Statistics

قسم الهندسة الكيمياوية

وزارة التعليم العالي والبحث العلمي
الجامعة التكنولوجية

Statistics

Second Year

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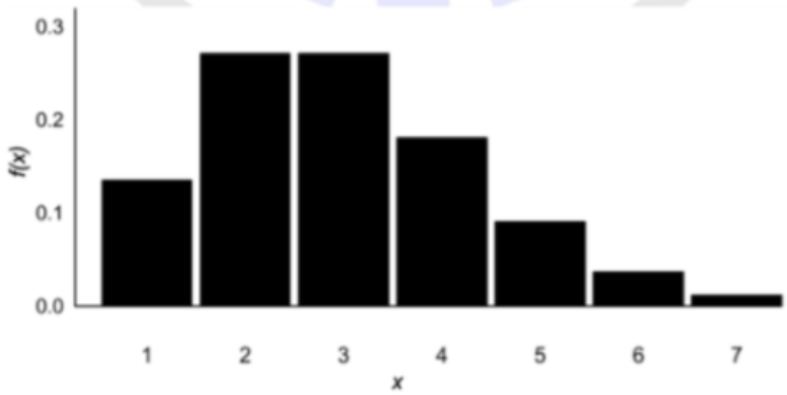


Statistics

Chapter (1) Introduction

- Statistics : Is concerned with scientific methods for collecting, organizing, summarizing, presenting and analysis data, as well as drawing valid conclusions and making reasonable decisions on the basis of such analysis.

- Population : Set of all possible measurements.

- Finite : All bots produced in a factory, catalyst pellets.

- Infinite : All possible outcomes (heads, tails) in successive tosses of a coin.

- Sample : A set of measurements taken to represent an infinite or large finite population, which is selected randomly.

- Random sample : Is selected so that all elements of the pop. have an equal chance of being measured.

- Sample array : Is the set of measurements of sample elements.

- Inductive or statistical inference : If a sample is representative of a pop., important conclusions about the pop. can often be inferred from the analysis of the sample. The phase of statistics dealing with the conditions under which such inference is valid is called inductive statistics.

- Deductive or descriptive statistics : The phase of statistics which seeks only to describe and analysis a given group without drawing any conclusion or inferences about a large group.

- Variable : Is a symbol, such as X , Y, which can assume any of a prescribed set of values, called the **domain** of the variables.

- If the variable can assume only one value is called a **constant**.

- A variable which can theoretically assume any value between two given values is called a **continuous variable**, otherwise it is called a **discrete variable**.

- The no. N of children in a family, which can assume any of the values 0,1,2,3,... but cannot be 2.5 or 3.842, is a **discrete variable**.

- The age A of an individual, which can be 62 years, 63.8, ... depending on accuracy of measurements is a **continuous variable.**

- **Size of data** : Number of measurements.

- **Range** : Highest – Lowest measurements.

Chapter (2)

Frequency Distribution

- When elements of population are unequal in a certain parameter and/or measurement error is involved a statistical estimation is needed. This involves:

1. Data sampling for repeated measurements.
2. Classification of data (frequency dist.).
3. Presentation of classified data.
4. Estimation of statistical parameters.
5. Analysis of statistical parameters and hypotheses.

- Frequency distribution into classes:

The sample range is sub-divided in to a number of classes. Usually:

For size
- > 50 : 10-20 classes
- ≤ 50 : 5-10 classes are used

Example :

The life of electric bulbs in hours was sampled :

690	701	722	684	680
728	705	693	691	688
740	663	676	738	714
698	687	703	726	699
694	705	717	682	717
712	733	705	673	694
679	680	664	691	669
689	702	710	696	697
685	724	726	698	688
702	696	708	696	710

- sample size = 50 measurements.

- sample range = 740 – 663 = 77 hr.

- class limits = highest and lowest measurements in the class.

- class interval = upper limit – lower limit.

- class boundaries = limits $\mp \frac{1}{2}$ unit in LSD.

- class width = upper – lower boundaries.

- class mark = mid point of class.

- e. g. if the class limits are 670 → 678.

* class interval = 678 – 670 = 8.

* class boundaries are : lower bound. = 670 – 0.5 = 669.5.

upper bound. = 678 + 0.5 = 678.5.

* Class width = 678.5 – 669.5 = 9

* class mark $= \frac{670+678}{2} = \underline{674}$

Or :

Class mark $= \frac{669.5+678.5}{2} = \underline{674}$

e. g. if class limits are $5.87 \rightarrow 6.32$:

* class interval $= 6.32 - 5.87 = \underline{0.45}$

* class boundaries are : lower bound. $= 5.87 - 0.005 = \underline{5.865}$.

upper bound. $= 6.32 + 0.005 = \underline{6.325}$.

* Class width $= 6.325 - 5.865 = \underline{0.46}$

* class mark $= \frac{5.87+6.32}{2} = \underline{6.095}$

Or :

Class mark $= \frac{5.865+6.325}{2} = \underline{6.095}$

Determination of classes :

1. Determine the range.
2. Determine the total width

   Total width = range + one unit in LSD.
3. Divided the total width into a convenient no. of classes

   Class width $= \frac{total\ width}{no.of\ classes}$

Note :

(Adjust the total width by adding one or two units in LSD if necessary, to select a suitable no. of classes, so that the class width is of a similar accuracy to the measurements).

4. Determine class interval :

Class interval = class width – one unit in LSD

5. Starting at lowest measurements, calculate the limits of successive classes.

Solution of example (electric bulb sample) :

1. range = 740 – 663 = 77 hr

2. one unit in LSD = 1

∴ total width = 77 +1 = 78

3. select no. of classes (for example take 5 classes) so the class width = $\frac{78}{5}$ = 15.6

Which is not the same accuracy as the data.

So take 6 classes :

class width = $\frac{78}{6}$ = 13

Which is the same accuracy as the data.

4. class interval = 13 – 1 = 12

Freq. dist. Table

	Class limit	Class bound.	Class mark	Freq.
1.	663-675	662.5-675.5	669	4
2.	676-688	675.5-688.5	682	10
3.	689-701	688.5-701.5	695	15
4.	702-714	701.5-714.5	708	11
5.	715-727	714.5-727.5	721	6
6.	728-740	727.5-740.5	734	4
				N = 50

Types of Frequency :

1. Numeric frequency : $f \rightarrow \sum f_i = N$

2. Relative frequency : $f_r = \frac{f}{N} \rightarrow \sum f_r = 1$

3. Percent frequency : $f_p = f_r * 100 \rightarrow \sum f_p = 100$

4. Cumulative frequency : The freq. is also expressed cumulatively of : f , f_r , f_p.

- Cumulative freq. of class K is the sum of frequencies of all classes up to K.

$$fc_K = \sum_{i=1}^{K} f_i \quad , \quad fc_{rK} = \sum_{i=1}^{K} f_{ri} = 1$$

$$fc_{pK} = \sum_{i=1}^{K} f_{pi} = 100$$

f	f_r	f_p
4	0.08	8
10	0.2	20
15	0.3	30
11	0.22	22
6	0.12	12
4	0.08	8
$\sum f = 50$	$\sum f_r = 1$	$\sum f_p = 100$

$$* fc = \sum_{i=1}^{i=6} f_i = 50$$

$$* fc_r = \sum_{i=1}^{6} f_{ri} = 1$$

$$* fc_p = \sum_{i=1}^{6} f_{pi} = 100$$

Graphical presentation of freq. dist. :

- Classified data may be presented as graphical plot with freq. as vertical axis Versus measurement as horizontal axis :

1) Histogram : Is a bar chart, in which each class is represent by a rectangle, whose base extends between the class boundaries and the area proportional to frequency.

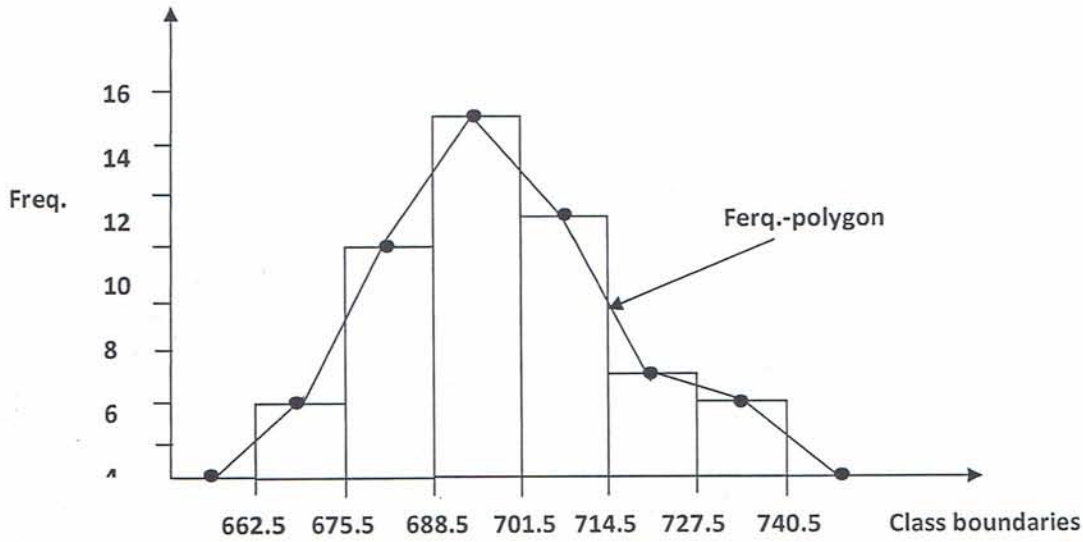
2) Frequency polygon : Consists of lines joining the class mark with freq., it may be obtained from the histogram by joining the mid-points of the bar tops.

*3) **Frequency curve :*** Is a smoothed frequency polygon in to a continuous curve.

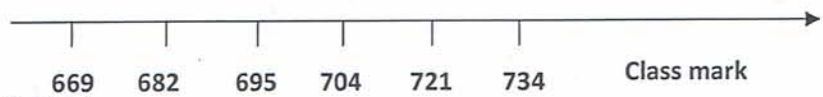
*4) **Cumulative freq. curve (Ogive) :*** Is a smoothed cumulative freq. polygon. It is usually S-shape.

Graphical presentation of frequency dist. Table :

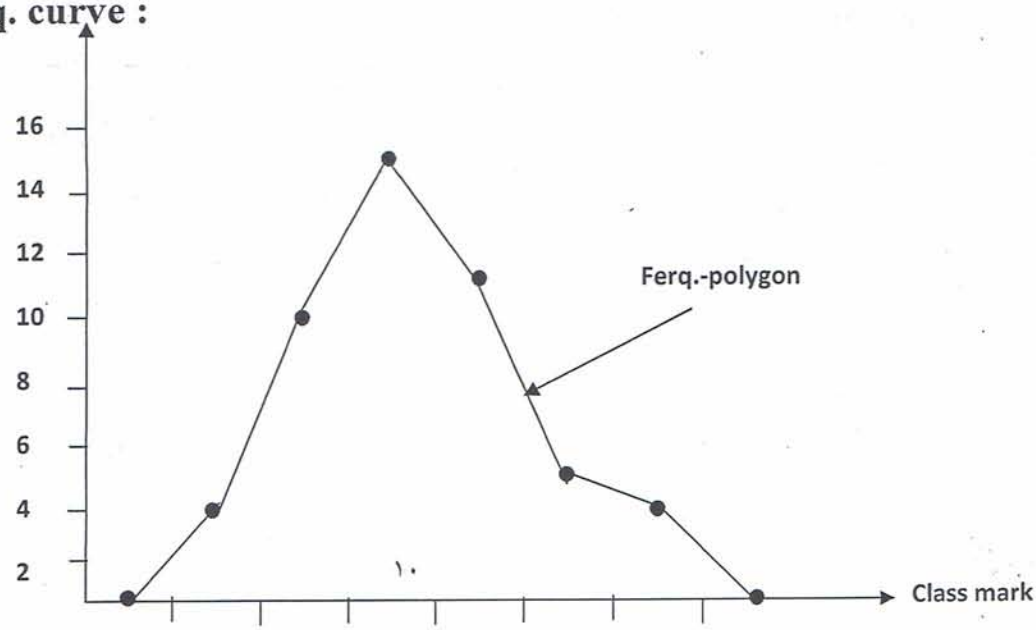
1) Histogram :



2) freq. polygon :

