الجامعة التكنولوجية

قسم الهندسة الكيمياوية

المرحلة الثالثة

انتقال كتلة

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Transport Phenomena

Boundary Layer

1 Introduction.

When a fluid flows over a surface, that part of the stream which is close to the surface suffers a significant retardation, and a velocity profile develops in the fluid. The velocity gradients are steepest close to the surface and become progressively smaller with distance from the surface. Although theoretically there is no outer limit at which the velocity gradient becomes zero, it is convenient to divide the flow into two parts for practical purposes.

1- A *boundary layer* close to the surface in which the velocity increases from zero at the surface itself to a near constant stream velocity at its outer boundary.

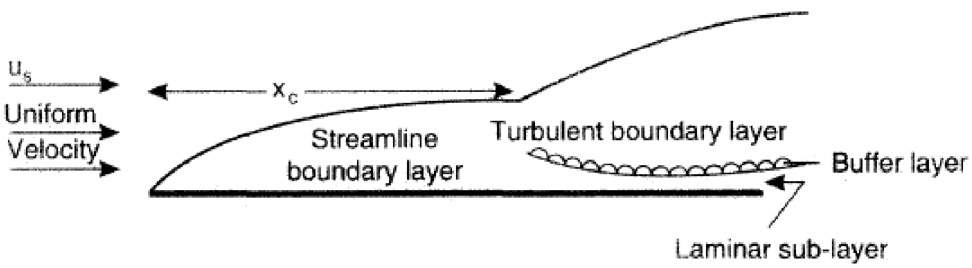
2- A region outside the boundary layer in which the velocity gradient in a direction perpendicular to the surface is negligibly small and in which the velocity is everywhere equal to the stream velocity.

The thickness of the boundary layer may be arbitrarily defined as the distance from the surface at which the velocity reaches some proportion of the undisturbed stream velocity. The flow conditions in the boundary layer are of considerable interest to chemical engineers because these influence, not only the drag effect of the fluid on the surface, but also the heat or mass transfer rates where a temperature or a concentration gradient exists.

It is convenient first to consider the flow over a thin plate inserted parallel to the flow of a fluid with a constant stream velocity u_s. It will be assumed that the plate is sufficiently wide for conditions to be constant across any finite width w of the plate which is being considered. Furthermore, the extent of the fluid in a direction perpendicular to the surface is considered as sufficiently large for the velocity of the fluid remote from the surface to be unaffected and to remain constant at the stream velocity u_s. Whilst part of the fluid flows on one side of the flat plate and part on the other, the flow on only one side is considered.

On the assumption that there is no slip at the surface, the fluid velocity at all points on the surface, where $y = 0$, will be zero. At some position a distance x from the leading edge, the velocity will increase from zero at the surface to approach the stream velocity u_s asymptotically. At the leading edge, that is where $x = 0$, the fluid will have been influenced by the surface for only an infinitesimal time and therefore only the molecular layer of fluid at the surface will have been retarded. At progressively greater distances (x) along the surface, the fluid will have been retarded for a greater time and the effects will be felt to greater depths in the fluid. Thus the thickness (δ) of the boundary layer will

increase, starting from a zero value at the leading edge. Furthermore, the velocity gradient at the surface $(du_x/dy)_{y=0}$(where $u_x$is the velocity in the x-direction at a distance y from the surface) will become less because the velocity will change by the same amount $u_x = 0$ at $y = 0$ to $u_x = u_s$, at $y = \delta$) over a greater distance. The development of the boundary layer is illustrated in Figure 1 below:-



I Figure 1 opment of the boundary layer

Near the leading edge of the surface where the boundary layer thickness is small, the flow will be streamline, or laminar, and the shear stresses will arise solely from viscous shear effects. When the boundary layer thickness exceeds a critical value, however, streamline flow ceases to be stable and turbulence sets in. the important flow parameter is the Reynolds number$Re_\delta(= u_s\delta\rho/\mu)$ Because δ can be expressed as a function of ***x,*** the distance from the leading edge of the surface, the usual criterion is taken as the value of the Reynolds number $Re_x(= u_s x\rho/\mu)$ .If the location of the transition point is at a distance xi; from the leading edge, then $Re_{x_c} = u_s x_c\rho/\mu$ is of the order of 10^5.

