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Unit Operation Branch.
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Typical Question and It's Solutions

Q1- Solve the 2nd order equation: $x^2 y'' - xy' - 3y = x^2 \ln x$.

Solⁿ: $[x^2 y'' - xy' - 3y = \ x^2 \ln x\ ]/x$

$$xy'' - y' - 3\frac{y}{x} = x\ \ln x\ -------------*$$

$$let\ \ v = \frac{y}{x}\ \ \Rightarrow y = v.x\,,\ \ \Rightarrow\ \ \frac{dy}{dx} = v + x\frac{dv}{dx}\ ,$$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\ ,$$

$$\therefore x\frac{d^2y}{dx^2} = 2x\frac{dv}{dx} + x^2\frac{d^2v}{dx^2}\ ,$$

Now substitute in eq. *

$$x^2\frac{d^2v}{dx^2} + 2x\frac{dv}{dx} - x\frac{dv}{dx} - v - 3v = x\ln x\ ,$$

$$x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} - 4v = x\ln x\ ---------**$$

Now let $t = \ln x$, $\therefore x = e^t$, $\frac{dt}{dx} = \frac{1}{x}$,& $\frac{d^2t}{dx^2} = -\frac{1}{x^2}$

$$Now\ \frac{dv}{dx} = \frac{dv}{dt}\ .\frac{dt}{dx} = \frac{dv}{dt}.\frac{1}{x}\,,\ \ \frac{d^2v}{dx^2} = -\frac{1}{x^2}\ \frac{dv}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dv}{dt}\right)$$

$$= -\frac{1}{x^2}\ \frac{dv}{dt} + \frac{1}{x}\frac{d}{dt}\frac{dv}{dt}\frac{dt}{dx} = \frac{1}{x^2}\frac{d^2v}{dt^2} - \frac{1}{x^2}\frac{dv}{dt}$$

*Now substitute in eq.***

Typical Question and It's Solutions

$\frac{d^2v}{dt^2} - \frac{dv}{dt} + \frac{dv}{dt} - 4v = te^t \ , \Rightarrow \frac{d^2v}{dt^2} - 4v = te^t , \Rightarrow \left(\frac{d^2}{dt^2} - 4\right) v = te^t$

$(D^2 - 4)v = te^t$,

$Now\ solve\ for\ v_c\ , (D^2 - 4)v_c = 0\ ,\ (D-2)(D+2)v_c = 0$

$\therefore\ r_1 = 2\ \&\ r_2 = -2\ ,\ \Rightarrow v_c = Ae^{2t} + Be^{-2t}$

$Now\ solve\ for\ v_p\ ,\ \ let\ v_p = (\alpha_0 + \alpha_1 t)e^t$,

$v'_p = (\alpha_0 + \alpha_1 t)e^t + \alpha_1 e^t$

$v''_p = (\alpha_0 + \alpha_1 t)e^t + \alpha_1 e^t + \alpha_1 e^t,\ now\ substitut\ for\ v''_p$

$(\alpha_0 + \alpha_1 t)e^t + 2\alpha_1 e^t - 4(\alpha_0 + \alpha_1 t)e^t = te^t$

$2\alpha_1 e^t - 3\alpha_0 e^t - 3\alpha_1 (t) e^t = te^t$

$2\alpha_1 - 3\alpha_0 = 0\ ,\ -3\alpha_1 = 1\ ,\ \ \alpha_1 = -\frac{1}{3}\ , \alpha_0 = -\frac{2}{9}$

$\therefore\ v_p = \left(-\frac{2}{9} - \frac{1}{3}t\right) e^t$,

$\therefore v = Ae^{2t} + Be^{-2t} - \frac{2}{9}e^t - \frac{1}{3}te^t$

$\frac{y}{x} = Ax^2 + Bx^{-2} - \frac{2}{9}x - \frac{1}{3}x \ln x$

$y = Ax^3 + Bx^{-1} - \frac{2}{9}x^2 - \frac{1}{3}x^2 \ln x$

Typical Question and It's Solutions

Q2: $Solve\ the\ 2nd\ order\ differential\ equation$:

$y'' - 2y' + y = x^3 e^x$.