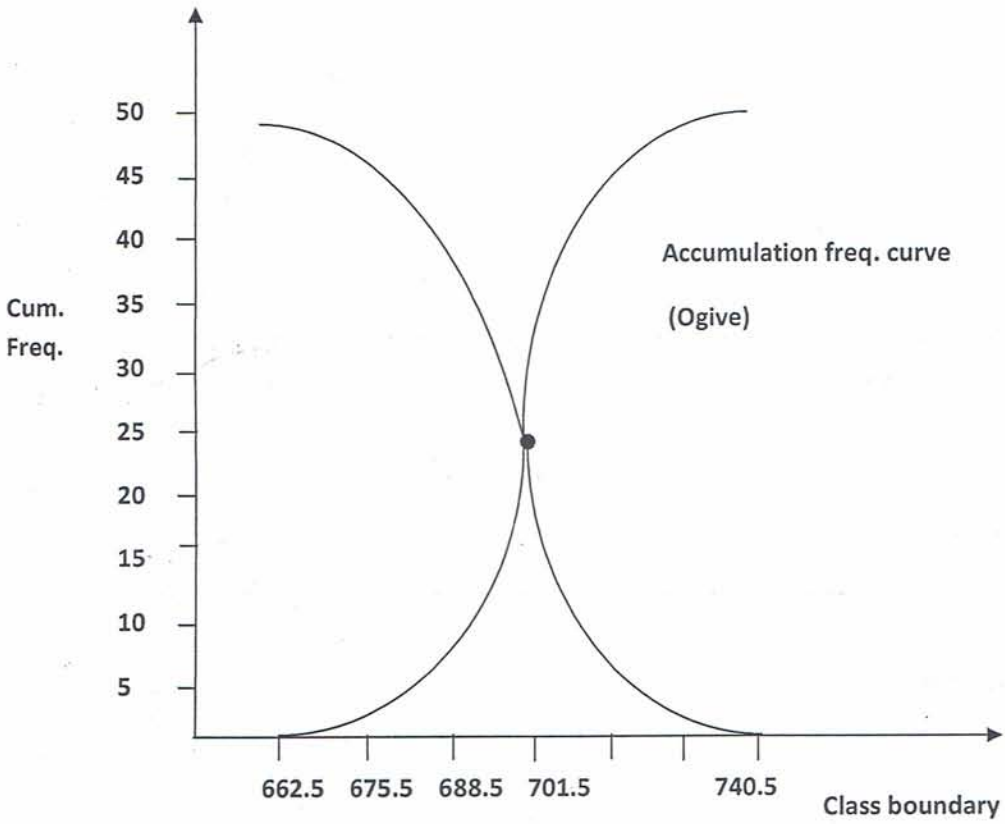


3) freq. curve :



4) Cumulative freq. :

Ascending cum. Freq.		Desending cum.	
Upper boundaries	cum. Freq.	Upper boundaries	cum. Freq.
Less than 662.5	0	Greater than 662.5	50
Less than 675.5	4	Greater than 675.5	46
Less than 688.5	14	Greater than 688.5	36
Less than 701.5	29	Greater than 701.5	21
Less than 714.5	40	Greater than 714.5	10
Less than 727.5	46	Greater than 727.5	4
Less than 740.5	50	Greater than 740.5	0



Tutorial Sheet No. (1)

For the following data groups obtain :

1) Frequency distribution table.

2) f_r , f_p , f_c

3) Histogram, freq. polygon, and Ogives

Data 1)

6.3	7.0	7.5	9.0	7.7	7.8	7.1	8.1
6.6	7.2	8.3	8.5	6.9	7.7	8.0	7.3
8.6	7.1	8.7	6.4	7.7	7.4	8.0	7.6
7.5	7.2	7.5	8.8	7.8	7.9	7.3	7.0
6.8	8.1	8.4	6.7	7.1	8.2	8.1	7.7

Data 2)

5.4	4.1	5.2	2.8	4.9	5.6	4.0	4.1	4.3
3.9	4.5	6.1	3.7	2.3	4.5	4.9	5.6	4.3
4.2	3.2	5.0	4.8	3.7	4.6	5.5	1.8	5.1
5.1	6.3	3.3	5.8	4.4	4.8	3.0	4.3	4.7

Data 3)

12.16	12.38	12.21	12.55	12.22	12.40	12.43	12.35
12.31	12.07	12.31	12.33	12.56	12.41	12.42	12.44
12.30	12.39	12.10	12.37	12.18	12.48	12.19	12.43
12.25	12.37	12.47	12.49	12.35	12.28	12.30	12.31
12.35	12.20	12.39	12.54	12.59	12.29	12.46	12.09

Chapter (3)

Measures of Location

When raw data is classified in to a frequency distribution table and presented graphically, the major features of the sample become apparent. However, to make quantitative decisions, further condensation in to a number of statistical parameters is needed.

Measures of location are statistical parameters, giving an estimate of the data centre, being typical of all measurement.

Mode : Is the measurement that occurs with the greatest frequency.

e. g. for sample :

14 , 19 , 16 , 21 , 19 , 24 , 18 , 19

Mode = 19

For sample : 6 , 7 , 7 , 3 , 8 , 3 , 9 , 5

Mode = 3 , 7

(bimodal)

For grouped data, the mode corresponds to the top of the frequency curve.

$$mode = L_m + \frac{\Delta L}{\Delta L + \Delta H} C_m$$

Where:

L_m is lower boundary of modal class

$\Delta L = f_m - f_{\text{lower class}}$

$\Delta H = f_m - f_{\text{higher class}}$

C_m = width of modal class

e. g. for electric bulbs sample :

$$mode = 688.5 + \frac{15 - 10}{(15 - 10) + (15 - 11)} (13) = 695.7$$

Median : Is the middle measurement of an ordered array (odd). Or the arithmetic mean of the two middle values (even).

e. g. for sample : 3 , 4 , 4 , 5 , 6 , 8 , 8 , 10 , 11

median = 6

for sample : 5 , 5 , 7 , 9 , 11 , 12 , 15 , 18

median = 10

* For grouped data, the median line halves the area under the frequency curve.

$$median = L_m + \frac{\frac{N}{2} - f_{CL}}{f_m} C_m$$

Where :

L_m is lower boundary of median class

N is sample size

F_{CL} is cumulative frequency of lower class

f_m is frequency of median class

C_m is width of median class

e. g. for electric bulbs sample :

3rd class is median class, since $f_c = 29 > \frac{N}{2}$

$$\text{median} = 688.5 + \frac{\frac{50}{2} - 14}{15}(13) = 689.0$$

Arithmetic Mean: is the sum of measurements divided by sample size.

$$\bar{x} = \frac{\sum x_i}{N}$$

For grouped data :

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

e. g. for electric bulbs sample:

$\bar{x}$ = [(4)(669) + (10)(682) + (15)(695) + (11)(708) + (6)(721) + (4)(734)] / 50

$\bar{x}$ = 699.4

Relation between mode/median/mean :

For symmetrical distributions, the three measures coincide. Else, the mean is further removed from mode than is the median. For moderately skewed unimodal distributions :

Mean – mode $\approx$ 3 (mean – median)

Other Mean Measures :

* ***Geometric Mean*** $G = (\pi x_i)^{\frac{1}{N}}$, $\log G = \frac{\sum f_i \log X_i}{N}$

* ***Harmonic Mean*** $H = \frac{N}{\sum \frac{1}{x_i}}$, $H = \frac{N}{\sum \frac{f_i}{x_i}}$

* ***Root Mean Square***

$$RMS = \sqrt{\frac{\sum x_i^2}{N}} \quad , \quad RMs = \sqrt{\frac{\sum f_i x_i^2}{N}}$$

For a sample of positive measurements,

$H \leq G \leq \bar{x} \leq RMS$

e. g. for electric bulbs sample :

$\bar{x} = 699.4$

$G = 699.2$

$H = 699.0$

$RMS = 699.6$

Properties of the Arithmetic Mean :

1. The sum of deviations of the data from their arithmetic mean is zero.

$\sum(x_i - \bar{x}) = 0$ (prove)

2. For several samples, the combined mean is given by:

$$\bar{x} = \frac{N_1\overline{x_1} + N_2\overline{x_2} + \ldots}{N_1 + N_2 + \ldots}$$

3. If the deviations (d_i) from any value (A) are available, then :

$\bar{x} = A + \frac{\sum d_i}{N}$ $where\ d_i = x_i - A$ (prove)

Or $\bar{x} = A + \frac{\sum f_i d_i}{N}$ (grouped data)

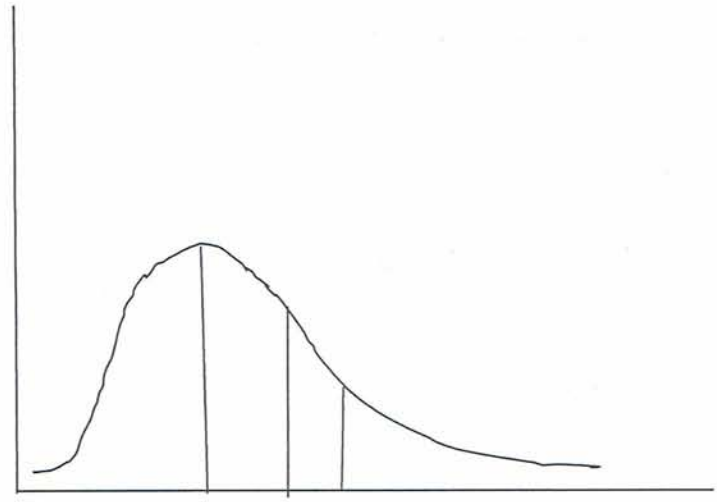
Empirical Relation between mean, median and mode :

For unimodal freq. curves which are moderately skewed (asymmetrical), where the empirical relation.

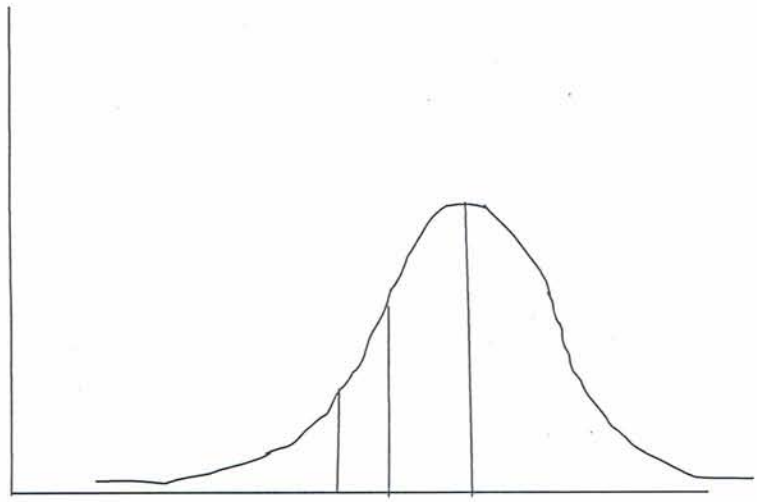
Mean – mode = 3 (mean – median)

In figs. Below are shown the relative position of the mean, median and mode for freq. curve which are skewed to the right and left resp.

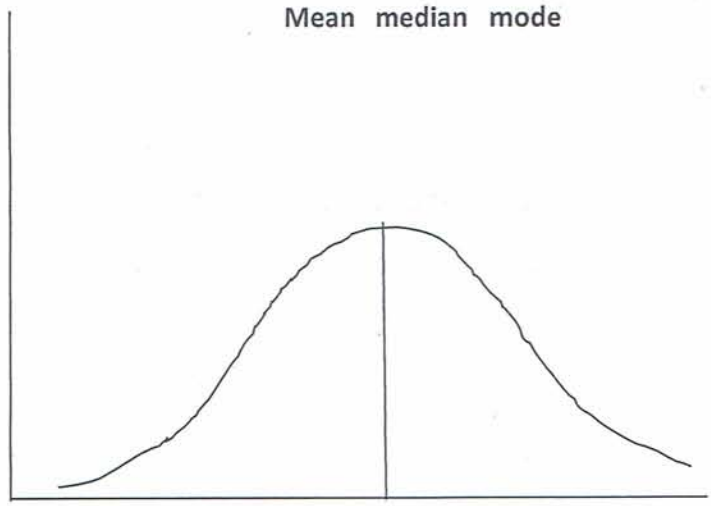
For symmetrical curves the mean, mode and median al coincide.



Mode median mean



Mean median mode



Mode
mean
Median

Chapter (4)

Measures of Dispersion

Dispersion is the degree of data spread about an average.
Several measures are used including:

Range, mean absolute deviation , standard deviation , variance and coefficient of variation.

Mean Absolute Deviation :

Is the arithmetic mean of the absolute deviations.

$$M.A.D = \frac{\sum|x_i - \bar{x}|}{N} \qquad \text{for raw data}$$

$$= \frac{\sum f_i\,|x_i - \bar{x}|}{N} \qquad \text{for grouped data}$$

Mean other than $\bar{x}$ may be used to obtain M.A.D from the respective mean.