When the flow in the boundary layer is turbulent, streamline flow persists in a thin region close to the surface called the *laminar sub-layer.* This region is of particular importance because, in heat or mass transfer, it is where the greater part of the resistance to transfer lies. High heat and mass transfer rates therefore depend on the laminar sub-layer being thin. Separating the laminar sub-layer from the turbulent part of the boundary layer is the *buffer layer,*in which the contributions of the viscous effects and of the turbulent eddies are of comparable magnitudes.

For flow against a pressure gradient *(dP/dx*positive in the direction of flow) the combined force due to pressure gradient and friction may be sufficient to bring the fluid completely to rest and to cause some backflow close to the surface. When this occurs the fluid velocity will be zero, not only at the surface, but also at a second position a small distance away. In these circumstances, the boundary layer is said to *separate* and circulating currents are set up as shown in Figure 2 below.

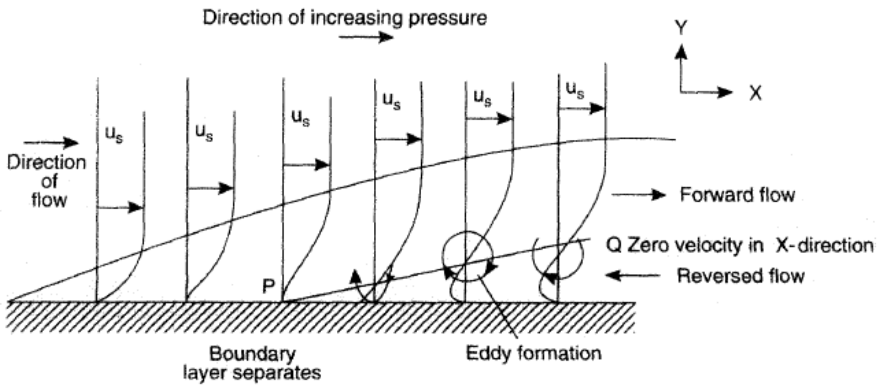


Figure 2 Boundary layer separation

The following treatment, based on the simplified approach suggested by PRANDTL1, involves the following three assumptions:

(1) That the flow may be considered essentially as unidirectional (x-direction) and that the effects of velocity components perpendicular to the surface within the boundary layer may be neglected (that is, $u_y << u_x$). This condition will not be met at very low Reynolds numbers where the boundary layer thickens rapidly.

(2) That the existence of the buffer layer may be neglected and that in turbulent flow the boundary layer may be considered as consisting of a turbulent region adjacent to a laminar sub-layer which separates it from the surface.

(3) That the stream velocity does not change in the direction of flow. On this basis, from Bernoulli's theorem, the pressure then does not change (that is, $dP/dx = 0$).

2 The momentum Equation.

It will be assumed that a fluid of density ϱ and viscosity μ flows over a plane surface and the velocity of flow outside the boundary layer is u_s. A boundary layer of thickness δ forms near the surface, and at a distance y from the surface the velocity of the fluid is reduced to a value u_x. The equilibrium is considered of an element of fluid bounded by the planes 1-2 and 3-4 at distances x and $x+dx$ respectively from the leading edge; the element is of length l in the direction of flow and is of depth w in the direction perpendicular to the plane 1-2 , 3-4. The distance l is greater than the boundary layer thickness δ(Figure 3), and conditions are constant over the width w. The velocities and forces in the x-direction are now considered.

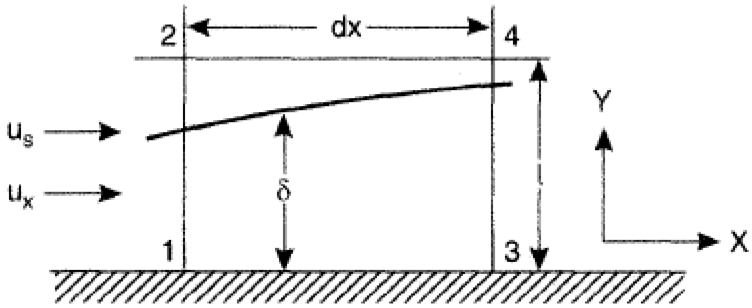


Figure 3 Element of boundary layer

At plane 1-2, mass rate of flow through a strip of thickness *dy*at distance y from the surface $= \rho u_x w \, dy$

The total flow through plane 1 -2 $= w \int_0^l \rho u_x \, dy$ ……(1)

