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On The Time Lag of A First-Order Process's Sinusoidal Response

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Abstract

The time lag (TL) of a linear first-order process's sinusoidal response has been analyzed for a process time constant (τ) range of 5 to 10^3 s and a radian frequency (ω) range of $1x10^{-4}$ to 50 rad/s. The analysis showed that at $\omega \leq 0.1/\tau$, TL has a maximum value equal to τ . As ω increases , TL decreases such that over the range $3 \leq \omega \leq 50$ rad/s , TL = 1.5456/ω ; being independent of τ . At $\omega \geq 50$ rad/sec , TL $\cong$ 0 ; i.e. the response is in phase with the sinusoidal forcing function (SFF) but with a non-existent amplitude. The obtained results are contrary to what had/has been reported in some relevant textbooks in which the response's out-of-phase condition , with respect to the SFF , is shown to be minimum at very low ω values , increasing to a maximum as $\omega \to \infty$.

Key words: Process dynamics; Frequency response; Time lag; Time constant.

1. **Introduction**

Textbooks on process dynamics and control abound with illustrations on the phase shift (φ) associated with the response of a linear 1st-order process to a sinusoidal forcing function (SFF). However, little is mentioned about the response's time lag (TL) and nothing about its significance. Furthermore, in some texts, φ is erroneously illustrated in lieu of TL on the time axis of graphical representation of SFFs and their corresponding responses, giving the wrong impression that as ω increases the response's time lag also increases up to a φ value of $\pi/2$ radians. References [1] to [6] are well-known textbooks in this field, published over a time span of more than four decades (1964-2011); cited here as an example that this is the case.

2. Time Lag

The steady state response of a linear 1st-order process to a SFF of the form:

$$SFF = A \sin(\omega t) \qquad (1) \quad ,$$

is given by:

$$Y(t) = \frac{AK}{\sqrt{\omega^2\tau^2+1}} \sin(\omega t - \varphi) \qquad (2)$$

This may be expressed as,

$$Y(t) = \frac{AK}{\sqrt{\omega^2\tau^2+1}} \sin \omega \left(t - \frac{\varphi}{\omega}\right) \qquad (3)$$

where (φ / ω) is the time lag (also called time delay or time shift). As $\omega \to \infty$, φ (being $\tan^{-1}\omega\tau$) increases to a limit of $\pi/2$ radians; however TL decreases to a limit of zero. In other words, increasing ω brings the response closer to the forcing function (on a time scale), albeit with a reduced amplitude.

Fig. 1 shows the variation of TL with ω over the range $1x10^{-4} \leq \omega \leq 50$ rad/s for the practically significant process time constant range of $5 \leq \tau \leq 10^3$ s.

Three aspects of Fig. 1 are noteworthy. The first is that for any value of τ there is an ω value below which TL is simply the process's τ. Table 1 gives a sample of such values.