Standard Deviation :

Is the root mean square of the deviation.

$$S = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N}} \qquad \text{for raw material}$$

$$S = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}} \qquad \text{for grouped data}$$

Means other than $\bar{x}$ may be used to obtain the standard deviation from the respective mean .

Standard deviation of a sample (S) is related to the standard deviation of the population (σ) by :

$$\sigma = S\sqrt{\frac{N}{N-1}} = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N-1}}$$

Variance :

Is the square of the standard deviation i.e. S^2 for sample, σ^2 for population.

Coefficient of Variation :

Is a relative dispersion measure (dimension less).

$$Relative\ Dispersion = \frac{absolute\ dispersion}{average}$$

$$Coefficient\ of\ Variation = \frac{S}{\bar{x}}$$

Properties of Standard Deviation :

* Of all standard deviations, the min. is that from the arithmetic mean.

* For ideal normal distributions:

With in $\bar{x} \mp S$ $68.27\%\ of\ data$

$\bar{x} \mp 2S$ $95.45\%\ of\ data$

$\bar{x} \mp 3S$ $99.73\%\ of\ data$

* For several samples, the combined S is given by :

$$S^2 = \frac{N_1 S_1^2 + N_2 S_2^2 + \ldots}{N_1 + N_2 + \cdots}$$

Standard Variable :

The dimensional measurements x_i may be expressed as dimension less standardized variables Z_i

$$Z_i = \frac{x_i - \bar{x}}{S} = \frac{(\bar{x} + S) - \bar{x}}{S} = 1$$

i.e. when Z=1 , the measurement is removed by one standard deviation from the mean.

Properties of Z :

1. The arithematic mean for the standard scores equal to zero.

$$\acute{Z} = \frac{\sum f_i Z_i}{N} = 0$$

2. The standard deviation (or variance) for the standard scores equal to **one**.

$$S_Z = \sqrt{\frac{\sum f_i (Z_i - \bar{Z})^2}{N}} = 1$$

$$S_Z^2 = \frac{\sum f_i (Z_i - \bar{Z})^2}{N} = 1$$

Tutorial Sheet No. (2)

Q 1) Prove the following

A. $S = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$

B. $S = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$

Where $d_i = x_i - A$ where A is constant

Q 2) For the following data :

Class limits	f
60 – 62	5
63 – 65	18
66 – 68	42
69 – 71	27
72 - 74	8

Obtain :

A) S , S^2 B) Z C) $\acute{Z}$, S_z

Chapter (5)

Probability Distribution

* Probability : When an event may happen in (x) ways out of a total of (n) possible equally likely ways, the probability of occurrence (success) is given by :

$$p = Pr(E) = \frac{x}{n}$$

Hence the prob. Of non-occurrence (failure) is :

$$q = \Pr(\tilde{E}) = \frac{n-x}{n} = 1 - \frac{x}{n} = 1 - p$$

Thus $p + q = 1$

Discrete Prob. Distribution :

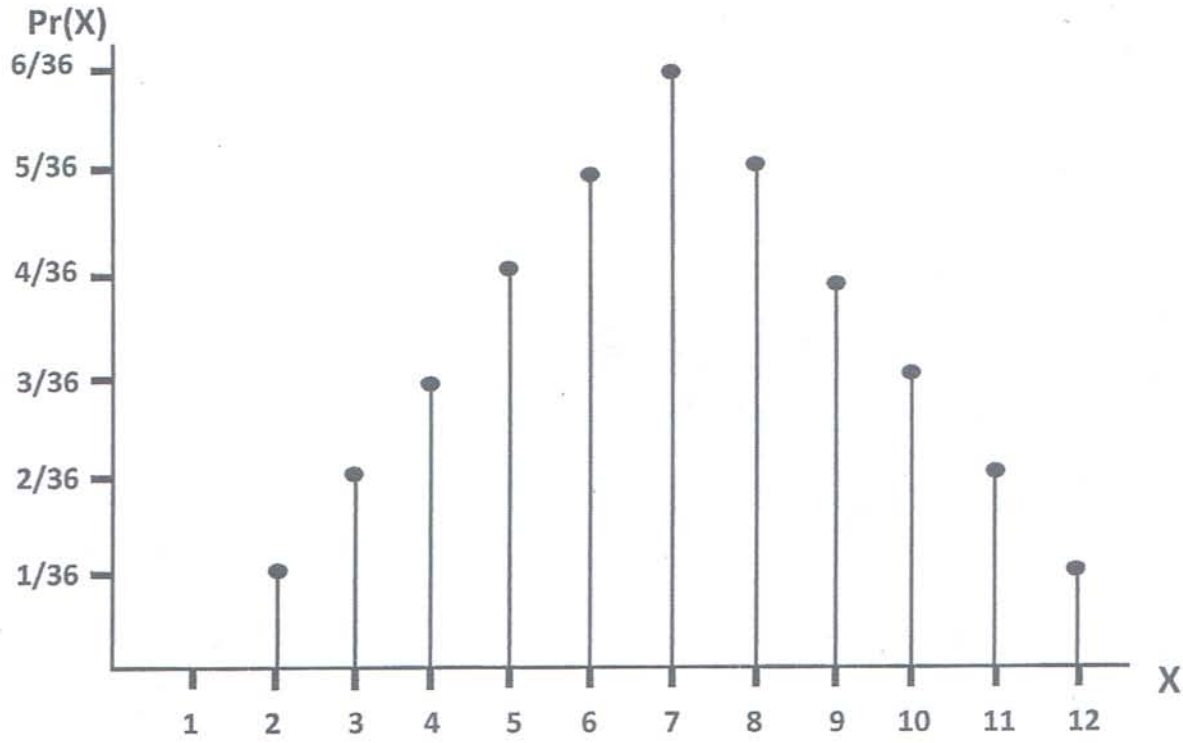
If a variable X may assume a set of discrete values (x_i) with respective prob. p_i , where $\sum p_i = 1$, this defines a discrete prob. distribution for X.

Example :

Let X be the sum of points obtained on a throw of two dice. The prob. or frequency distribution is given as :

X :	2	3	4	5	6	7	8	9	10	11	12
Pr(x):	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\sum \Pr(X) = 1$$



* Relative frequency distribution of (N) throws is thus related to a sample of size (N) drawn out of an infinite population.

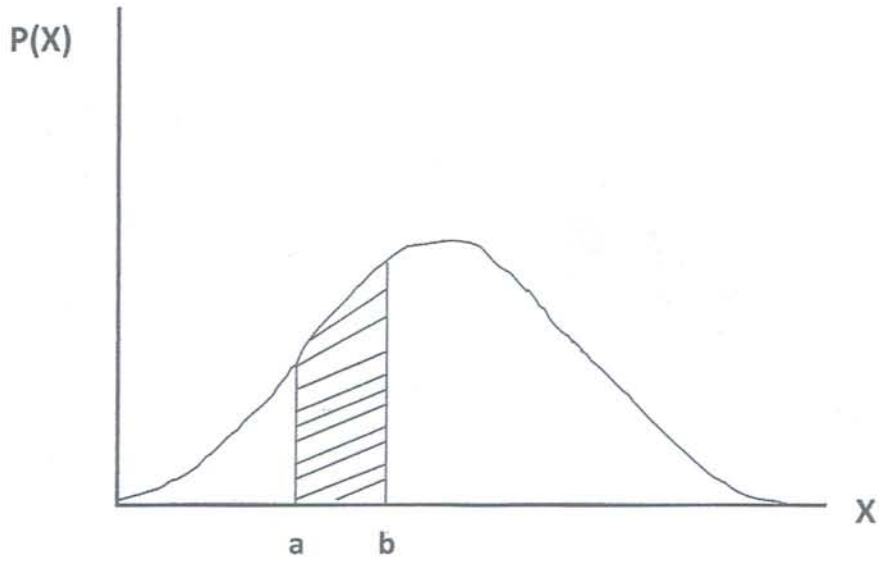
As $N \rightarrow \infty$, the relative freq. dist. Approaches the prob. dist. Of the population .

Continuous Probability Distribution :

* If a variable X may assume a continuous set of values, the prob. dist. Is a frequency curve where p(x)=fr .

Total area under the curve = $\sum fr = \sum p = 1$.

* Prob. that X may lie between a and b ; Pr [a < X < b] = area under curve from a to b .



The Normal Distribution

* The normal distribution is the most important of all probability distribution. It is applied directly to many practical problems, and several very useful distributions are based on it .

It is some times called the Gaussin dist. .

Characteristics :

Many empirical freq. dist. Have the following characteristics :

1. They are approximately symmetrical, and the mode is close to the centre of the dist.
2. The mean, median, and mode are close together.
3. The shape of the dist. Can be approximated by a bell.

*** The prob. density function for the normal dist. Is given by:**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where :

μ : is the mean of the theoretical dist.

σ : is the standard deviation, and $\pi = 3.14$

*** This function extends from $-\infty$ to ∞**

Let $z = \frac{X-\mu}{\sigma}$,

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

* The total area bounded by the curve and the X axis is one. Hence, the area under the curve between X=a and X=b , Where a < b represent the prob. that X lies between a and b.

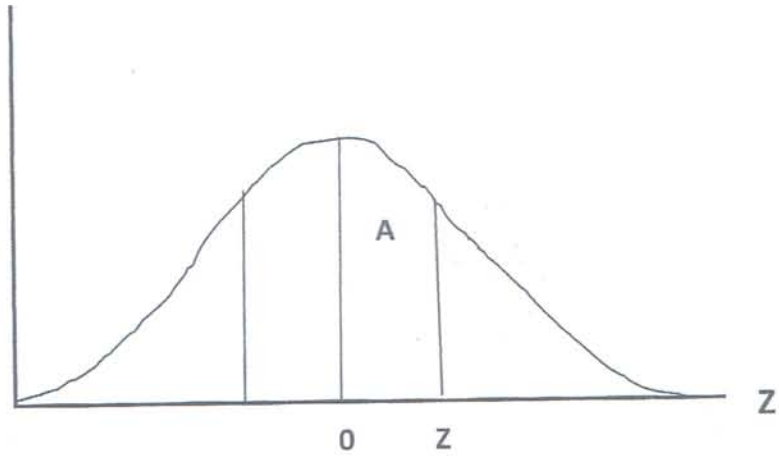
* Areas under the normal dist. Curve between 0 and Z are given in a :

Table (*). (Given below)

The prob. that Z lies between 0 and Z :

Pr [0 < Z < z] = A

From table (*) the area between any two ordinates can be found by using the symmetry of the curve about Z=0 .



*** Some properties of the normal dist. :**

$Mean = \mu$

$Variance = \sigma^2$

$Standard\ dev. = \sigma$

$$Mean\ dev. = \sigma \sqrt{\frac{2}{\pi}} = 0.7979\,\sigma$$

*** Areas under the normal curve :**

When $Pr[0 < Z < Z_1] = A$

$Pr[-Z_1 < Z < 0] = A$ (symmetrical curve).

$Pr[Z_1 < Z] = 0.5 - A$
$Pr[-Z_1 < Z] = 0.5 + A$
} Tatal area=1 so that area from $0 \rightarrow \infty$ is 0.5

When Z_1 and Z_2 are of same signs :

$Pr[Z_1 < Z < Z_2] = A_{Z_2} - A_{Z_1}$

When Z_1 and Z_2 are of different signs :

$Pr[Z_1 < Z < Z_2] = A_{Z_2} + A_{Z_1}$

* For bound of measurements, the bound in actual value is ∓ 0.5 units in L.S.D.