The rate of transfer of momentum through the elementary strip

$$= \rho u_x w \, dy \, u_x = w \rho u_x^2 \, dy$$

The total rate of transfer of momentum through plane 1 -2

$$M_i = w \int_0^l \rho u_x^2 \, dy \qquad \text{.. ……….. (2)}$$

In passing from plane 1-2 to plane 3-4, the mass flow changes by:

$$w \frac{\partial}{\partial x} \left(\int_0^l \rho u_x \, dy \right) dx \qquad \text{…………(3)}$$

and the momentum flux changes by:

$$M_{ii} - M_i = w \frac{\partial}{\partial x} \left(\int_0^l \rho u_x^2 \, dy \right) dx \qquad \text{……….(4)}$$

whereMi is the momentum flux across the plane 3-4.

A mass flow of fluid equal to the difference between the flows at planes 3-4 and 1-2 (equation 3) must therefore occur through plane 2-4, as it is assumed that there is uniformity over the width of the element. Since plane 2-4 lies outside the boundary layer, the fluid crossing this plane must have a velocity $u_s$in the x-direction. Because the fluid in the boundary layer is being retarded, there will be a smaller flow at plane 3-4 than at 1-2, and hence the flow through plane 2-4 is outwards, and fluid leaves the element of volume. Thus the rate of transfer of momentum through plane 2-4 out of the element is:

$$M_{iii} = -wu_s \frac{\partial}{\partial x}\left(\int_0^l \rho u_x \,\mathrm{d}y\right)\mathrm{d}x \qquad \text{.........(5)}$$

It will be noted that the derivative is negative, which indicates a positive outflow of momentum from the element.

Steadystate momentum balance over the element 1-2-3-4

The terms which must be considered in the momentum balance for the x-direction are:

(i) The momentum flux through plane 1 -2 *into* the element.
(ii) The momentum flux through plane 3-4 *out of the* element.
(iii) The momentum flux through plane 2-4 *out of the* element.

The net rate of change of momentum in the x-direction on the element must be equal to the momentum added from outside, through plane 2-4, together with the net force acting on it. The forces in the x-direction acting on the element of fluid are:

(1) A shear force resulting from the shear stress R_o acting at the surface. This is a retarding force and therefore R_o is negative.

(2) The force produced as a result of any difference in pressure *dP*between the planes 3-4 and 1-2. However, if the velocity u_s outside the boundary layer remains constant, from Bernoulli's theorem, there can be no pressure gradient in the x-direction and $dP / dx = 0$.

The net force = Shear force + Pressuredrop
Ro.1.dx + (*-dP/dx).1.dx*

Thus, the net momentum flux out of the element to the net retarding force is given

$$\frac{\partial}{\partial x}\int_0^l \rho(u_s - u_x)u_x \,\mathrm{d}y = -R_0 \qquad \text{......(6)}$$

This expression, known as the *momentum equation,* may be integrated provided that the relation between *u,*and y is known. It is used for compressible or in compressible fluid for laminar and turbulent regime. If the velocity of the main stream remains constant at *us* and the density may be taken as constant, equation 6 then becomes:

$$\rho\frac{\partial}{\partial x}\int_0^l (u_s - u_x)u_x \,\mathrm{d}y = -R_0 \qquad \text{........(7)}$$

It may be noted that no assumptions have been made concerning the nature of the flow within the boundary layer and therefore this relation is applicable to both the streamline and the turbulent regions. The relation between u_x and y is derived for streamline and turbulent flow over a plane surface and the integral in equation 7 is evaluated.

2.1 The Streamline Portion of Boundary Layer.

In the streamline boundary layer the only forces acting within the fluid are pure viscous forces and no transfer of momentum takes place by eddy motion. Assuming that the relation between u, and y can be expressed approximately by:

$$u_x = u_0 + ay + by^2 + cy^3$$

The coefficients a, *b, c* and u_o,may be evaluated because the boundary conditions which the relation must satisfy are known, as shown in Figure 4.