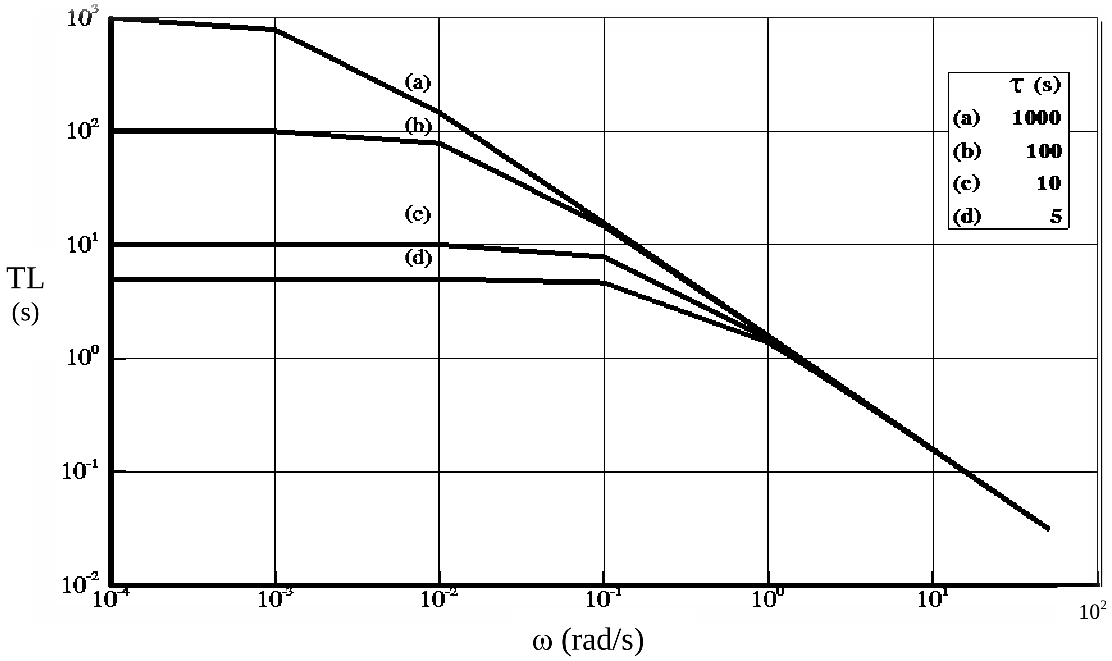

Fig. 1 . Variation of TL with ω for τ values of 5, 10, 100, and 10^3 s

Table 1 ω values at/below which TL = τ with corresponding τ value

ω [rad/sec]	1×10^{-5}	1×10^{-4}	1×10^{-3}	1×10^{-2}	0.1
τ [sec]	1×10^{4}	1×10^{3}	1×10^{2}	10	1

Or , generally for $\omega \le 0.1/\tau$, TL = τ .

This fact can be mathematically proven as follows :

Since TL = φ/ω = $(\tan^{-1}\omega\tau)/\omega$ then for ω = 0.1/τ , TL becomes τ [$(\tan^{-1} 0.1)/0.1$] = τ . Values of ω smaller than 0.1/τ (say x/τ) will also lead to this result because the expression for TL will always be τ [$(\tan^{-1}x)/x$] in which the numerator is an angle and the denominator is its tangent which are virtually equal for small angles in the radian measure.

The second aspect of Fig.1 is that over the range $3 \le \omega \le 50$ rad/s, TL is virtually a function of ω only; being independent of τ. Fig.2 enlarges the range $1 \le \omega \le 10$ rad/s of Fig.1 to elucidate this fact.

The third aspect of Fig.1 is that at a ω value of 50 rad/s, TL is equal to the negligibly small value of 0.03 s relative to the process τ values considered. Over the virtually τ-independent ω range $3 \le \omega \le 50$ rad/s, TL is related to ω by the following correlation:

$$TL = \frac{1.5456}{\omega} \qquad (R^2 = 0.99998) \qquad (4)$$

With a maximum error of 2.75% over its specified applicability range.