Example :

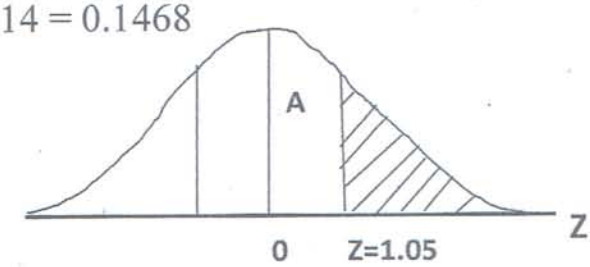
For measurements of $\mu = 160$, $\sigma = 10$, obtain the following :

1. Pr[X greater than 170] = Pr[X >170]

$$Z = \frac{X-\mu}{\sigma} = \frac{170.5-160}{10} = 1.05$$

$\therefore$ Pr[Z > 1.05]= = 0.5 − 0.35314 = 0.1468

Where $A_{Z=1.05} = 0.35314$

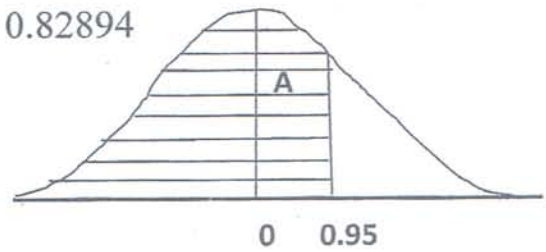


2. Pr[X less than 170] = Pr[X<170]

$x = 169.5 \rightarrow Z = 0.95$

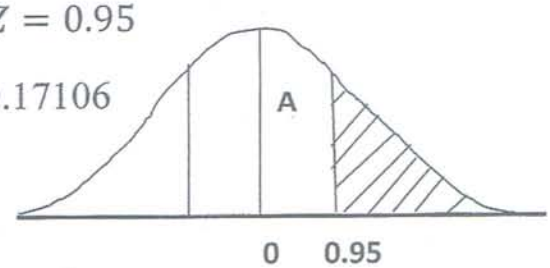
Pr [Z < 0.95] = 0.5 + 0.32894 = 0.82894

Where $A_{Z=0.95} = 0.32894$



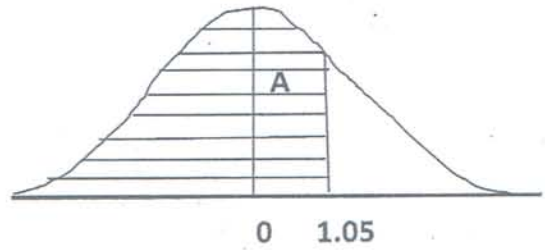
3. Pr[X ≥ 170] → $x = 169.5 \rightarrow Z = 0.95$

Pr [Z > 0.95] = 0.5 − 0.32894 = 0.17106



4. $\Pr[X \leq 170] \rightarrow x = 170.5 \rightarrow Z = 1.05$

$\Pr[Z > 1.05] = 0.5 + 0.35314 = 0.85314$



Linear inter polation :

When Z lies between successive Z_1 and Z_2 with respective A_1 and A_2 , A is obtained by linear interpolation "

$$\frac{Z - Z_1}{Z_2 - Z_1} = \frac{A - A_1}{A_2 - A_1} \qquad , Z_1 < Z < Z_2$$

e.g. for $Z_1 = 2.32$ $A_1 = 0.48983$

$Z_2 = 2.33$ $A_2 = 0.49010$

Then when $Z = 2.327 \rightarrow A = ?$

$$A = \frac{Z - Z_1}{Z_2 - Z_1}(A_2 - A_1) + A_1 = 0.49002$$

Example 1)

For a measurement of size (N)=500

$\mu = 151$, $\sigma = 15$, assuming normal dist. Find how many measurements :

a) between 120 and 155 = $\Pr[120 \leq X \leq 155]$

$x_1 = 119.5 \rightarrow z_1 = -2.1 \rightarrow A_1 = 0.4821$

$x_2 = 155.5 \rightarrow z_2 = 0.31 \rightarrow A_2 = 0.1179$

Pr $[-2.1 < Z < 0.3] = 0.4821 + 0.1179 =$ No. of meas. = 500[0.4821+0.1179]=300

b) more than 185 = Pr[Z>185]

$x = 185.5 \rightarrow Z = 2.3 \rightarrow A = 0.4893$

Pr $[Z > 2.3] = 0.5 - 0.4893 =$ No. of meas. = 500[0.51−0.4893]=5.

c) Less than 128 = Pr [X < 128]

$x = 127.5 \rightarrow Z = -1.57 \rightarrow A = 0.4418$

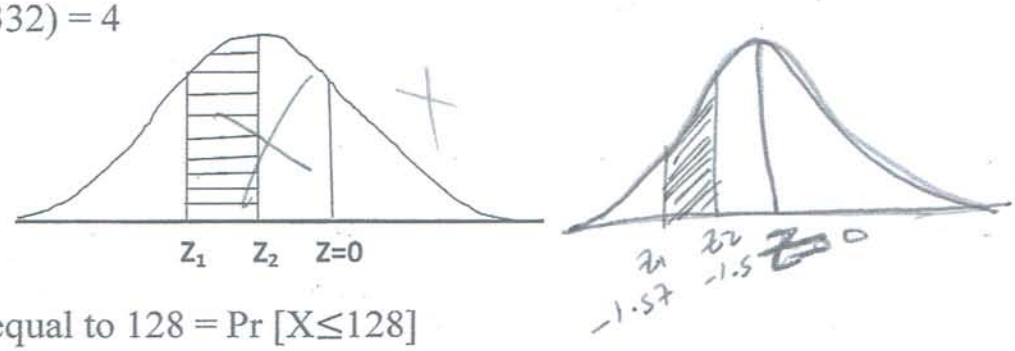
No. of meas. = 500[0.5−0.4418]=29

d) equal to 128 = Pr [X=128]

$x_1 = 127.5 \rightarrow z_1 = -1.57 \rightarrow A_1 = 0.4418$

$x_2 = 128.5 \rightarrow z_2 = -1.5 \rightarrow A_2 = 0.4332$

Pr $[-1.57 < Z < -1.5] = 0.4418-0.4332 =$ No. of meas. = 500(0.4418−0.4332) = 4



e) Less than or equal to 128 = Pr [X≤128]

$x = 128.5 \rightarrow Z = -1.5 \rightarrow A = 0.4332$

No. = 500[0.5−0.4332]=33

f) Less than or equal to 185 = Pr [X≤185]

$x = 185.5 \rightarrow Z = 2.3 \rightarrow A = 0.4893$

No. = 500[0.5 + 0.4893] = 495

Example 2)

For a sample of washers produced by a machine the mean inside dia. (μ) is 5.02 mm and the standard deviation is 0.05 mm. The max. useful tolerance in the dia. Is 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine % of defective washers.

Solu.)

Pr of max. to lerance = Pr $(4.96 \leq X \leq 5.08)$

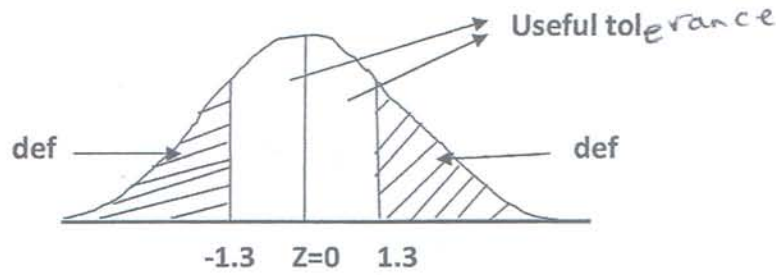
One unit in L.S.D. = 0.01

$x_1 = 4.955 \rightarrow z_1 = -1.3 \rightarrow A_1 = 0.4032$

$x_2 = 5.085 \rightarrow z_2 = +1.3 \rightarrow A_2 = 0.4032$

Pr $[-1.3 < Z < 1.3] = 2 * 0.4032 = 0.8064$

$\therefore$ % of defective washers = $(1-0.8064) *$ = 19.4 %



Example 3)

Out of a large No. of examination applicant a sample of size 50 gave a mean mark of 64 and a standard dev. of 14 . What is the expected % of applicants achieving a min. pass mark of 50 ?

Solu. :

one unit = 1

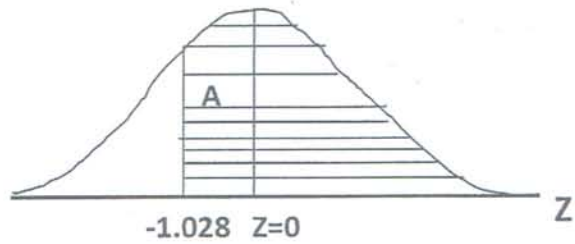
$\mu = 64$

$$\sigma = S\sqrt{\frac{N}{N-1}} = 14\sqrt{\frac{50}{50-1}} = 14.1$$

Pr [app. Have a min. pass mark of 50]= Pr [50≤X]

$x = 49.5 \rightarrow Z = -1.028 \rightarrow A = 0.3480$

Pr [−1.028 < Z] = 0.348 + 0.5 = 0.848 = 84.8 %



Example 4)

The strength of individual bars made by a certain manufacturing process are approximate normally distributed with mean 28.4 and standard dev. 2.95 . To ensure safety, a customer requires at least 95% of the bars to be stronger than 24.0 . (one unit =0.1)

a) Do the bars meet the specification ?

b) By improved manufacturing techniques, the manufacturer make the bars more uniform (that is, decrease the standard dev.) what value of standard dev. will just meet the specification if the mean stays the same ?

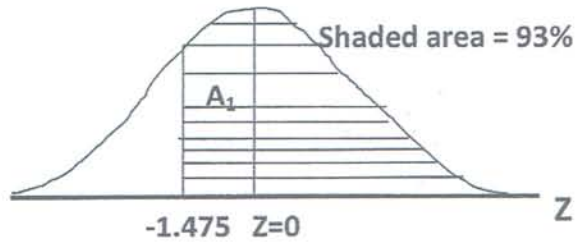
Solu. :

Pr [X > 24.0]→ $Z_1 = \frac{X-\mu}{\sigma} = \frac{24.05-28.4}{2.95}$

$$Z_1 = -1.475 \rightarrow A_1 = 0.4299$$

Pr [Z > - 1.475] = 0.5 + 0.4299 = 0.9299 ≃93%

Since (93%) less than 95% , the bars do not meet the specification.



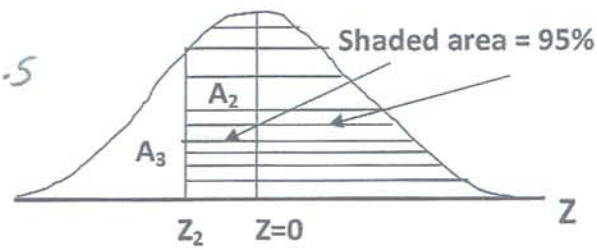
b) The specification is at least 95% of bars > 24.0

$A_3 = 1 - 0.95 = 0.05$

$\therefore \quad A_2 = 0.5 - 0.05 = 0.45$

OR: $A_2 = 0.95 - 0.5$

$A_2 = 0.45$



At $A_2 = 0.45 \rightarrow Z_2 = -1.645$ (from table)

$$Z_2 = \frac{X - \mu}{\sigma} \rightarrow -1.645 = \frac{24.05 - 28.4}{\sigma}$$

$\sigma = 2.644$ (if the σ can be reduced to 2.644 while keeping the mean constant, the specification will just be met)

Normal Distribution

Tutorial sheet (4)

Q. 1) Diameters of bolts produced by a machine are normally distributed with $\mu = 0.76$ *and* $\sigma = 0.012$ cm. Specifications call for dia. From 0.72 cm to 0.78 cm.

a) What percentage of bolts will meet there specification ?

b) What percentage of bolts will be smaller than 0.73 cm. ?

Q. 2) The diameters of screws are normally distribution with $\mu = 2.1$ *and* $\sigma = 0.15$ cm .

a) What proportion of screws are expected to have dia. Greater than 2.5 cm. ?

b) A specification calls for screw dia. Between 1.75 cm and 2.5 cm . What proportion of screws will meet the specification ?

Q. 3) Diameters of ball bearings produced by a company follow a normal distribution. If the mean dia. is 0.4 cm and stand. dev. is 0.001 cm.

a) What percentage of the bearings can be used o a machine specifying a size of 0.399 ∓ 0.0015 cm. ?

b) What is the upper bound of the size range that has a lower bound of 0.398 cm . and include 80% of the bearings ?

Q. 4) The probability that a river flow exceeds 2000 m^3/sec is 15% . The coefficient of variation of these flows is 20% . Assuming a normal distribution, calculate :

a) The mean of the flow ?

b) The stand. dev. of the flow ?

c) The prob. That the flow will be between 1300 or 1900 m^3/sec?

Q. 5) A water quality parameter monitored in a lake is normally dist. With $\mu = 24.3$. It is also known that there is 70% probability that the parameter will exceed 17.6 :

a) Find the stand. dev. of the parameter ?

b) If the parameter exceeds the 95% an investigation of a local industry begins. What is this critical value ?