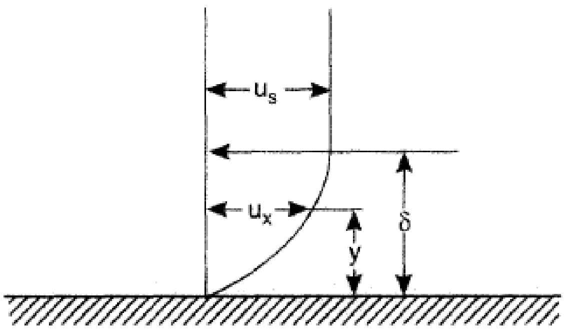


Figure 4 Velocity distribution in streamline boundary layer

It is assumed here that the fluid in contact with the surface is at rest and therefore u_o must be zero. Furthermore, all the fluid close to the surface is moving at very low velocity and therefore any changes in its momentum as it flows parallel to the surface must be extremely small. Consequently, the net shear force acting on any element of fluid near the surface is negligible, the retarding force at its lower boundary being balanced by the accelerating force at its upper boundary. Thus the shear stress R_o in the fluid near the surface must approach a constant value.

The shear stress in the fluid at the surface is given by:

$$R_0 = -\mu \left(\frac{\partial u_x}{\partial y}\right)_{y=0}$$

and ($\frac{\partial u_x}{\partial y}$ tant at small values of y .

$\frac{\partial u_x}{\partial y}$ = constant at y=0 ……………..and $\frac{\partial^2 u_x}{\partial y^2} = 0$

At the distant edge of the boundary layer it is assumed that the velocity just equals the main stream velocity and that there is no discontinuity in the velocity profile.

Thus, when $y = \delta$: $u_x = u_s$ and $\frac{\partial u_x}{\partial y} = 0$ $\frac{\partial^2 u_x}{\partial y^2} = 0$

with $u_0 = 0$

$$u_x = ay + by^2 + cy^3$$

$$\frac{\partial u_x}{\partial y} = a + 2by + 3cy^2$$

$$\frac{\partial^2 u_x}{\partial y^2} = 2b + 6cy$$

At $y = 0$: $\frac{\partial^2 u_x}{\partial y^2} = 0$

Thus: $b = 0$

At $y = \delta$: $u_x = a\delta + c\delta^3 = u_s$

and: $\frac{\partial u_x}{\partial y} = a + 3c\delta^2 = 0$

Thus: $a = -3c\delta^2$

Hence: $c = -\frac{u_s}{2\delta^3}$ and $a = \frac{3u_s}{2\delta}$

The velocity profile is therefore

$$u_x = \frac{3u_s}{2}\frac{y}{\delta} - \frac{u_s}{2}\left(\frac{y}{\delta}\right)^3$$

$$\frac{u_x}{u_s} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Velocity profiles in a laminar boundary layer applies over the range $0 < y < \delta$

When $y > \delta$, then: $u_x = u_s$

The integral in the momentum equation can now be evaluated for the streamline boundary layer by considering the ranges $0 < y < \delta$ and $\delta < y < L$ separately

$$\int_0^l (u_s - u_x)u_x \, dy = \int_0^\delta u_s^2 \left(1 - \frac{3}{2}\frac{y}{\delta} + \frac{y^3}{2\delta^3}\right)\left(\frac{3}{2}\frac{y}{\delta} - \frac{y^3}{2\delta^3}\right) dy + \int_\delta^l (u_s - u_s)u_s \, dy$$

$$= u_s^2 \int_0^\delta \left(\frac{3}{2}\frac{y}{\delta} - \frac{9}{4}\frac{y^2}{\delta^2} - \frac{1}{2}\frac{y^3}{\delta^3} + \frac{3}{2}\frac{y^4}{\delta^4} - \frac{1}{4}\frac{y^6}{\delta^6}\right) dy$$

$$= u_s^2 \delta \left(\frac{3}{4} - \frac{3}{4} - \frac{1}{8} + \frac{3}{10} - \frac{1}{28}\right)$$

$$= \frac{39}{280}\delta u_s^2$$

$$R_0 = -\mu \left(\frac{\partial u_x}{\partial y}\right)_{y=0} = -\frac{3}{2}\mu\frac{u_s}{\delta}$$

