of objects in an image. That is, the subject matter should not change under interpolation.

2- **Contrast Invariance**: The method should preserve the luminance values of objects in an image and the overall contrast of the image.

3- **Noise**: The method should not add noise or other artifacts to the image, such as ringing artifacts near the boundaries.

4- **Edge Preservation**: The method should preserve edges and boundaries, sharpening them where possible.

5- **Aliasing**: The method should not produce jagged or "staircase" edges.

6- **Texture Preservation**: The method should not blur or smooth textured regions.

7- **Over-smoothing**: The method should not produce undesirable piecewise constant or blocky regions.

8- **Application Awareness**: The method should produce results appropriate to the type of image and order of resolution. For example, the interpolated results should appear realistic for photographic images, but for medical images the results should have crisp edges and high contrast. If the interpolation is for general images, the method should be independent of the type of image.

9- **Sensitivity to Parameters**: The method should not be too sensitive to internal parameters that may vary from image to image.

1.4 Project Layout

This project includes:

Chapter One: presents general introduction of image interpolation and its methods. In addition, all mathematical techniques for these methods are mentioned in this chapter.

Chapter Two presents the basic concepts of Lagrange polynomial. Also, some examples of applying this interpolation method are presented in this chapter.

Chapter Three presents the general block diagram of proposed image interpolation approach.

Chapter Four presents the results of implementing proposed image interpolation approach on some digital image.

CHAPTER 2

LAGRANGE INTERPOLATION

2.1 Introduction

In general, interpolation is the process of estimating the intermediate values of a continuous event from discrete samples Interpolation is used extensively in digital image processing to magnify or reduce images and to correct spatial distortions. Because of the amount of data associated with digital images. An efficient interpolation algorithm is essential. Cubic convolution Interpolation was developed in response to this requirement. An interpolation function is a special type of approximating function. a fundamental property of interpolation functions is that they must coincide with the sampled data at the Interpolation, nodes, or sample points.

2.2 Lagrange Interpolation

In numerical analysis, Lagrange polynomials are used for polynomial interpolation. For a given set of distinct points x_j and numbers y_j, the Lagrange polynomial is the polynomial of the least degree that at each point x_j assumes the corresponding value y_j (i.e. the functions coincide at each point). The interpolating polynomial of the least degree is unique, however, and it is therefore more appropriate to speak of "the Lagrange form" of that unique polynomial rather than "the Lagrange interpolation polynomial", since the same polynomial can be arrived at through multiple methods. Although named after Joseph Louis Lagrange, who published it in 1795, it was first discovered in 1779 by Edward Waring and it is also an easy consequence of a formula published in 1783 by Leonhard Euler [6].

Lagrange interpolation is susceptible to Runge's phenomenon, and the fact that changing the interpolation points requires recalculating the entire interpolation can make Newton polynomials easier to use. Lagrange polynomials are used in the Newton–Cotes method of numerical integration and in Shamir's secret sharing scheme in cryptography.

2.2.1 Lagrange Definition

Given a set of k + 1 data points:

(Xo,Yo), …. , (Xj, Yj), … , (Xk, Yk)

Where no two are the same, the interpolation polynomial in the Lagrange form is a linear combination of Lagrange basis polynomials

$$L(x) := \sum_{j=0}^{k} y_j l_j(x)$$

$$l_j(x) := \prod_{\substack{0<m<k \\ m\neq j}} \frac{x-x_m}{x_j-x_m} = \frac{x-x_0}{x_j-x_0} \cdots \frac{x-x_{j-1}}{x_j-x_{j-1}} \frac{x-x_{j+1}}{x_j-x_{j+1}} \cdots \frac{x-x_k}{x_j-x_k}$$