The Binomial Distribution

* If P is the prob. That an event will happen in any single trial (called the prob. Of succeed)and q=1−P is the prob. That it will fail to happen in any single trial (called the prob. Of a failure), then the prob. That the event will happen exactly X times in N trials (X success and N−X failures will occur) is given by :

$$P(X) = {}_NC_X\, P^X\, q^{N-X} = \frac{N\,!}{X\,!\,(N-X)!}\; P^X\, q^{N-X}$$

Where :

X = 0 , 1 , 2 , … , N

N ! = N (N−1) (N−2) …1

0 ! = 1

Example 1)

The prob. Dist. For getting heads in N tossed of a coin is :

$$P(X) = {}_NC_X \left(\frac{1}{2}\right)^X \left(\frac{1}{2}\right)^{N-X}$$

For example N=3 ?

$$\text{Pr (0 heads)} = {}_3C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{3-0}$$

$$= \frac{3\,!}{0\,!\,(3-0)!} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{3-0} = \frac{1}{8}$$

$$\text{Pr (1 heads)} = {}_3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{3-1} = \frac{3}{8}$$

Pr (2 heads) = $_3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} = {}_3C_2 \left(\frac{1}{2}\right)^3 = \frac{3}{8}$

Pr (3 heads) = $_3C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{3-3} = {}_3C_3 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

H = head , T = tail

HHH , HHT , HTT , TTT , THH , TTH , THT , HTH

Total out coins = 8 = $(2)^N = 2^3 = 8$

Where 2 = P(head + tail)

Example 2)

Find the prob. Of getting :

a) Pr (2 H in 6 tosses) = $_6C_2 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{6-2}$

$$= \frac{6!}{2!(6-2)!} \left(\frac{1}{2}\right)^6 = \frac{15}{64}$$

Total out coins : $(2)^6 = 64$

b) Pr (at least 4 H in 6 tosses) = Pr ($X \geq 4$)

= Pr (X=4) + Pr (X=5) + Pr (X=6)

$\therefore$ $Pr(X \geq 4) = [{}_6C_4 + {}_6C_5 + {}_6C_6] [\frac{1}{2}]^6 = \frac{22}{64}$

Example 3)

If 10% of items produced by a machine are defective, find the prob. that out of 5 items :

a) None are defective

P = prob. of success (defection).

P = 0.1 → q + p =1 → q=0.9

q = 0.9 = prob. of failure (non def.)

Pr (o def.) = Pr (X=0) = ${}_5C_0\ (0.1)^0\ (0.9)^5$

$$\Pr(X=0) = \frac{5!}{0!\,(5-0)!}\ (0.1)^0\ (0.9)^5 = 0.5905$$

b) All are def.

Pr (5 def.) = Pr (X=5) = ${}_5C_5\ (0.1)^5\ (0.9)^5$

$$= \frac{5!}{5!\,(5-5)!}\ (0.1)^5\ (0.9)^0 = 0.00001$$

c) At most 2 def.

Pr (at most 2 def.) = Pr (X≤2) = Pr (X=2) + Pr (X=1)+ Pr (X=0)

$$\Pr(X\leq 2) = {}_5C_2\ (0.1)^2\ (0.9)^3 + {}_5C_1\ (0.1)^1\ (0.9)^4 + {}_5C_0\ (0.1)^0\ (0.9)^5$$

$$= 0.00729 + 0.32805 + 0.59049 = 0.9258$$

Some properties of Binomial dist :

Mean = μ = NP

Variance = σ^2 = NPq

Stand. dev. = $\sigma = \sqrt{N\ P\ q}$

The Binomial

Tutorial sheet (5)

Q. 1) Out of 800 families with 5 children howmany would you expect to have : a) 3 boys , b) 5 girls , c) either 2 or 3 boys . Assume equal probabilities for boys and girls ?

Q. 2) Find the prob. of getting a total of 11 : a) once , b) twice , in two tosses of a pair dice ?

Q. 3) What is the prob. of getting a 9 exactly once in 3 throws with a pair of dice ?

Q. 4) Find the prob. of guessing correctly at least 6 of the 10 answers on a atrue – false examination ?

Q. 5) An insurance salesman sells policies to 5 men the prob. that a man will be a live in 30 years is 2/3. Find the prob. that in 30 years :

a) all 5 men , b) at least 3 men , c) only 2 men , d) at least 1 man will be a live ?

Relation between Binomial and Normal Distributions

* If N is large and if neither P nor q is too close to zero, the binomial distribution can be closely approximated by a normal distribution with standardized variable given by :

$$Z = \frac{X - \mu}{\sigma} = \frac{X - NP}{\sqrt{N\,P\,q}}$$

The approximation becomes better with increasing N , . In practice, the approximate is good when : NP > 5 , Nq > 5 .

Example 1)

Find the prob. of obtaining 3 – 6 heads in 10 tosses of a coin :

a) using the Binomial dist.

Pr(3-6 heads) = Pr(3) + Pr(4) + Pr(5) + Pr(6)

$$= \left[{}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5 + {}_{10}C_6\right]\left[\frac{1}{2}\right]^{10} = 0.7734$$

b) using the normal dist.

one unit = 1.0

Pr (3-6 heads) = Pr $(3 \le X \le 6)$

$$Z = \frac{X - \mu}{\sigma} = \frac{X - NP}{\sqrt{N\,P\,q}}$$

μ = NP = 10*0.5 = 5

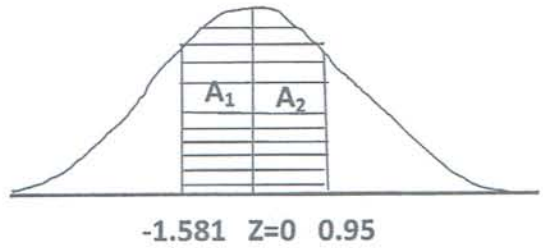
$\sigma = \sqrt{N\,P\,q} = \sqrt{10 * 0.5^2} = 1.581$

For $X_1 = 2.5 \rightarrow Z_1 = -1.581 \rightarrow A_1 = 0.4431$

$X_2 = 6.5 \rightarrow Z_2 = 0.949 \rightarrow A_2 = 0.3287$

Pr (-1.581 < Z < 0.949) = $A_1 + A_2 = 0.7718$

% of error between the binomial and normal = 0.0016



Example 2)

What is the prob. That at most 90% of 20 students will graduate ? Given % of graduate 70% ?

a) using the Binomial dist. :

Pr (at most $\frac{90}{100} * 20$) = Pr (at most 18)

Pr ($X \leq 18$) = Pr(0) + Pr(1) + … + Pr(18)

= 1 − [Pr(19) + Pr(20)]

Pr (X=19) = $_{20}C_{19}$ $(0.7)^{19}$ $(0.3)^{1}$ = $6.84*10^{-3}$

Pr (X=20) = $_{20}C_{20}$ $(0.7)^{20}$ $(0.3)^{0}$ = $7.98*10^{-4}$

Pr (X≤18) = 1 − ($6.84*10^{-3} + 7.98*10^{-4}$) = 0.992

b) using the normal dist.

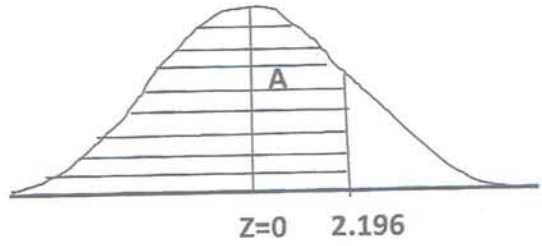
$$Z = \frac{X - \mu}{\sigma} = \frac{X - NP}{\sqrt{N\,P\,q}}$$

μ = NP = 20*0.7 = 14

$\sigma = \sqrt{N\,P\,q}$ = 2.049

Pr (X≤18) → X = 18.5 → Z = 2.196 → A = 0.48

Pr (Z < 2.196) = 0.5 + 0.486 = 0.986



Relation between Binomial and Poisson :

Poisson dist. :

Is a discrete prob. dist. Defined by :

$$Pr(X) = \frac{\lambda^x \, e^{-\lambda}}{x\,!}$$

Where :

X = 0 , 1 , 2, … , N

e = 2.71828

With properties :

$\mu = \lambda \quad , \ \sigma = \sqrt{\lambda}$

Example 1)

Product of a machine is 10% defective, Find the prob. of obtaining at most 2 def. items out of 10 :

a) using Binomial :

Pr [at most 2] = Pr($X \leq 2$)=Pr(0) + Pr(1) + Pr(2)

$= {}_{10}C_0 \, (0.1)^0 \, (0.9)^{10} + {}_{10}C_1 \, (0.1)^1 \, (0.9)^9 + {}_{10}C_2 \, (0.1)^2 \, (0.9)^8$

$\Pr(X \leq 2) = 0.9298$

b) using Poisson :

Poisson is applicable for large N , While P is close to zero.

In practice , $N \geq 50$, $NP < 5$.

Use $\mu = NP = \lambda = 10 * 0.1 = 1$

Pr (at most 2) = Pr($X \leq 2$) = Pr(0) + Pr (1) +Pr(2)

$$= \frac{1^0 e^{-1}}{0!} + \frac{1^1 e^{-1}}{1!} + \frac{1^2 e^{-1}}{2!} = \left[1 + 1 + \frac{1}{2}\right] e^{-1}$$

$$= 0.9197$$

Example 2)

The prob. of failure of a certain process is 3% . Determine the prob. of 3 failures at most in 100 repetition of the process :

a) using binomial dist.

Pr (at most 3) = Pr($X \leq 3$)=

$$_{100}C_0 \, (0.03)^0 \, (0.97)^{100} + \cdots + {}_{100}C_3 \, (0.03)^3 \, (0.97)^{97}$$

= 0.6474

b) using Poisson dist. :

N = 100 , NP = = $100 * \frac{3}{100} = 3 = \lambda$

$$\Pr(X \leq 3) = \frac{3^0 e^{-3}}{0!} + \cdots + \frac{3^3 e^{-3}}{3!}$$

$$= \left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right] [e^{-3}] = 0.6472$$

Use Normal probability plots to assess normality :

- A normal prob. plot is a graph that plots observed data versus normal scores, (expected Z – scores).

- Drawing a normal prob. plot requires the following step :

1. Arrange the data in ascending order.

2. Compute $\left(f_i = \frac{i-0.375}{n+0.25}\right)$ where i is the index of the data, n . number of observation.

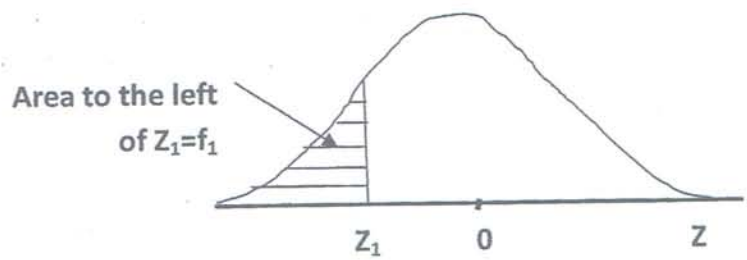
3. Find the Z – score corresponding to f_i from the table of the normal curve.

4. plot the observed values on the horizontal axis and the corresponding expected Z-scores on the vertical axis.

- The value of f_i represents the expected area to the left of the ith observation when the data come from a population that is normally distributed.

- For example, f_1 is the expected area to the left of the smallest data value.

- Values of normal random variables and their Z-scores are linearly related $(X = \mu + Z\sigma)$, so a plot of observation of normal variable against their expected Z-scores will be linear. We conclude the following :



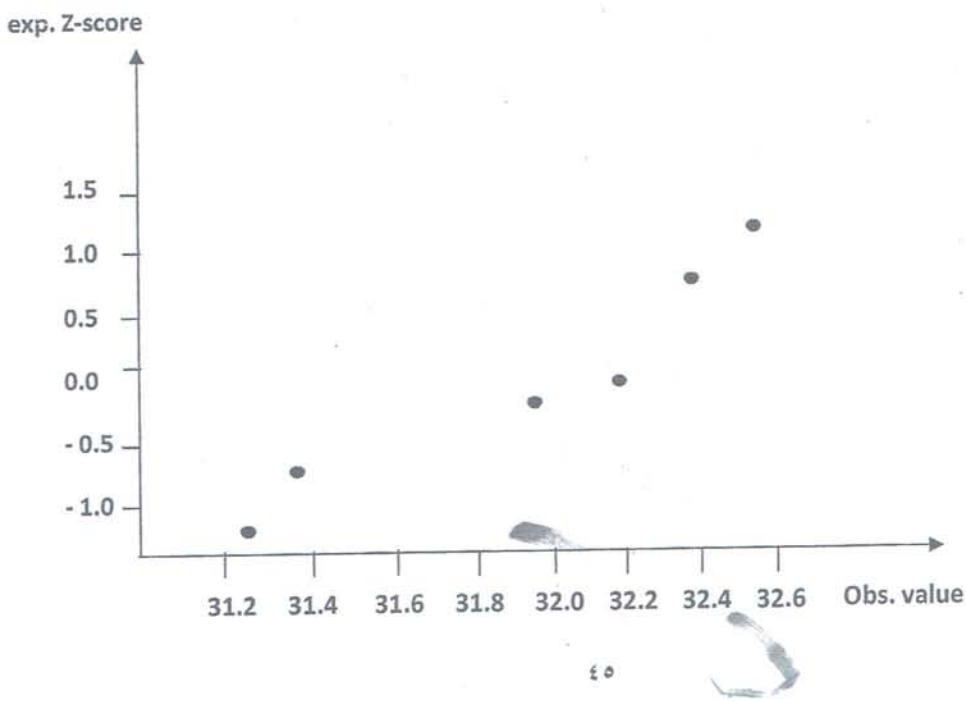
(If the sample data are taken from a population that is normally distributed, a normal prob. plot of the observed values vs. The expected Z-scores will be approximately linear).