Sol^{n}: $solve\ for\ y_c$,

$(D^2 - 2D + 1)y_c = 0$, $\therefore (D-1)(D-1) = 0$, $\therefore\ r_1 = r_2 = 1$

$y_c = (Ax + B)e^x$

$Now\ solve\ for\ y_p$

$let\ y_p = (\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)e^x$, $But\ \ \alpha_0 e^x \& \alpha_1 x e^x\ exist\ in\ y_c$

$so\ \ y_p = (\alpha_0 x^2 + \alpha_1 x^3 + \alpha_2 x^4 + \alpha_3 x^5)e^x$

$y'_p = (\alpha_0 x^2 + \alpha_1 x^3 + \alpha_2 x^4 + \alpha_3 x^5)e^x + (2\alpha_0 x^1 + 3\alpha_1 x^2 + 4\alpha_2 x^3 + 5\alpha_3 x^4)e^x$

$y''_p = (\alpha_0 x^2 + \alpha_1 x^3 + \alpha_2 x^4 + \alpha_3 x^5)e^x + (2\alpha_0 x + 3\alpha_1 x^2 + 4\alpha_2 x^3 +$
5α3x4ex+2α0x+3α1x2+4α2x3+5α3x4ex+(2α0+6α1x+12α2x2+
$20\alpha_3 x^3)e^x$

$\therefore (2\alpha_0 + 6\alpha_1 x + 12\alpha_2 x^2 + 20\alpha_3 x^3)e^x = x^3 e^x$

$20\alpha_3 = 1 \rightarrow \alpha_3 = \frac{1}{20}$, $\therefore\ \ y_p = \frac{1}{20}\ x^5 e^x$

$\therefore y = y_c + y_p$, $y = (Ax + B + \frac{1}{20} x^3)e^x$

Typical Question and It's Solutions

Q3:Solve the 2nd order differential equation: $x^2y'' + xy' = 1$.

$Sol^n: x^2y'' + xy' = 1\,]\frac{1}{x^2}$, $\therefore y'' + \frac{1}{x}y' = \frac{1}{x^2}$, $let\ \ y' = p\ \&\ y'' = \frac{dp}{dx}$

$\frac{dp}{dx} + \frac{1}{x}p = \frac{1}{x^2}$, $linear\ diffe.eq.\ \ I = e^{\int p dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$x.p = \int Q\,.I\,dx = \int \frac{1}{x^2}\,x\,dx = \ln x + c$

$\therefore p = \frac{\ln x}{x} + \frac{c}{x}$

$\because\ \frac{dy}{dx} = p$, $\therefore \frac{dy}{dx} = \frac{\ln x}{x} + \frac{c}{x}$, $\therefore \int dy = \int \frac{\ln x}{x}dx + \int \frac{c}{x}dx$

$y = \frac{(\ln x)^2}{2} + c\ln x + c_1$

Q4:Solve the 2nd order differential equation: $y'' - 2y' + 2y = e^x \cos x$

$Sol^n: (D^2 - 2D + 2)y_c = 0$, $\therefore D = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$D = \frac{2\pm\sqrt{2^2-8}}{2} = 1 \pm i$, $\therefore\ \alpha = 1\ \ \&\ \beta = 1$

$\therefore\ y_c = e^x(A\cos x + B\sin x)$

$Now\ solve\ for\ y_p$, $let\ \ y_p = e^x(D\cos x + E\sin x), but\ it\ is\ exist$

$so\ \ y_p = e^x(D\,\mathrm{x}\cos x + E\,\mathrm{x}\sin x)$

$y'_p = e^x(D\,\mathrm{x}\cos x + E\,\mathrm{x}\sin x) + (D\sin x + D\,x\cos x + E\cos x -$
$E\,x\sin x)e^x$