Where $0 < j < k$. Note how, given the initial assumption that no two xj are the same xj – xm ≠ 0 , so this expression is always well-defined. The reason pairs xi=xj with yi ≠ yj are not allowed is that no interpolation function L such that yi=L(xi) would exist; a function can only get one value for each argument xi . On the other hand, if also yi=yj, then those two points would actually be one single point. For all i ≠ j, Lj includes the term (x-xi) in the numerator, so the whole product will be zero at x=xi:

$$l_{j\neq i}(x_i) := \prod_{\substack{0<m<k\\ m\neq j}} \frac{x_i - x_m}{x_j - x_m} = \frac{x_i - x_0}{x_j - x_0} \cdots \frac{x_i - x_i}{x_j - x_i} \cdots \frac{x_i - x_k}{x_j - x_k} = 0$$

On the other hand

$$l_{j\neq i}(x_i) := \prod_{m\neq i} \frac{x_i - x_m}{x_j - x_m} = 1$$

In other words, all basis polynomials are zero at x=xi, except , for Li(x) which it holds that Li(x)=1, because it lacks the (x-xi) term.

It follows that yi Li(xi)=yi, so at each point xi, Li(xi)=yi+0+0+….+0=yi, showing that L interpolates the function exactly.

2.2.2 Main Idea

Solving an interpolation problem leads to a problem in linear algebra amounting to inversion of a matrix. Using a standard monomial basis for our interpolation polynomial $L(x) = \sum j{=}0^k\ x^j\ m_j$, we must invert the Vander monde matrix $(xi)^j$ to solve L(xi) = yi for the coefficients mj of L(x). By choosing a better basis, the Lagrange basis, $L(x) = \sum j{=}0^k$ lj(x) yj, we merely get the identity matrix, δij, which is its own inverse: the Lagrange basis automatically inverts the analog of the Vander monde matrix.

This construction is analogous to the Chinese Remainder Theorem. Instead of checking for remainders of integers modulo prime numbers, we are checking for remainders of polynomials when divided by linear.

Example 1

We wish to interpolate $f(x) = x^2$ over the range $1 \le x \le 3$, given these three points:

$$\begin{aligned} x_0 &= 1 & f(x_0) &= 1 \\ x_1 &= 2 & f(x_1) &= 4 \\ x_2 &= 3 & f(x_2) &= 9. \end{aligned}$$

The interpolating polynomial is:

$$\begin{aligned} L(x) &= 1 \cdot \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} + 4 \cdot \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} + 9 \cdot \frac{x-1}{3-1} \cdot \frac{x-2}{3-2} \\ &= x^2. \end{aligned}$$

Example 2

We wish to interpolate *ƒ*(x) = x3 over the range 1 ≤ x ≤ 3, given these three points:

$$x_0 = 1 \quad f(x_0) = 1$$

$$x_1 = 2 \quad f(x_1) = 8$$

$$x_2 = 3 \quad f(x_2) = 27$$

The interpolating polynomial is:

$$L(x) = 1 \cdot \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} + 8 \cdot \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} + 27 \cdot \frac{x-1}{3-1} \cdot \frac{x-2}{3-2}$$

$$= 6x^2 - 11x + 6.$$

2.2.3 Linear Characteristics of Lagrange Interpolation

The Lagrange form of the interpolation polynomial shows the linear character of polynomial interpolation and the uniqueness of the interpolation polynomial. Therefore, it is preferred in proofs and theoretical arguments. Uniqueness can also be seen from the invariability of the Vander monde matrix, due to the non-vanishing of the Vander monde determinant. But, as can be seen from the construction, each time a node xk changes, all Lagrange basis polynomials have to be recalculated. A better form of the interpolation polynomial for practical (or computational) purposes is the barycentric form of the Lagrange interpolation (see below) or Newton polynomials. Lagrange and other interpolation at equally spaced points, as in the example above, yield a polynomial oscillating above and below the true function. This behavior tends to grow with the number of points, leading to a divergence known as Runge's phenomenon; the problem may be eliminated by choosing interpolation points at Chebyshev nodes [7].