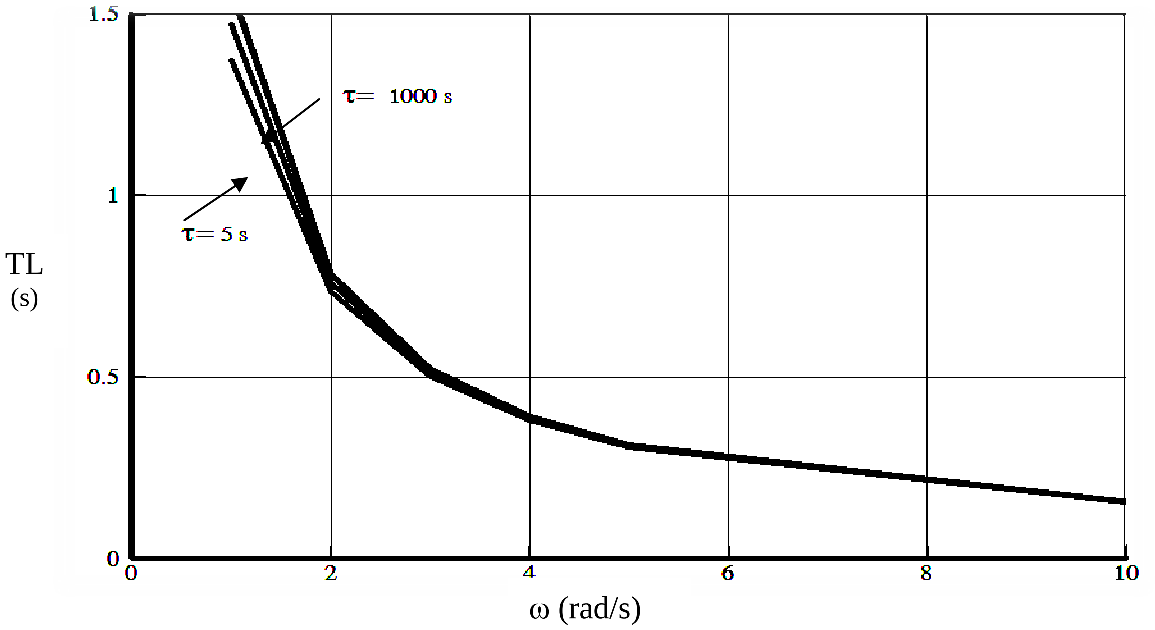

Fig. 2 .TL vs. ω for $1 \le \omega \le 10$ rad/s

3. Verification

The aforementioned treatise was verified by the simulation arrangement using the Simulink of MATLAB (version 2012) as in shown in Fig. 3. Figs. 4 and 5 show samples of the on-line simulated results, which provide and confirm the presented analysis.