Then sub in momentum equation $$\rho\frac{\partial}{\partial x}\left(\frac{39}{280}\delta u_s^2\right) = \frac{3}{2}\mu\frac{u_s}{\delta}$$

$$\delta\,\mathrm{d}\delta = \left(\frac{140}{13}\right)\frac{\mu}{\rho}\frac{1}{u_s}\mathrm{d}x$$

$$\frac{\delta^2}{2} = \left(\frac{140}{13}\right)\left(\frac{\mu x}{\rho u_s}\right) \quad (\text{since } \delta = 0 \text{ when } x = 0)$$

$$\delta = 4.64\sqrt{\frac{\mu x}{\rho u_s}}$$

$$\frac{\delta}{x} = 4.64\sqrt{\frac{\mu}{x\rho u_s}} = 4.64 Re_x^{-1/2}$$

Shear stress at the surface

The shear stress in the fluid at the surface is given by:

$$\begin{aligned} R_0 &= -\mu \left(\frac{\partial u_x}{\partial y}\right)_{y=0} \\ &= -\frac{3}{2}\mu\frac{u_s}{\delta} \\ &= -\frac{3}{2}\mu u_s\frac{1}{x}\frac{1}{4.64}\sqrt{\frac{x\rho u_s}{\mu}} \\ &= -0.323\rho u_s^2\sqrt{\frac{\mu}{x\rho u_s}} \\ &= -0.323\rho u_s^2 Re_x^{-1/2} \end{aligned}$$

The shear stress R acting on the surface itself is equal and opposite to the shear stress on the fluid at the surface; that is, $R = -R_0$

$$\frac{R}{\rho u_s^2} = 0.323 Re_x^{-1/2}$$

Above Equation of shear stress gives the point values of R and $R/\rho u_s^2$ at $x = x$. In order to calculate the total frictional force acting at the surface, it is necessary to multiply the average value of R between $x = 0$ and $x = x$ by the area of the surface.

The average value of $R/\varrho u_s^2$ denoted by the symbol $R/\varrho u_s^2$:

$$\left(\frac{R}{\rho u_s^2}\right)_m x = \int_0^x \frac{R}{\rho u_s^2}\mathrm{d}x$$

$$= \int_0^x 0.323\sqrt{\frac{\mu}{x\rho u_s}}\,\mathrm{d}x$$

$$= 0.646x\sqrt{\frac{\mu}{x\rho u_s}}$$

$$\left(\frac{R}{\rho u_s^2}\right)_m = 0.646\sqrt{\frac{\mu}{x\rho u_s}}$$

$$= 0.646Re_x^{-0.5} \approx 0.65Re_x^{-0.5}$$

2.2 The TurbulentPortion of Boundary Layer.

In the simplified treatment of the flow conditions within the turbulent boundary layer the existence of the buffer layer, shown in Figure 1, is neglected and it is assumed that the boundary layer consists of a laminar sub-layer, in which momentum transfer is by molecular motion alone, outside which there is a turbulentregion in which transfer is effected entirely by eddy motion (Figure 5). The approachis based on the assumption that the shear stress at a plane surface can be calculated fromthe simple power law developed by Blasius equation.

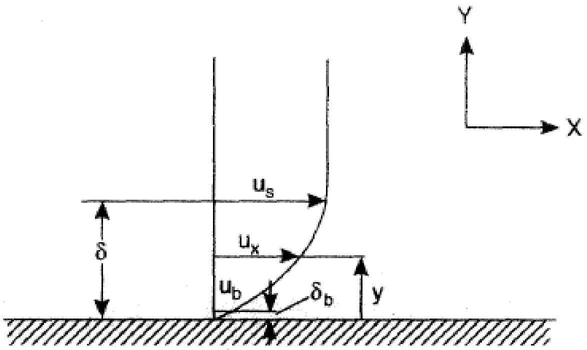


Figure 5Turbulent boundary layer

The shear stress at a plane smooth surface over which a fluid is flowing with a velocity us, for conditions where Rex < 10^7.

$$\frac{R}{\rho u_s^2} = 0.0228\left(\frac{\mu}{u_s\delta\rho}\right)^{0.25}$$

The shear stress is expressed as a function of the boundary layer thickness$\delta$and it is therefore implicitly assumed that a certain velocity profile exists in the fluid. As a first assumption, it may be assumed that a simple power relation exists between the velocity and the distance from the surface in the boundary layer, or

$$\frac{u_x}{u_s} = \left(\frac{y}{\delta}\right)^f$$

$$\text{Hence} \quad R = 0.0228\rho u_s^2\left(\frac{\mu}{u_s\delta\rho}\right)^{0.25}$$