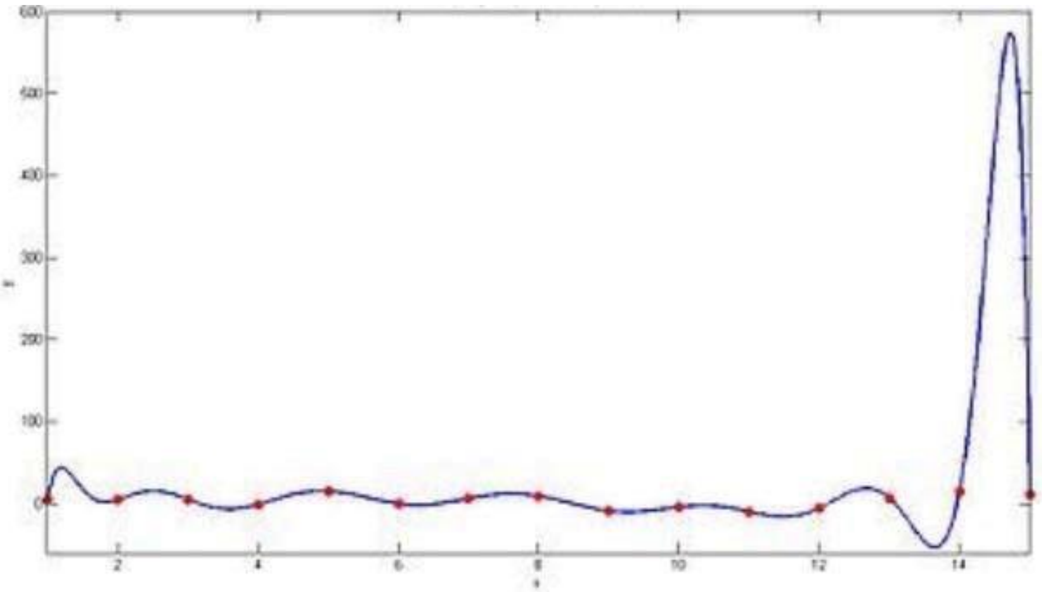


Figure 2-1: Example of Lagrange polynomial interpolation divergence.

CHAPTER 3

PROPOSED METHODOLOGY

3.1 Introduction

As it is known, the Lagrange polynomial method is used originally to compute the interpolation of multi point with XY coordinate. While, the main idea of this project is to apply this method of interpolation on digital image for zooming purposes. Thus some changes must be done on the Lagrange method to apply it on digital image. A new approach of image interpolation has been proposed in this chapter using Lagrange method.

3.2 Proposed Image Interpolation Approach

The new approach presents a facility of zooming in any small segment inside digital image with different increasing factor. The small segment inside in located by two points (Xmin, Ymin) and (Xmax, Ymax) as marked by the red rectangle in Figure 3.1.

The proposed interpolation approach is applied on selected segment to compute the new enlarged segment after applying Lagrange interpolation on all pixels which are limited inside original segment. The proposed image interpolation is applied through many steps, which are discussed in more detail in following text.