Typical Question and It's Solutions

$y''_p = e^x(D \text{ x} \cos x + E \text{ x} \sin x) + (D \sin x + D\, x \cos x + E \cos x -$
$E\, x \sin x e x + D \sin x + D\, x \cos x + E \cos x - E\, x \sin x e x + (D \cos x + D \cos x - D$
$x \sin x - E \sin x - E \sin x - E\, x \cos x)e^x$

$now\ \ subsititute\ for\ \ y_p\ \&\, y'_p\ \&\, y''_p\ \ in\ main\ eq.to\ get:$

$(2D \cos x - 2E \sin x)e^x\ = e^x \cos x\ \ , \to 2D = 1 \to D = \frac{1}{2}\ , \&\, E = 0$

$\therefore\ y_p = \frac{1}{2}\, x \sin x\ e^x\ \ ,\quad \therefore y = (A \cos x + B \sin x + \frac{1}{2} x \sin x)e^x$

Q5:Find the Fourier series and the sine and cosine half range expansion of each one of the following function:-

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$Sol^{n}: period: 2p = 4 \to p = 2$

$a_0 = \frac{1}{2p} \int f(x)dx = \frac{1}{4}[\int_{-2}^{-1} 0dx + \int_{-1}^{0} dx + \int_{0}^{1} -dx + \int_{1}^{2} 0dx$

$a_0 = \frac{1}{4}[x|_{-1}^{0} - x|_{0}^{1}] = \frac{1}{4}[0 - (-1) - 1] = 0$

$a_n = \frac{1}{p} \int f(x) \cos \frac{n\pi x}{p}\ dx = \frac{1}{2}[\int_{-2}^{-1} 0 \cos \frac{n\pi x}{2}\ dx + \int_{-1}^{0} \cos \frac{n\pi x}{2}\ dx +$
$\int_{0}^{1} - \cos \frac{n\pi x}{2}\ dx + \int_{1}^{2} 0 \cos \frac{n\pi x}{2}\ dx$

$a_n = \frac{1}{n\pi}\left[\sin \frac{n\pi x}{2}\ |_{-1}^{0} - \sin \frac{n\pi x}{2} |_{0}^{1}\right] = [0 - \sin -\frac{n\pi}{2} - \sin \frac{n\pi}{2} + 0]$

Typical Question and It's Solutions

$$for\ \ n = \begin{Bmatrix} odd & \sin -\frac{n\pi}{2} = -1 \& \sin\frac{n\pi}{2} = 1 \\ even & \sin -n\pi = 0 \& \sin n\pi = 0 \end{Bmatrix}\ so\ \ a_n = 0$$

$$b_n = \frac{1}{p}\int f(x)\sin\frac{n\pi x}{p}\ dx = \frac{1}{2}[\int_{-2}^{-1} 0\sin\frac{n\pi x}{2}\ dx + \int_{-1}^{0}\sin\frac{n\pi x}{2}\ dx + \int_{0}^{1} -\sin\frac{n\pi x}{2}\ dx + \int_{1}^{2} 0\sin\frac{n\pi x}{2}\ dx$$

$$b_n = -\frac{1}{n\pi}[\cos\frac{n\pi x}{2}]_{-1}^{0} + \frac{1}{n\pi}[\cos\frac{n\pi x}{2}]_{0}^{1}$$

$$b_n = \frac{1}{n\pi}\left[-\cos 0 + \cos -\frac{n\pi}{2} + \cos\frac{n\pi}{2} - \cos 0\right] = \frac{1}{n\pi}[-2 + 2\cos\frac{n\pi}{2}]$$

$$b_n = \frac{2}{n\pi}\left[-1 + \cos\frac{n\pi}{2}\right] = \begin{Bmatrix} -\frac{2}{n\pi} & for\ n\ \ odd \\ 0 & n\ \ even\ ,\ \ 2,4,8,\ldots \\ -\frac{4}{n\pi} & n\ \ even\ , 6,10,14,\ldots \end{Bmatrix}$$

$$\because f(x) = a_0 + \sum_{n=1}^{n=\infty}(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p})$$

$$\therefore f(x) = -\frac{2}{n\pi}[\sin\frac{\pi x}{2} + 0 + \sin\frac{3\pi x}{2} + 0 + \sin\frac{5\pi x}{2} + 2\sin 3\pi x + \cdots$$