$$= 0.0228\rho^{0.75}\mu^{0.25}\delta^{-0.25}u_s^{1.75}$$

$$= 0.0228\rho^{0.75}\mu^{0.25}\delta^{-0.25}u_x^{1.75}\left(\frac{\delta}{y}\right)^{1.75f}$$

$$= 0.0228\rho^{0.75}\mu^{0.25}u_x^{1.75}y^{-1.75f}\delta^{1.75f-0.25}$$

If the velocity profile is the same for all stream velocities, the shear stress must be defined by specifying the velocity u_x at any distance y from the surface. The boundary layer thickness, determined by the velocity profile, is then no longer an independent variable so that the index of δ , must be zero ,

$$1.75f - 0.25 = 0$$

$$f = \frac{1}{7}$$

$$\frac{u_x}{u_s} = \left(\frac{y}{\delta}\right)^{1/7}$$ (The Prandtl seventh power law equation)

Integrating momentum equation with limits $0 < y < \delta$

$$\int_0^l (u_s - u_x)u_x \,\mathrm{d}y = u_s^2 \left\{ \int_0^\delta \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right] \left(\frac{y}{\delta}\right)^{1/7} \mathrm{d}y \right\} + \int_\delta^l (u_s - u_s)u_s \,\mathrm{d}y$$

$$= u_s^2 \int_0^\delta \left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right] \mathrm{d}y$$

$$= u_s^2 \delta \left(\frac{7}{8} - \frac{7}{9}\right)$$

$$= \frac{7}{72} u_s^2 \delta$$

From the Blasius equation

$$-R_0 = R = 0.0228 \rho u_s^2 \left(\frac{\mu}{u_s \delta \rho}\right)^{1/4}$$

$$\rho \frac{\partial}{\partial x} \left[\frac{7}{72} u_s^2 \delta\right] = 0.0228 \rho u_s^2 \left(\frac{\mu}{u_s \delta \rho}\right)^{1/4}$$

$$\delta^{1/4} \,\mathrm{d}\delta = 0.235 \left(\frac{\mu}{u_s \rho}\right)^{1/4} \mathrm{d}x$$

$$\frac{4}{5} \delta^{5/4} = 0.235x \left(\frac{\mu}{u_s \rho}\right)^{1/4} + \text{constant}$$

$$\delta = 0.376 x^{0.8} \left(\frac{\mu}{u_s \rho}\right)^{0.2}$$

$$= 0.376x \left(\frac{\mu}{u_s \rho x}\right)^{0.2}$$

$$\frac{\delta}{x} = 0.376 Re_x^{-0.2}$$

2.3 The Laminar Sub-Layer.

If at a distance x from the leading edge the laminar sub-layer is of thickness δ_b and the total thickness of the boundary layer is δ, the properties of the laminar sub-layer can be found by equating the shear stress at the surface as given by the Blasius equation to that obtained from the velocity gradient near the surface. It has been noted that the shear stress and hence the velocity gradient are almost constant near the surface. Since the laminar sub-layer is very thin, the velocity gradient within it may therefore be taken as constant.

Thus the shear stress in the fluid at the surface,

$$R_0 = -\mu\left(\frac{\partial u_x}{\partial y}\right)_{y=0} = -\mu\frac{u_x}{y}, \quad \text{where } y < \delta_b$$

Equating this to the value obtained from Blasiusequation gives:

$$0.0228\rho u_s^2\left(\frac{\mu}{u_s\delta\rho}\right)^{1/4} = \mu\frac{u_x}{y}$$

$$u_x = 0.0228\rho u_s^2\frac{1}{\mu}\left(\frac{\mu}{u_s\delta\rho}\right)^{1/4} y$$

If the velocity at the edge of the laminar sub-layer is u_b, that is, if $u_x = u_b$, when $y = \delta_b$

$$u_b = 0.0228\rho u_s^2\frac{1}{\mu}\left(\frac{\mu}{u_s\delta\rho}\right)^{1/4} \delta_b$$

$$= 0.0228\frac{\rho u_s^2}{\mu}\frac{\mu}{u_s\delta\rho}\delta_b\left(\frac{\mu}{u_s\delta\rho}\right)^{-3/4}$$

$$\frac{\delta_b}{\delta} = \frac{1}{0.0228}\left(\frac{u_b}{u_s}\right)\left(\frac{\mu}{u_s\delta\rho}\right)^{3/4}$$