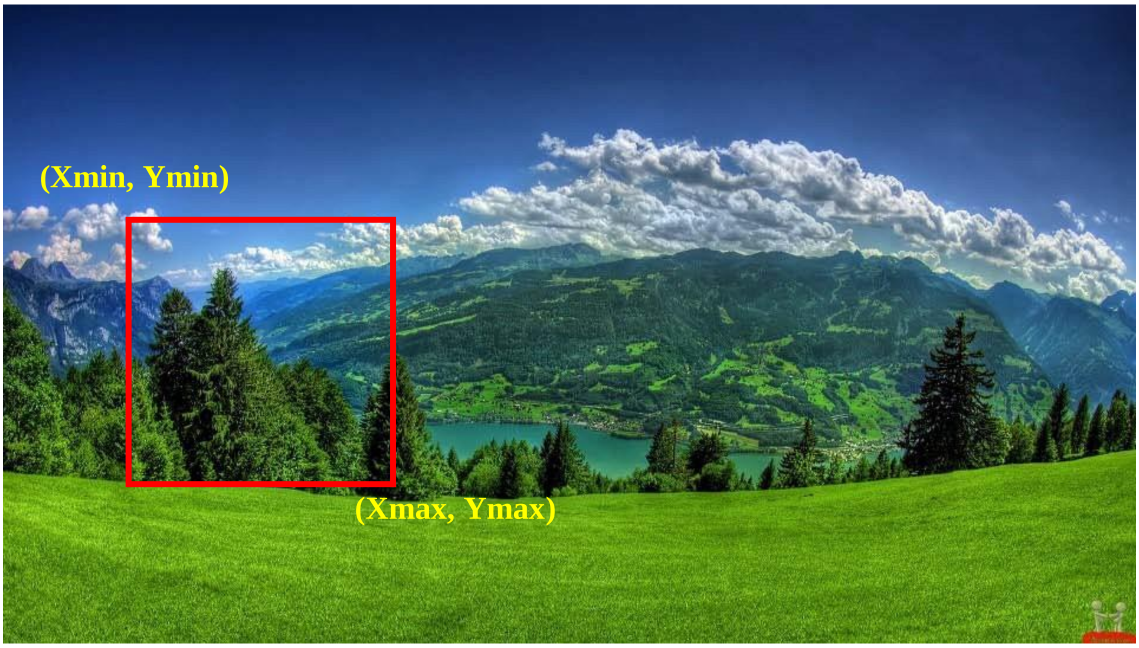


Figure 3.1: Assigning XY coordinates of Zooming Segment inside digital Image.

3.2.1 Segmentation of Image Segment

The selected image segment is divided into small interval with same size in rows and columns. Thus, the implementation of interpolation is applied separately on each small segment. The main idea of segmentation comes from deceasing set of data that are computed per each interpolation cycle, thus the overall execution time of interpolation process is decreased.

3.2.2 First Cycle of Applying Lagrange Interpolation

On each small segment resulted from previous step, the Lagrange interpolation is applied sequentially for all rows. The result from this cycle of interpolation generates enlarge segment in horizontal direction.

3.2.3 Second Cycle of Applying Lagrange Interpolation

The resulted image segment from the previous step is transposed, and then the second cycle of Lagrange interpolation is applied to generate final enlarge segment with a scaling factor that is defined previously (X2, X4, etc).