Q6: Find the Fourier series and the sine and cosine half range expansion of each one of the following function:-

$$f(x) = \left\{\cos x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$$

$$period:\ 2p = \pi \rightarrow p = \pi/2$$

$$a_0 = \frac{1}{2p}\int f(x)dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\, dx = \frac{1}{\pi}[\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi}[\sin\frac{\pi}{2} - \sin -\frac{\pi}{2}]$$

Typical Question and It's Solutions

$$and \because \sin -\frac{\pi}{2} = -\sin\frac{\pi}{2}\ ,\ \therefore a_0 = \frac{1}{\pi}[\sin\frac{\pi}{2} + \sin\frac{\pi}{2}] = \frac{2}{\pi}$$

$$a_n = \frac{1}{p}\int f(x)\cos\frac{n\pi x}{p}\ dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ \cos\frac{n\pi x}{\frac{\pi}{2}}\ dx$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos x \cos 2nx\, dx$$

$$\because \cos x \cos 2nx = \frac{1}{2}[\cos(1-2n)\, x + \cos(1+2n)x]$$

$$\therefore \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \cos x \cos 2nx\, dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}[\cos(1-2n)x \cos(1+2n)x]\, dx$$

$$a_n = \frac{1}{\pi}[\frac{\sin(1-2n)x}{1-2n} + \frac{\sin(1+2n)x}{1+2n}]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\pi}[\frac{-2\sin\frac{n\pi}{2}}{1-2n} + \frac{2\sin\frac{n\pi}{2}}{1+2n}]$$

$$\because \sin(A-B) = \sin A\cos B - \cos A \sin B$$

$$and\ \sin(A+B) = \sin A\cos B + \cos A \sin B$$

$$\therefore \sin(\frac{\pi}{2} - n\pi) = \sin\frac{\pi}{2}\cos n\pi - \cos\frac{\pi}{2}\sin n\pi = \cos n\pi$$

$$and\ \ \sin(\frac{\pi}{2} + n\pi) = \sin\frac{\pi}{2}\cos n\pi + \cos\frac{\pi}{2}\sin n\pi = \cos n\pi$$

$$\sin(-\frac{\pi}{2} + n\pi) = \sin(-\frac{\pi}{2})\cos n\pi + \cos(-\frac{\pi}{2})\sin n\pi = -\sin\frac{\pi}{2}\cos n\pi$$

$$= -\cos n\pi\ \ , \&\ \sin(-\frac{\pi}{2} - n\pi) = -\cos n\pi$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos n\pi}{1-2n} + \frac{\cos n\pi}{1+2n}\right] - \frac{1}{\pi}\left[\frac{-\cos n\pi}{1-2n} + \frac{-\cos n\pi}{1+2n}\right] =$$

$$a_n = \frac{2\cos n\pi}{\pi}\left[\frac{1}{1-2n} + \frac{1}{1+2n}\right] = \frac{2\cos n\pi}{\pi}[\frac{1+2n+1-2n}{1-4n^2}]$$