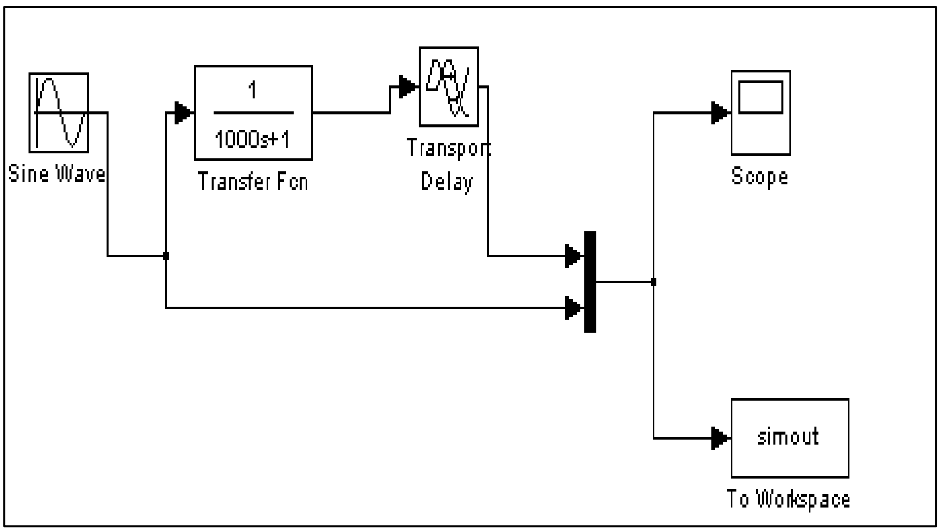

Fig. 3. Simulation arrangement for first-order process's sinusoidal response

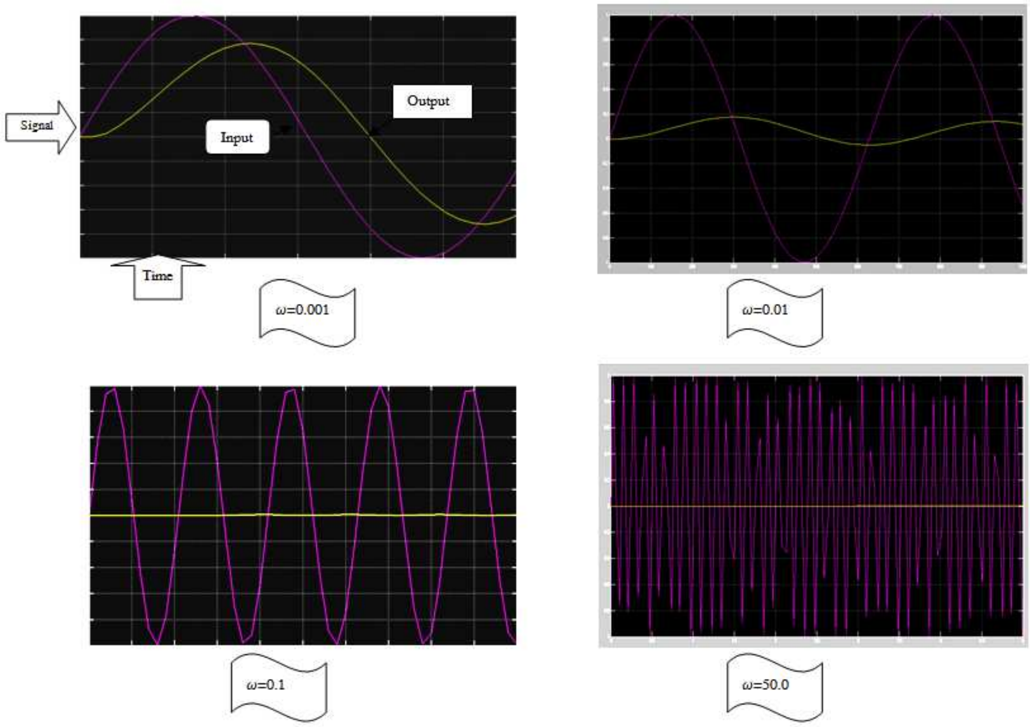

Fig. 4. On-line samples of simulated results at τ=10sec for different ω.

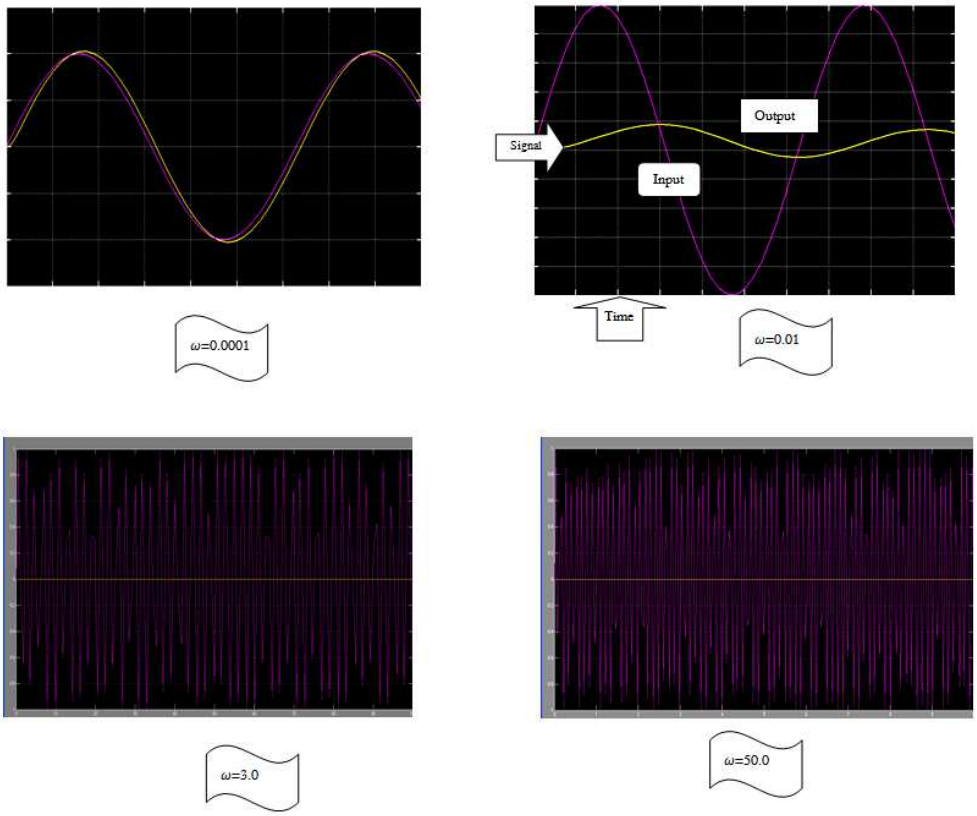

Fig. 5. On-line samples of simulated results at τ=1000 sec for different ω.

4. Conclusions

The time lag of the steady state sinusoidal response of a linear first- order process has a maximum value equal to its time constant at the low ω range of $\omega \leq 0.1/\tau$ rad/s. This value decreases as ω increases such that over the range $3 \leq \omega \leq 50$ rad/s, TL = 1.5456/ω , being independent of τ for all $\tau \geq 5$ s. The response's TL practically vanishes at $\omega \geq 50$ rad/s rendering it in phase with the SFF but with a virtually non-existent amplitude.

Analysis of TL in this article indicates that what had/has been reported (explicitly or implied) in some renowned textbooks on process dynamics and control , over more than four decades (e.g. the ones listed below as references) that at very small ω values the steady state response is virtually in phase with the SFF; becoming increasingly out of phase as ω increases is incorrect.

Nomenclature

A SFF amplitude
K Process steady state gain
R^2 Coefficient of determination

Greek

τ Process time constant (s)
φ Phase shift (rad)
ω Radian frequency (rad/s)

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