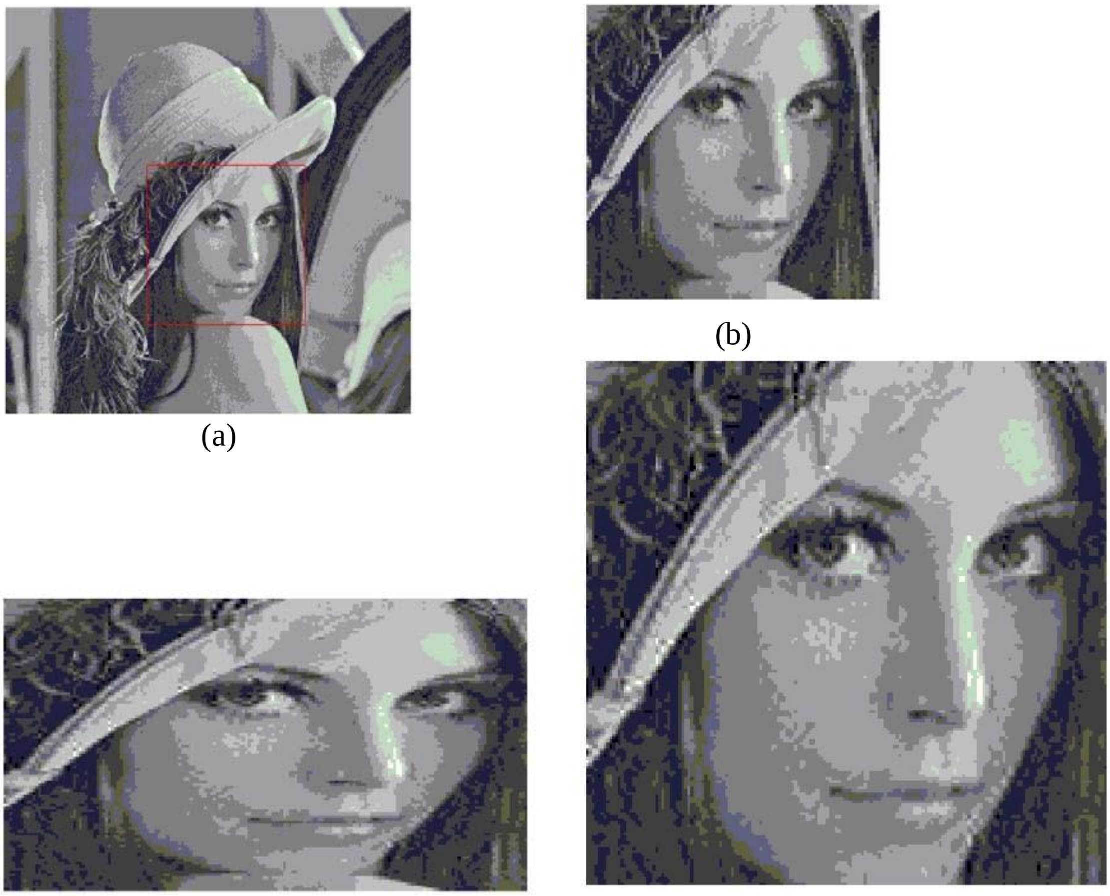
3.3 Introduction

In this chapter, results of applying the proposed image interpolation approach are presented. The proposed approach is applied on three different images with grayscale color. In each case, the input image, scaling factor, and size of segment are selected different to improve the capability of the proposed approach to make interpolation for general cases.

The implantation results for the three cases are obtained by MATLAB program which is written by the team work of this project to simulate the implementation of the proposed image interpolation approach according the steps that are discussed in previous chapter.

3.4 Interpolation Results of First Case

In this implementation, the input image is (LENA) of size (256x256) pixels in grayscale color mode. The size of segment is (100 x 100) pixels and scaling factor is 200% (X2). The simulation results of these implementation are shown in Figure 4.1.a-d.