Typical Question and It's Solutions

$$a_n = \frac{4\cos n\pi}{\pi(1-4n^2)}, \cos n\pi = \begin{Bmatrix} -1 & n\ odd \\ 1 & n\ even \end{Bmatrix}, a_n = \begin{Bmatrix} \frac{-4}{\pi(1-4n^2)} & n\ odd \\ \frac{4}{\pi(1-4n^2)} & n\ even \end{Bmatrix}$$

$$a_n\ ;\ \ a_1 = \frac{4}{3\pi}\ , a_2 = \frac{-4}{15\pi}\ , a_3 = \frac{4}{35\pi}\ , \dots\dots$$

$$b_n = \frac{1}{p}\int f(x)\sin\frac{n\pi x}{p}\ dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin\frac{n\pi x}{\frac{\pi}{2}} dx =$$

$$\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin 2nx\ dx$$

$$b_n = \frac{1}{p}\int f(x)\sin\frac{n\pi x}{p}\ dx = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin\frac{n\pi x}{\frac{\pi}{2}} dx$$

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin 2nx\ dx\ ,$$

$$and\ \ \because \sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]$$

$$\therefore\ \sin mx \cos nx = \frac{1}{2}[\sin(2n-1)x + \sin(2n+1)x]$$

$$\therefore b_n = \frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\sin(2n-1)x + \sin(2n+1)x]dx$$

$$b_n = -\frac{1}{\pi}\left[\frac{\cos(2n-1)x}{2n-1} + \frac{\cos(2n+1)x}{2n+1}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\cos(A-B) = \cos A\cos B + \sin A \sin B$$

$$\therefore \cos(n\pi - \tfrac{\pi}{2}) = \cos n\pi \cos\frac{\pi}{2} + \sin n\pi \sin\frac{\pi}{2} = 0$$

Typical Question and It's Solutions

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\therefore \cos(n\pi + \frac{\pi}{2}) = \cos n\pi \cos\frac{\pi}{2} - \sin n\pi \sin\frac{\pi}{2} = 0$

$and\ so \cos(-n\pi + \frac{\pi}{2}) = 0 , \& \cos(-n\pi - \frac{\pi}{2}) = 0$

$\therefore\ b_n = 0$

$\because f(x) = a_0 + \sum_{n=1}^{n=\infty}(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p})$

$\therefore f(x) = \frac{2}{\pi} + \frac{4}{\pi}[\frac{\cos\ 2x}{3} - \frac{\cos\ 4x}{15} + \frac{\cos\ 6x}{35} - \frac{\cos\ 8x}{63} + \cdots$

Q7:Evaluate the following integrals:-

1- $I = \int_0^\infty e^{-\sqrt[3]{x}}\, dx$

$Sol^n: let\ \sqrt[3]{x} = t , \rightarrow x = t^3 , and\ \ dx = 3t^2 dt,$

$\therefore I = \int_0^\infty e^{-t}\, 3t^2 dt = 3\int_0^\infty t^2\, e^{-t} dt = 3\int_0^\infty t^{(3-1)}\, e^{-t} dt$

$I = 3\Gamma 3 = 3 \times 2 \times 1 = 6$

2- $I = \int_0^\infty (x+1)^2\, e^{-x^3} dx$

$Sol^n: let\ x^3 = t , \rightarrow x = t^{\frac{1}{3}},\ \ dx = \frac{1}{3} t^{-2/3} dt$

$I = \int_0^\infty \left(t^{\frac{1}{3}} + 1\right)^2 e^{-t}\ .\frac{1}{3} t^{-\frac{2}{3}} dt$

$I = \frac{1}{3}\int_0^\infty (t^{2/3} + 2t^{1/3} + 1) e^{-t} t^{-2/3} dt$

$I = \frac{1}{3}\int_0^\infty e^{-t}\, dt + \frac{2}{3}\int_0^\infty t^{-1/3}\, e^{-t} dt + \frac{1}{3}\int_0^\infty t^{-2/3} e^{-t}\, dt$

$$I = \frac{1}{3} + \frac{2}{3}\int_0^\infty t^{(\frac{2}{3}-1)} e^{-t}\, dt + \frac{1}{3}\int_0^\infty t^{(\frac{1}{3}-1)} e^{-t}\, dt$$

$$I = \frac{1}{3} + \frac{2}{3}\, \Gamma\left(\frac{2}{3}\right) + \frac{1}{3}\, \Gamma\,(\frac{1}{3})$$