Example : For the following data, construct the normal prob. plot. :

31.35 35.52 32.06 31.26 31.91
32.37

Solution :

Index, i	Observed value	Fi	Expected Z-score
1	31.26	$\frac{1-0.375}{6+0.25}=0.1$	- 1.28
2	31.35	$\frac{2-0.375}{6+0.25}=0.26$	- 0.64
3	31.91	0.42	- 0.20
4	32.06	0.58	0.20
5	32.37	0.74	0.64
6	35.52	0.9	1.28



* Although the normal prob. plot does show some curvature, it is roughly linear. We conclude that the data are approximately normally distributed.

Examples :

A random sample of college students aged 18 to 24 years was obtaind, the no. of hours of television watched was recorded:

36.1	30.5	2.9	17.5	21.0
23.5	25.6	16.0	28.9	29.6
7.8	20.4	33.8	36.8	0.0
9.9	25.8	19.5	19.1	18.5
22.9	9.7	39.2	19.0	8.6

Determine if the data come from a normal dist.

Data for a normal prob. plot. :

1)

0.276	0.274	0.275	0.274	0.277
0.273	0.276	0.276	0.279	0.274
0.273	0.277	0.275	0.277	0.277
0.276	0.277	0.278	0.275	0.276

2)

26	24	22	25	23
24	25	23	25	22
21	26	24	23	24
25	24	25	24	25
26	21	22	24	24

3)

24.0	7.9	1.5	0.0	0.3
0.4	8.1	4.3	0.0	0.5
3.6	2.9	0.4	2.6	0.1
16.6	1.4	23.8	25.1	1.6
12.2	14.8	0.4	3.7	4.2

Examples :

1) Steel rods are manufactured with a mean length of 25 cm, and standard deviation of 0.07 cm.

a) What proportion of rods has a length less than 24.9?
b) Any rods that are shorter than 24.85 cm or longer than 25.15 cm are discarded. What proportion of rods will be discarded?
c) Using the results of part (b), if 5000 rods are manufactured in a day, how many should the plant manager expected to discard?
d) If an order comes for 10000 steel rods, how many rods should the plant manager manufacture if the order states that all rods must be between 24.9 cm and 25.1 cm?

2) Ball bearing are manufactured with a mean dia. of 5 mm and stand. dev. of 0.02 mm.

a) What proportion of ball bearings ~~gas~~ has a dia. more than 5.03 mm?
b) Any ball bearing that have a dia. less than 4.95 mm or greater than 5.05 mm are discarded. What proportion of ball bearing will be discarded?

c) Using the results of (b) if 30000 ball bearings are manufactured in a day, how many should the plant manager expected to discard?

d) If any order comes in for 50000 ball beaning, how many bearing should the plant manager manufacture if the order stares that all ball bearing must be between 4.97 and 5.03 mm?

Tutorial Sheet No. (2)

Q 1) Prove the following

A. $S = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$

B. $S = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$

Where $d_i = x_i - A$ where A is constant

Q 2) For the following data :

Class limits	f
60 – 62	5
63 – 65	18
66 – 68	42
69 – 71	27
72 - 74	8

Obtain :

A) S , S^2 B) Z C) $\acute{Z}$, S_z

Chapter 6

The Chi – Square test

Definition :

* Results obtained in samples denote always agree exactly with theoretical results, expected according to rules of probability.

- Suppose that a set of possible events, E_1 , E_2 , … , E_K occure with observed frequencies, O_1 , O_2 , … , O_K, and according to prob. rules the expected frequencies e_1 , e_2 , …, e_K.

- Chi – square measure the discrepancy existing between observed and expected frequencies.

$$\chi^2 = \frac{(O_1 - e_1)^2}{e_1} + \frac{(O_2 - e_2)^2}{e_2} + \ldots + \frac{(O_k - e_k)^2}{e_k}$$

$$\chi^2 = \sum_{j=1}^{k} \frac{(O_j - e_j)^2}{e_j} \quad \ldots\ldots (1)$$

If the total freq. is N :

$$\sum O_j = \sum e_j = N \quad \ldots\ldots (2)$$

$$\therefore \quad \chi^2 = \sum_{j=1}^{k} \left(\frac{O_j^2}{e_j}\right) - N \quad \ldots\ldots (3)$$

Where K : number of possible events.

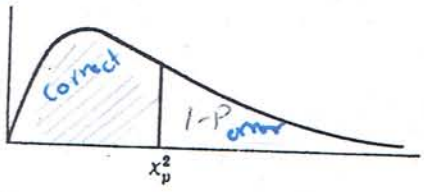
$$\chi^2 = \sum \left(\frac{(O-e)^2}{e}\right)$$

$$= \sum \left(\frac{O^2 - 2Oe + e^2}{e}\right) = \sum \left(\frac{O^2}{e} - \frac{2Oe}{e} + \frac{e^2}{e}\right)$$

Appendix IV

صدول جديد لقيم الحرجة لـ Chi-sq χ^2_p

PERCENTILE VALUES (χ^2_p)
for
THE CHI-SQUARE DISTRIBUTION
with ν degrees of freedom
(shaded area = p)



ν	$\chi^2_{0.995}$	$\chi^2_{0.99}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.90}$	$\chi^2_{0.75}$	$\chi^2_{0.50}$	$\chi^2_{0.25}$	$\chi^2_{0.10}$	$\chi^2_{0.05}$	$\chi^2_{0.025}$	$\chi^2_{0.01}$	$\chi^2_{0.005}$
1	7·88	6·63	5·02	3·84	2·71	1·32	0·455	0·102	0·0158	0·0039	0·0010	0·0002	0·0000
2	10·6	9·21	7·38	5·99	4·61	2·77	1·39	0·575	0·211	0·103	0·0506	0·0201	0·0100
3	12·8	11·3	9·35	7·81	6·25	4·11	2·37	1·21	0·584	0·352	0·216	0·115	0·072
4	14·9	13·3	11·1	9·49	7·78	5·39	3·36	1·92	1·06	0·711	0·484	0·297	0·207
5	16·7	15·1	12·8	11·1	9·24	6·63	4·35	2·67	1·61	1·15	0·831	0·554	0·412
6	18·5	16·8	14·4	12·6	10·6	7·84	5·35	3·45	2·20	1·64	1·24	0·872	0·676
7	20·3	18·5	16·0	14·1	12·0	9·04	6·35	4·25	2·83	2·17	1·69	1·24	0·989
8	22·0	20·1	17·5	15·5	13·4	10·2	7·34	5·07	3·49	2·73	2·18	1·65	1·34
9	23·6	21·7	19·0	16·9	14·7	11·4	8·34	5·90	4·17	3·33	2·70	2·09	1·73
10	25·2	23·2	20·5	18·3	16·0	12·5	9·34	6·74	4·87	3·94	3·25	2·56	2·16
11	26·8	24·7	21·9	19·7	17·3	13·7	10·3	7·58	5·58	4·57	3·82	3·05	2·60
12	28·3	26·2	23·3	21·0	18·5	14·8	11·3	8·44	6·30	5·23	4·40	3·57	3·07
13	29·8	27·7	24·7	22·4	19·8	16·0	12·3	9·30	7·04	5·89	5·01	4·11	3·57
14	31·3	29·1	26·1	23·7	21·1	17·1	13·3	10·2	7·79	6·57	5·63	4·66	4·07
15	32·8	30·6	27·5	25·0	22·3	18·2	14·3	11·0	8·55	7·26	6·26	5·23	4·60
16	34·3	32·0	28·8	26·3	23·5	19·4	15·3	11·9	9·31	7·96	6·91	5·81	5·14
17	35·7	33·4	30·2	27·6	24·8	20·5	16·3	12·8	10·1	8·67	7·56	6·41	5·70
18	37·2	34·8	31·5	28·9	26·0	21·6	17·3	13·7	10·9	9·39	8·23	7·01	6·26
19	38·6	36·2	32·9	30·1	27·2	22·7	18·3	14·6	11·7	10·1	8·91	7·63	6·84
20	40·0	37·6	34·2	31·4	28·4	23·8	19·3	15·5	12·4	10·9	9·59	8·26	7·43
21	41·4	38·9	35·5	32·7	29·6	24·9	20·3	16·3	13·2	11·6	10·3	8·90	8·03
22	42·8	40·3	36·8	33·9	30·8	26·0	21·3	17·2	14·0	12·3	11·0	9·54	8·64
23	44·2	41·6	38·1	35·2	32·0	27·1	22·3	18·1	14·8	13·1	11·7	10·2	9·26
24	45·6	43·0	39·4	36·4	33·2	28·2	23·3	19·0	15·7	13·8	12·4	10·9	9·89
25	46·9	44·3	40·6	37·7	34·4	29·3	24·3	19·9	16·5	14·6	13·1	11·5	10·5
26	48·3	45·6	41·9	38·9	35·6	30·4	25·3	20·8	17·3	15·4	13·8	12·2	11·2
27	49·6	47·0	43·2	40·1	36·7	31·5	26·3	21·7	18·1	16·2	14·6	12·9	11·8
28	51·0	48·3	44·5	41·3	37·9	32·6	27·3	22·7	18·9	16·9	15·3	13·6	12·5
29	52·3	49·6	45·7	42·6	39·1	33·7	28·3	23·6	19·8	17·7	16·0	14·3	13·1
30	53·7	50·9	47·0	43·8	40·3	34·8	29·3	24·5	20·6	18·5	16·8	15·0	13·8
40	66·8	63·7	59·3	55·8	51·8	45·6	39·3	33·7	29·1	26·5	24·4	22·2	20·7
50	79·5	76·2	71·4	67·5	63·2	56·3	49·3	42·9	37·7	34·8	32·4	29·7	28·0
60	92·0	88·4	83·3	79·1	74·4	67·0	59·3	52·3	46·5	43·2	40·5	37·5	35·5
70	104·2	100·4	95·0	90·5	85·5	77·6	69·3	61·7	55·3	51·7	48·8	45·4	43·3
80	116·3	112·3	106·6	101·9	96·6	88·1	79·3	71·1	64·3	60·4	57·2	53·5	51·2
90	128·3	124·1	118·1	113·1	107·6	98·6	89·3	80·6	73·3	69·1	65·6	61·8	59·2
100	140·2	135·8	129·6	124·3	118·5	109·1	99·3	90·1	82·4	77·9	74·2	70·1	67·3

Source: Catherine M. Thompson, *Table of percentage points of the χ^2 distribution*, Biometrika, Vol. 32 (1941), by permission of the author and publisher.

If $\sum O = \sum e = N$

$$\chi^2 = \sum_{j=1}^{k} \left(\frac{O_j^2}{e_j} \right) - N$$

- If $\chi^2 = 0$, observed and expected freq. agree exactly.
- If $\chi^2 > 0$, Do not agree exactly.
- The greater $\chi^2 \rightarrow$ the greater the discrepancy.

Sampling distribution of χ^2 :

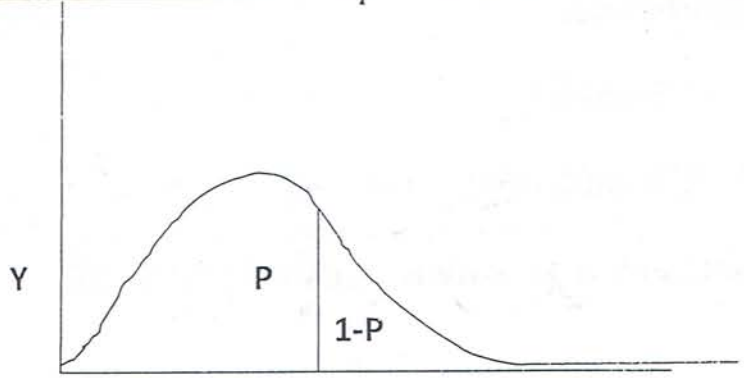
$$Y = Y_0 (\chi)^{v-2} e^{-\frac{1}{2}\chi^2}$$

Where the number of degree of freedom (v) is given by :

a. $v = k - 1$ (for hypothesises, where the expected freq. can be computed with out having to estimate pop. parameters from sample statistics). And K: no. of possible events.