$$\left(\frac{\delta_b}{\delta}\right)^{1/7} = \frac{u_b}{u_s}$$

$$\left(\frac{u_b}{u_s}\right)^7 = \frac{1}{0.0228}\left(\frac{u_b}{u_s}\right)\left(\frac{\mu}{u_s\delta\rho}\right)^{3/4}$$

$$\frac{u_b}{u_s} = 1.87\left(\frac{\mu}{u_s\delta\rho}\right)^{1/8}$$

$$= 1.87Re_\delta^{-1/8}$$

$$\delta = 0.376x^{0.8}\left(\frac{\mu}{u_s\rho}\right)^{0.2}$$

$$\frac{u_b}{u_s} = 1.87\left(\frac{u_s\rho}{\mu}0.376\frac{x^{0.8}\mu^{0.2}}{u_s^{0.2}\rho^{0.2}}\right)^{-1/8}$$

$$= \frac{1.87}{0.376^{1/8}}\left(\frac{u_s^{0.8}x^{0.8}\rho^{0.8}}{\mu^{0.8}}\right)^{-1/8}$$

$$= 2.11Re_x^{-0.1} \approx 2.1Re_x^{-0.1}$$

The thickness of the laminar sub-layer is given by

$$\frac{\delta_b}{\delta} = \left(\frac{u_b}{u_s}\right)^7 = \frac{190}{Re_x^{0.7}}$$

$$\frac{\delta_b}{x} = \frac{190}{Re_x^{0.7}}\frac{0.376}{Re_x^{0.2}}$$

$$= 71.5Re_x^{-0.9}$$

Thus $\delta_b \propto x^{0.1}$; that is, δ_b it increases very slowly as x increases. Further, $\delta_b \propto u_s^{-0.9}$ and therefore decreases rapidly as the velocity is increased, and heat and mass transfercoefficients are therefore considerably influenced by the velocity. The shear stress at the surface, at a distance x from the leading edge, is given by:

$$R_0 = -\mu\frac{u_b}{\delta_b}$$

Since $R_0 = -R$, then:

$$R = \mu\, 2.11\, u_s\, Re_x^{-0.1}\,\frac{1}{x}\,\frac{1}{71.5}\,Re_x^{0.9}$$

$$= 0.0296Re_x^{0.8}\frac{\mu u_s}{x}$$

$$= 0.0296\rho u_s^2 Re_x^{-0.2} \approx 0.03\rho u_s^2 Re_x^{-0.2}$$

$$\frac{R}{\rho u_s^2} = 0.0296Re_x^{-0.2}$$

$$\frac{R}{\rho u_s^2} = 0.03Re_x^{-0.2}$$

The mean value of $R/\rho u_s^2$ over the range x =0 to x = x is given by:

$$\left(\frac{R}{\rho u_s^2}\right)_m x = \int_0^x \left(\frac{R}{\rho u_s^2}\right) \mathrm{d}x$$

$$= \int_0^x 0.0296 \left(\frac{\mu}{u_s x \rho}\right)^{0.2} \mathrm{d}x$$

$$= 0.0296 \left(\frac{\mu}{u_s x \rho}\right)^{0.2} \frac{x}{0.8}$$

$$\left(\frac{R}{\rho u_s^2}\right)_m = 0.037 Re_x^{-0.2}$$

The total shear force acting on the surface is found by adding the forces acting in the streamline

($x < x_c$) and turbulent ($x > x_c$) regions. This can be done provided the critical value Rex_c is known.

In the streamline region: $$\left(\frac{R}{\rho u_s^2}\right)_m = 0.646 Re_x^{-0.5}$$

In the turbulent region: $$\left(\frac{R}{\rho u_s^2}\right)_m = 0.037 Re_x^{-0.2}$$