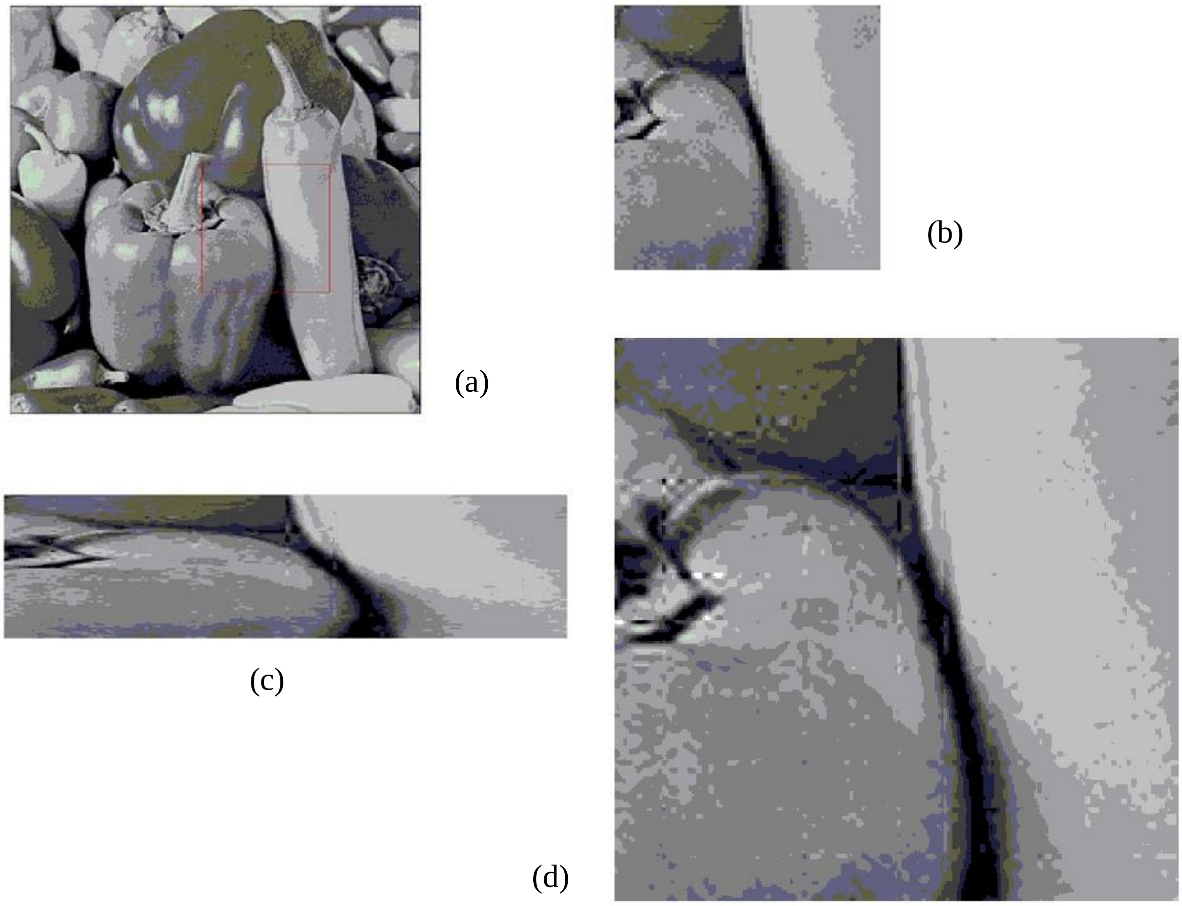


(a) (b) (c) (d)

Figure 4-1: Implementation results of applying Proposed interpolation approach on LENA Image, (a) Original Image. (b) Selected Segment, (c) Horizontal Interpolation X2, (d) Final Interpolation X2.

3.5 Interpolation Results of Second Case

In this implementation, the input image is (FRUIT) of size (256x256) pixels in grayscale color mode. The size of segment is (80 x 80) pixels and scaling factor is 400% (X4). The simulation results of these implementation are shown in Figure 4.2.a-d.