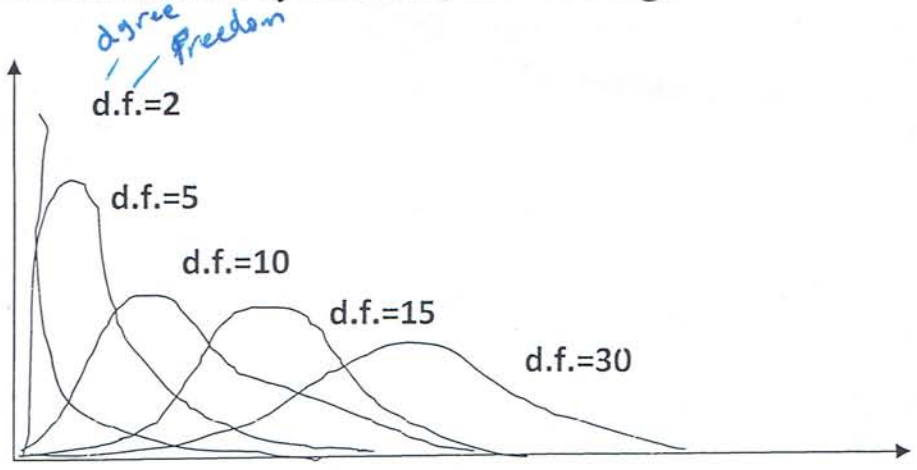
b. $v = k - 1 - m$ (if the expected freq. can be computed only by estimating (m) parameters).
m : no. of pop. parameters required to calculate the expected freq.

*** The critical values of χ_p^2 from the following table :**



Characteristic of the chi-square distribution :

1. It is not symmetric.
2. The shape of χ^2 depends on the degree of freedom.
3. As the no. of degree of freedom increases the χ^2 dist. Becomes more symmetric, as in the fig :



4. The values of χ^2 are nonnegative. That is the values of χ^2 are greater than or equal to 0.

Significance tests using χ^2 :

1. Expected freq. are computed on the basis of hypothesis, or theoretical distributions.

2. Determine the χ^2.

3. From table determine χ_p^2 (the critical values)

$\chi_{0.95}^2 \Rightarrow 0.05$ significance level

$\chi_{0.99}^2 \Rightarrow 0.01$ significance level

$\chi_p^2 \Rightarrow 1-p$ significance level

4. Compare the calculated χ^2 with the critical values χ_p^2 .

- If $\chi^2 > \chi_p^2 \rightarrow$ hypo. Or theo. Prob. dist. Is rejected at (1-p) sig. level.

Where :

P : prob. of being correct.

1-p : prob. of being error (sig. level)

5. If χ^2 is too small (too close to zero).

$\chi^2 > \chi_{0.95}^2 \rightarrow$ suspicious data.

Test if $\chi^2 < \chi_{0.95}^2$ or $\chi_{0.01}^2$, if so can not depend on data.

Example 1 : (test of hypothesis)

In 200 tosses of a coin, 115 heads, 85 tails, were observed. Test fairness of coin at a sig. level of

a) 0.05 b) 0.01

Soln. : the observed freq. of heads, and tails

O_1=115 , O_2=85

Expected freq. of heads and tails if the coin is fair are

e_1=100 , e_2=100

$$\therefore \quad \chi^2 = \frac{(115-100)^2}{100} + \frac{(85-100)^2}{100} = 4.5$$

ν=k-1 = 2-1 =1

$$\therefore \quad \left.\begin{array}{l} at \ \ \nu = 1 \rightarrow \chi_{0.95}^2 = 3.84 \\ at \ \ \nu = 1 \rightarrow \chi_{0.99}^2 = 6.63 \end{array}\right\} \ from\ table$$

$\therefore$ a) $\chi^2 > \chi_{0.95}^2 \rightarrow$ reject hypothesis

b) $\chi^2 \leq \chi^2_{0.99} \rightarrow$ accept hypothesis

Example 2 : (test of hypothesis)

In 120 throws of a die the following data were observed :

events :	1	2	3	4	5	6
obs. freq.	25	17	15	23	24	16

Test fairness of die at a sig. level of 0.05.

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Soln. :

If the die is fair, then the expected freq. are :

$$e_1 = e_2 = e_3 = \cdots = e_6 = \frac{\sum O_f}{6} = \frac{120}{6} = 20$$

$$\therefore \quad \chi^2 = \sum \frac{(O_f - e_f)^2}{e_f} = \frac{(25-20)^2}{20} + \frac{(17-20)^2}{20} + \cdots$$

$$\therefore \quad \chi^2 = 5.0$$

ν=k-1 = 6-1 =5

$at \quad \nu = 5 \rightarrow \chi^2_{0.95} = 11.1$ (from table)

$\therefore \quad \chi^2 < \chi^2_{0.95} \rightarrow$ accept the hypothesis but test of data is required.

$at \quad \nu = 5 \rightarrow \chi^2_{0.05} = 1.15$ (from table)

$\chi^2 > \chi^2_{0.95}$, $\therefore$ data is acceptable, and the die is fair at 0.05 sig. level

Example 3 : (test of hypothesis)

A survey of 320 families with 5 children revealed the dist. below; is the result consistent with the hypothesis that the male and female births are equally probable:

No. of boys & girls	5 boys 0 girls	4 1	3 2	2 3	1 4	0 5	Total
No. of families	18	56	110	88	40	8	320
Exp.	10	50	100	100	50	10	320

Soln. :

If $p = q = \frac{1}{2}$, then :

$$pr\binom{5\ boys}{0\ girls} = \frac{5!}{5!\,(5-5)!}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = \frac{1}{32}$$

$$pr\binom{4}{1} = \frac{5}{32} \quad , \quad pr\binom{3}{2} = \frac{10}{32} \quad , \quad pr\binom{2}{3} = \frac{10}{32}$$

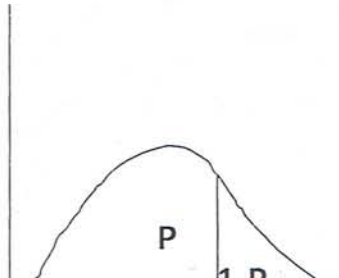
$$pr\binom{1}{4} = \frac{5}{32} \quad , \quad pr\binom{0}{5} = \frac{1}{32}$$

Then the expected no. of families : 10,50,100,100,50,10, hence

$$\chi^2 = \sum \frac{(O-e)^2}{e} = 12.0$$

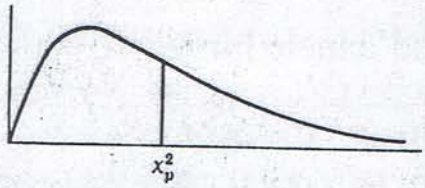
Since $\chi^2_{0.95} = 11.1$ at ν=k-1 = 6-1 =5

We can reject the hypo. at 0.05 significance level, (We conclude that male and female births are not equally).



Appendix IV

PERCENTILE VALUES (χ_p^2)
for
THE CHI-SQUARE DISTRIBUTION
with ν degrees of freedom
(shaded area = p)



ν	$\chi^2_{0.995}$	$\chi^2_{0.99}$	$\chi^2_{0.975}$	$\chi^2_{0.95}$	$\chi^2_{0.90}$	$\chi^2_{0.75}$	$\chi^2_{0.50}$	$\chi^2_{0.25}$	$\chi^2_{0.10}$	$\chi^2_{0.05}$	$\chi^2_{0.025}$	$\chi^2_{0.01}$	$\chi^2_{0.005}$
1	7·88	6·63	5·02	3·84	2·71	1·32	0·455	0·102	0·0158	0·0039	0·0010	0·0002	0·0000
2	10·6	9·21	7·38	5·99	4·61	2·77	1·39	0·575	0·211	0·103	0·0506	0·0201	0·0100
3	12·8	11·3	9·35	7·81	6·25	4·11	2·37	1·21	0·584	0·352	0·216	0·115	0·072
4	14·9	13·3	11·1	9·49	7·78	5·39	3·36	1·92	1·06	0·711	0·484	0·297	0·207
5	16·7	15·1	12·8	11·1	9·24	6·63	4·35	2·67	1·61	1·15	0·831	0·554	0·412
6	18·5	16·8	14·4	12·6	10·6	7·84	5·35	3·45	2·20	1·64	1·24	0·872	0·676
7	20·3	18·5	16·0	14·1	12·0	9·04	6·35	4·25	2·83	2·17	1·69	1·24	0·989
8	22·0	20·1	17·5	15·5	13·4	10·2	7·34	5·07	3·49	2·73	2·18	1·65	1·34
9	23·6	21·7	19·0	16·9	14·7	11·4	8·34	5·90	4·17	3·33	2·70	2·09	1·73
10	25·2	23·2	20·5	18·3	16·0	12·5	9·34	6·74	4·87	3·94	3·25	2·56	2·16
11	26·8	24·7	21·9	19·7	17·3	13·7	10·3	7·58	5·58	4·57	3·82	3·05	2·60
12	28·3	26·2	23·3	21·0	18·5	14·8	11·3	8·44	6·30	5·23	4·40	3·57	3·07
13	29·8	27·7	24·7	22·4	19·8	16·0	12·3	9·30	7·04	5·89	5·01	4·11	3·57
14	31·3	29·1	26·1	23·7	21·1	17·1	13·3	10·2	7·79	6·57	5·63	4·66	4·07
15	32·8	30·6	27·5	25·0	22·3	18·2	14·3	11·0	8·55	7·26	6·26	5·23	4·60
16	34·3	32·0	28·8	26·3	23·5	19·4	15·3	11·9	9·31	7·96	6·91	5·81	5·14
17	35·7	33·4	30·2	27·6	24·8	20·5	16·3	12·8	10·1	8·67	7·56	6·41	5·70
18	37·2	34·8	31·5	28·9	26·0	21·6	17·3	13·7	10·9	9·39	8·23	7·01	6·26
19	38·6	36·2	32·9	30·1	27·2	22·7	18·3	14·6	11·7	10·1	8·91	7·63	6·84
20	40·0	37·6	34·2	31·4	28·4	23·8	19·3	15·5	12·4	10·9	9·59	8·26	7·43
21	41·4	38·9	35·5	32·7	29·6	24·9	20·3	16·3	13·2	11·6	10·3	8·90	8·03
22	42·8	40·3	36·8	33·9	30·8	26·0	21·3	17·2	14·0	12·3	11·0	9·54	8·64
23	44·2	41·6	38·1	35·2	32·0	27·1	22·3	18·1	14·8	13·1	11·7	10·2	9·26
24	45·6	43·0	39·4	36·4	33·2	28·2	23·3	19·0	15·7	13·8	12·4	10·9	9·89
25	46·9	44·3	40·6	37·7	34·4	29·3	24·3	19·9	16·5	14·6	13·1	11·5	10·5
26	48·3	45·6	41·9	38·9	35·6	30·4	25·3	20·8	17·3	15·4	13·8	12·2	11·2
27	49·6	47·0	43·2	40·1	36·7	31·5	26·3	21·7	18·1	16·2	14·6	12·9	11·8
28	51·0	48·3	44·5	41·3	37·9	32·6	27·3	22·7	18·9	16·9	15·3	13·6	12·5
29	52·3	49·6	45·7	42·6	39·1	33·7	28·3	23·6	19·8	17·7	16·0	14·3	13·1
30	53·7	50·9	47·0	43·8	40·3	34·8	29·3	24·5	20·6	18·5	16·8	15·0	13·8
40	66·8	63·7	59·3	55·8	51·8	45·6	39·3	33·7	29·1	26·5	24·4	22·2	20·7
50	79·5	76·2	71·4	67·5	63·2	56·3	49·3	42·9	37·7	34·8	32·4	29·7	28·0
60	92·0	88·4	83·3	79·1	74·4	67·0	59·3	52·3	46·5	43·2	40·5	37·5	35·5
70	104·2	100·4	95·0	90·5	85·5	77·6	69·3	61·7	55·3	51·7	48·8	45·4	43·3
80	116·3	112·3	106·6	101·9	96·6	88·1	79·3	71·1	64·3	60·4	57·2	53·5	51·2
90	128·3	124·1	118·1	113·1	107·6	98·6	89·3	80·6	73·3	69·1	65·6	61·8	59·2
100	140·2	135·8	129·6	124·3	118·5	109·1	99·3	90·1	82·4	77·9	74·2	70·1	67·3

Source: Catherine M. Thompson, *Table of percentage points of the χ^2 distribution*, Biometrika, Vol. 32 (1941), by permission of the author and publisher.

χ^2 test for goodness of fit :

χ^2 test can be used to determine how well theoretical distributions, such as (normal, binomial, poisson), fit distributions which obtained from sample data.