In calculating the mean value of $(R/\rho u_s^2)_m$ in the turbulent region, it was assumed that the turbulent boundary layer extended to the leading edge. A more accurate value for the mean value of $R/\rho u_s^2$ over the whole surface can be obtained by using the expression for streamline conditions over the range from x = 0 to $x = x_c$ (where x_c is the critical distance from the leading edge) and the expression for turbulent conditions in the range $x = x_c$ to x = x :-

$$\left(\frac{R}{\rho u_s^2}\right)_m = \frac{1}{x}(0.646 Re_{x_c}^{-0.5} x_c + 0.037 Re_x^{-0.2} x - 0.037 Re_{x_c}^{-0.2} x_c)$$

$$= 0.646 Re_{x_c}^{-0.5} \frac{Re_{x_c}}{Re_x} + 0.037 Re_x^{-0.2} - 0.037 Re_{x_c}^{-0.2} \frac{Re_{x_c}}{Re_x}$$

$$= 0.037 Re_x^{-0.2} + Re_x^{-1}(0.646 Re_{x_c}^{0.5} - 0.037 Re_{x_c}^{0.8})$$

2.4 Boundary Layer Theory Applied to Pipe Flow.

When a fluid flowing with a uniform velocity enters a pipe, a boundary layer forms at the walls and gradually thickens with distance from the entry point. Since the fluid in the boundary layer is retarded and the total flow remains constant, the fluid in the central stream is accelerated. At a certain distance from the inlet, the boundary layers, which have formed in contact with the walls, join at the axis of the pipe, and, from that point onwards, occupy the whole cross-section and consequently remain of a constant thickness. Fully developed flow then exists. If the boundary layers are still streamline when fully developed flow commences, the flow in the pipe remains streamline. On the other hand, if the boundary layers are already turbulent, turbulent flow will persist as shown in Figure 6.

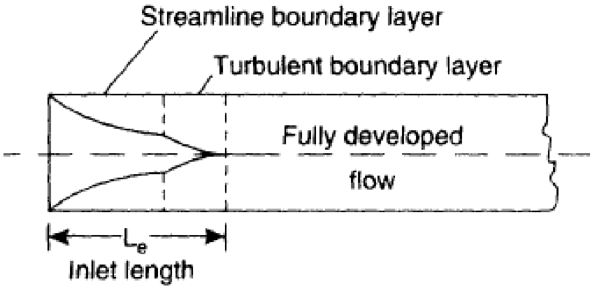


Figure 6:Conditions at entry to pipe

When the fluid is turbulent, for this purpose it is assumed that the boundary layer expressions may be applied to flow over a cylindrical surface and that the flow conditions in the region of fully developed flow are the same as those when the boundary layers first join. The thickness of the boundary layer is thus taken to be equal to the radius of the pipe and the velocity at the outer edge of the boundary layer is assumed to be the velocity at the axis.

The velocity of the fluid may be assumed to obey the Prandtl one-seventh power law. If the boundary layer thickness δ is replaced by the pipe radius r, this is then given by:

$$\frac{u_x}{u_s} = \left(\frac{y}{r}\right)^{1/7}$$

The shear stress at the walls is given by the Blasius equation:-

$$\frac{R}{\rho u_s^2} = 0.0228\left(\frac{\mu}{u_s r \rho}\right)^{1/4}$$

Whereu = 0.817u_s, andd = 2r:-

$$\frac{R}{\rho u^2} = 0.0386\left(\frac{\mu}{ud\rho}\right)^{1/4} = 0.0386Re^{-1/4}$$

$$\frac{R}{\rho u^2} = 0.0396Re^{-1/4}$$

The velocity at the edge of the laminar sub-layer is given by:

$$\frac{u_b}{u_s} = 1.87\left(\frac{\mu}{u_s r\rho}\right)^{1/8}$$

$$\frac{u_b}{u} = 2.49\left(\frac{\mu}{ud\rho}\right)^{1/8}$$

$$= 2.49Re^{-1/8}$$

$$\frac{u_b}{u_s} = 2.0Re^{-1/8}$$

The thickness of the laminar sub-layer is given by:-

$$\frac{\delta_b}{r} = \left(\frac{u_b}{u_s}\right)^7$$

$$= (1.87)^7\left(\frac{\mu}{u_s r\rho}\right)^{7/8}$$

$$\frac{\delta_b}{d} = 62\left(\frac{\mu}{ud\rho}\right)^{7/8}$$

$$= 62Re^{-7/8}$$

The mean velocity u is shown to be 0.82 times the velocity u_s at the axis, although in this calculation the thickness of the laminar sub-layer was neglected and the Prandtl velocity distribution assumed to apply over the whole cross-section. The result therefore is strictly applicable only at very high Reynolds numbers where the thickness of the laminar sub-layer is very small. At lower Reynolds numbers the mean velocity will be rather less than 0.82 times the velocity at the axis.