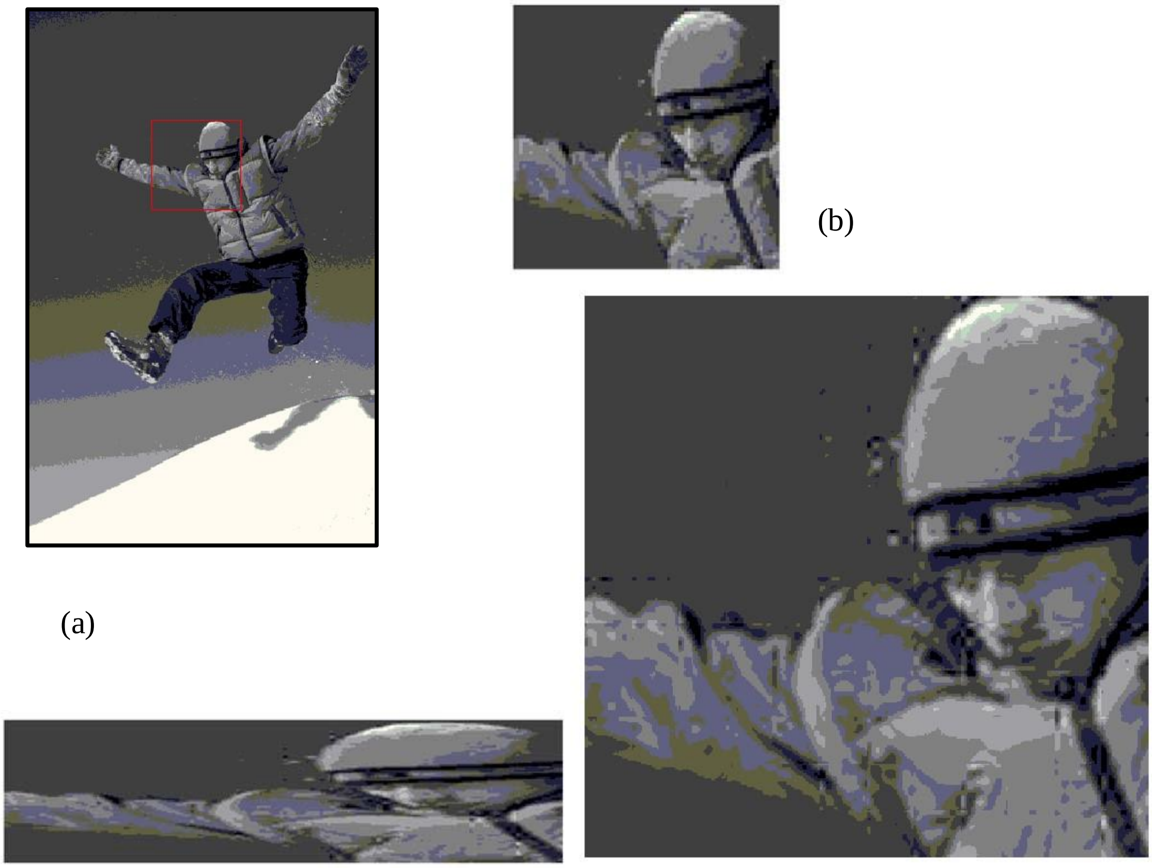


(a) (b) (c) (d)

Figure 4-2: Implementation results of applying proposed interpolation approach on FRUIT Image, (a) Original Image. (b) Selected Segment, (c) Horizontal Interpolation X4, (d) Final Interpolation X4.

3.6 Interpolation Results of Third Case

In this implementation, the input image is (JUMPMAN) of size (482 x 321) pixels in grayscale color mode. The size of segment is (100 x 100) pixels and scaling factor is 400% (X4). The simulation results of these implementation are shown in Figure 4.3.a-d.



(a) (b) (c) (d)

Figure 4.3: Implementation results of applying proposed interpolation approach on JUMPMAN Image, (a) Original Image. (b) Selected Segment, (c) Horizontal Interpolation X4, (d) Final Interpolation X4.

CHAPTER 4

CONCLUSION & FUTURE WORKS

4.1 Introduction

In this chapter, the results obtained by the proposed approach of interpolation are concluded. Also, in the end of this chapter, some important jobs which are related to the main idea of image interpolation are suggested as the future works.

4.2 Conclusion

As mentioned in the previous chapter, the proposed approach of image interpolation is tested by three different cases. In each case, different image, image slice, and scaling factor is selected to prove the capability of the proposed image interpolation approach to apply interpolation for any digital image.

The zooming segment in three cases take smooth view and all points are considered without missing any one. According the results obtained, the proposed approach success to apply interpolation for zooming out. Also these results take the smoothed view and not need for any further process.

4.3 Future works

According the implementation of the proposed image interpolation approach, some jobs are suggested to implement them as the future works as follows:

1- Apply same proposed image interpolation approach using Lagrange technique for color image.

2- Make a comparison study including some standard metrics for interpolation to validate the proposed approach with another image interpolation approach.

3- Evaluate performance of the proposed image interpolation approach for zooming out high resolution digital image like medical image, microscopic image, etc,

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