Example 3 : (test goodness of fit of Binomial dist.)

(5) pennies were tossed 1000 times, and at each toss the no. of heads was observed. Determine the goodness of fit of binomial dist. Of the following data at a sig. level of 0.05

No. of heads:	0	1	2	3	4	5
O_f :	38	144	342	287	164	25

$\sum O_f = 1000$

Soln. :

The expected freq. is obtained from the binomial dist. :

$$p(x) = \frac{N!}{X!\,(N-X)!} p^X q^{N-X}$$

p : is obtained as fellows :

the true mean $\mu = \frac{\sum f_i x_i}{\sum f_i} = \frac{0*38+1*144+2*342+\cdots}{1000} = 2.47$

$\mu_{true} = \mu_{binomial} = N * P = 2.47 = 5 * p \rightarrow p = 0.494$

p+q=1 → q = 0.506

∴ The binomial dist. Eqn. is given by :

$$p(x) = \underset{5}{C}\, x\,(0.494)^x (0.506)^{5-x}$$

Or

$$p(x) = \frac{5!}{x!\,(5-x)!}(0.494)^x (0.506)^{5-x}$$

$$\sum O_f = \sum e_f = N = 1000$$

No. of heads x :	0	1	2	3	4	5
Pr(x) :	0.0332	0.1619	0.13162	0.3087	0.1507	0.0294
e_f :	33.2	161.9	316.2	308.7	150.7	29.4
O_f :	38	144	342	287	164	25

$$\chi^2 = \sum \frac{(O_f - e_f)^2}{e_f} = 7.54$$

ν=k-1-m = 6-1-1 =4

$at \quad \nu = 4 \rightarrow \chi^2_{0.95} = 9.49$ (from table)

$\chi^2 < \chi^2_{0.95} \rightarrow$ thefit is good.

$at \quad \nu = 4 \rightarrow \chi^2_{0.05} = 0.711$ (from table)

$\chi^2 > \chi^2_{0.05} \rightarrow$ can depend on data.

Example 4 : (test goodness of fit of normal dist.)

The distribution of masses with $\mu = 67.45$ kg and $= 2.92$, were observed as fellows :

Mass (class limits)	Observed freq. O_f
60-62	5
63-65	18
66-68	42
69-71	27
72-74	8
	$\sum O_f = 100$

Determine the goodness of fit of normal dist. At a sig. level of 0.05.

Soln. :

The class boundaries converted to standard scores $(Z)^s$, then the area of each class is obtained as fraction between Z_1 and Z_2 :

Boundaries	Standard scroses Z_1 Z_2	Fraction
59.5 – 62.5	-2.72 to -1.7	0.0413
62.5 – 65.5	-1.7 to -0.67	0.2078 68
65.5 – 68.5	-0.67 to 0.36	0.3892
68.5 – 71.5	0.36 1.39	0.2771
71.5 – 74.5	1.39 2.41	0.0743

Then each fraction is converted to e_f by multiplying each fraction by $\sum O_f$

Event :	1	2	3	4	5
O_f :	5	8	42	27	8
e_f :	4.13	20.68	38.92	27.71	7.43

$$\left(\sum O_f = \sum e_f = N\right)$$

$$\therefore \quad \chi^2 = \sum \frac{(O_f - e_f)^2}{e_f} = 0.959$$

ν=k-1-m =6-1-2 =2

$$\therefore \quad at \quad \nu = 2 \rightarrow \chi^2_{0.95} = 5.99 \quad \text{(from table)}$$

$\chi^2 < \chi^2_{0.95} \rightarrow$ the fit is good at a sig. level of 0.05.

$$at \quad \nu = 2 \rightarrow \chi^2_{0.05} = 0.103 \quad \text{(from table)}$$

$\chi^2 > \chi^2_{0.05} \rightarrow$ we can depend on data.

مطلوب حل

Tutorial sheet (6)

The Chi – square test

Q. 1) The number of book borrowed from a public library during a particular week is given below, Test the hypothesis that the number of books borrowed does not depend on the day of the week, using a significance level of

a) 0.05 , b) 0.01

	Mon.	Tues.	Wed.	Thur.	Fri.
Number of book borrowed	135	108	120	114	146

Q. 2) Two hundred bolts were selected at random from the production of each (4) machines. The numbers of defective bolts found were 2 , 9 , 10 , 3 . determine whether there is no a significant difference between the machines using a significance level of 0.05.

Q. 3) Determine the goodness of fit of a binomial distribution to the following data, using a significance level of 0.05. Is the fit :too good" :

X :	0	1	2	3	4
f :	30	62	46	10	2

Where X : is no. of heads in tossing 4 coins 150 throws.

Q. 4) Determine the goodness of fit of normal distribution to the following data, using a significance level of 0.05. Is the fit is "too good".

Class limit	f
93-97	2
98-102	5

103-107	12
108-112	17
113-117	14
118-122	6
123-127	3
128-132	1
	Total = 60

Q. 5) Determine the goodness of fit of poisson distribution to the following data, using a significance level of 0.05.

X :	0	1	2	3	4
f :	109	65	22	3	1

Where X : no. of heads in tossing 4 coins 200 throws :

Chi – Square test for independence :

To test the association between (or independence) two variables in a table, we use the steps that follow :

1. Determine the null and alternative hypotheses .

Null hypo. = H_0 : The row variable and column variable are independent.

Alter. Hypo. = H_1 : The row and column variables are dependent

2. Determine the critical value at a specified level of significance (1-p).

χ^2_p calculated from the table of χ^2_p at a degree of freedom $\nu = (r-1)(c-1)$

Where : r : no. of rows

c : no. of columns

3. calculate the expected frequency for each cell in the table

4. compute the test statistic (χ^2).

5. compare the critical value to the test statistic.

If $\chi^2 > \chi^2_p$ →reject the null hypothesis

Example : (test the independence)

The following table contains observed frequency for two variables. X and Y.

	X_1	X_2	X_3	total
y_1	87	74	34	195
y_2	12	32	18	62
total	99	106	52	257

a) compute the value of χ^2

b) Test the hypothesis that X and Y are independent at the 0.05 of significance level (null hypothesis).

soln. : (R Total)(C Total) / Table Total

1. $expected\ freq. = \frac{(row\ total)(column\ total)}{table\ total}$

	X_1	X_2	X_3
y_1	75.12	80.43	39.46
y_2	23.88	25.57	12.54

For y_1 :

$$x_1 = \frac{195 * 99}{257} = 75.12$$

$$x_2 = \frac{195 * 106}{257} = 80.43$$

$$x_3 = \frac{195 * 52}{257} = 39.46$$

For y_2 :

$$x_1 = \frac{62 * 99}{257} = 23.88$$

$$x_2 = \frac{62 * 106}{257} = 25.57$$

$$x_3 = \frac{62 * 52}{257} = 12.54$$

2. calculate χ^2

O	e	$(O-e)^2/e$
87	75.12	1.878
74	80.43	0.514
34	39.46	0.755
12	23.88	5.91
32	25.57	1.62
18	12.54	2.377

$\chi^2 = 13.054$

Compute $\chi^2_{0.95}$ at $\nu = (r-1)(c-1) = (2-1)(3-1) = 2$

$\chi^2_{0.95} = 0.103$ at $\nu = 2$

5.99

4. compare χ^2 with $\chi^2_{0.95}$

$\chi^2 > \chi^2_p$ reject the null hypothesis H_0

(H_0) → the row and column variables are dependent. (H_1)

Ex :

	X_1	X_2	X_3
y_1	34	43	52
y_2	18	21	17

Test the null hy pothesis (H_0) at the significance level of 0.05

Chapter (7)

Curve fitting and method of Least – squares

* Relation ship between variables :

Very often in practice a relationship is found to exist between two (or more) variables. For example circumferences of circles depend on their radii; and the pressure of a given mass of gas depends on its temp. and volume.

It is frequently desirable to express this relationship in mathematical form by determining an equation connecting the variables.

Curve fitting procedure :

1. Plot set of data points (x,y).
2. Suggest a form of relation defining y=f(x)
   From : a. Theoretical considerations.
   b. observation of the trend of data points.
3. Evalute constants in the suggested function, so that the deviations of data points from the function are minimized.
4. Calculate statistical measures of the degree of fit.
5. Others functions may be proposed, and procedure is repeated.

Method of Least Squares :

The simplest situation is a linear or straight – line relation between a single input and the response :

$$E(y) = \alpha + \beta x$$

Where α and β are constants parameters that we want to estimate, (regression coefficients). For a sample of n pairs of data (x_i , y_i) we calculate a , for α and b for β .

If at $x = x_i$, $\hat{y}_i$ is the estimated value of (y) , we have the fitted regression line :

$$\hat{y}_i = a + bx_i$$

Let $e_i = y_i - \hat{y}_i$ be the deviation in the Y – direction of any data pt^s. from the fitted regression line. Then the estimates a and b are chosen so that the sum of the squares of deviations of all the $pt^s.\sum e_i^2$ is smaller than for any other choice of a and b . so that :

$\sum e_i^2 = \sum(y_i - \hat{y}_i)^2$ has a min. value.

This is called the **method of Least Squares** and the resulting eqn. Called the **regression line** of y on x , where y is the response (dependent) and x is the input (independent variable).

If the estimated eqn. $\hat{y} = a + bx$ then $e_i = y_i - (a + bx)$ these deviation called **residuals**

$$e_i^2 = [y_i - (a + bx)]^2$$

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - (a + bx)]^2$$

This sum of the squares of the deviations or errors or residuals for all n pt^s. is abbreviated as SSE. So the principle of L.S.M. is to minimize

$$\text{SSE} = \sum e_i^2 = \sum(y_i - \hat{y}_i)^2 = \sum[y_i - (a + bx_i)]^2$$

To minimize a quantity we take the derivative with respect to the independent variable and set it equal to zero.

$$\frac{\partial}{\partial a}(SSE) = \frac{\partial}{\partial a}\sum[y_i - (a + bx_i)]^2$$

$$= -2[\sum y_i - n\,a - b\,\sum x_i] = 0 \quad \ldots\ldots(1)$$

And

$$\frac{\partial}{\partial b}(SSE) = \frac{\partial}{\partial b}\sum[y_i - (a + bx_i)]^2 \Rightarrow \sum 2[y_i - (a+bx_i)] * x_i$$

$$= -2\left[\sum x_i y_i - a\sum x_i - b\sum x_i^2\right] = 0 \quad \ldots\ldots (2)$$

Eqn^s (1) and (2) are called the least squares eqn^s. (or normal equations).

Eqn. (1) and (2) can be solved simultaneously, the results are :

$$\frac{S_{xy}}{S_{xx}} \quad : \quad b = \frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n}[\sum x_i]^2} \quad \text{---(3)}$$

$$b = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

$$a = \frac{\sum y_i - b\sum x_i}{n} = \bar{y} - b\bar{x} \quad \ldots\ldots (4)$$

Then we have :

The sum of squares for $x = S_{xx} = \sum(x_i - \bar{x})^2$

1. $S_{xx} = \sum x_i^2 - \frac{1}{n}[\sum x_i]^2 \quad \ldots\ldots (5)$ ✓

2. $S_{yy} = \sum(y_i - \bar{y})^2 = \sum y_i^2 - \frac{1}{n}[\sum y_i]^2 \quad \ldots\ldots (6)$ ✓

3. $S_{x,y} = \sum(x - \bar{x})(y - \bar{y}) = \sum x_i y_i - \frac{1}{n}[\sum x_i][\sum y_i] \quad \ldots (7)$ ✓

Eqn^s (3) and (4) can be written compactly as :

$$b = \frac{S_{x,y}}{S_{xx}} \quad \ldots\ldots (8)$$ ✓

And

$$a = \bar{y} - b\bar{x} \quad \ldots\ldots (9)$$ ✓

If we subst. in the eqn. (9)

$$\hat{y} = a + b\,x_i$$

We get

$$(\hat{y}_i - \bar{y}) = b(x_i - \bar{x})$$

This is indicates that the best-fit line passes through the pt. $(\bar{x}, \bar{y})$, which is called the centroidal pt. and is the centre of mass of the data pt^s.

Example 1)

Data for simple linear regression :

x :	0	1	2	3	4	5	6	7	8	9	10	11	12
y :	3.85	0.03	3.50	6.13	4.07	7.07	8.66	11.65	15.23	12.29	14.74	16.02	16.86

Soln. :

N = 13 , $\sum x_i = 78$, $\sum y_i = 120.1$

$$\sum x_i^2 = 650 \quad , \quad \sum y_i^2 = 1483.0828 \quad , \quad \sum x_i y_i = 968.95$$

The centroidal pt. $(\bar{x}, \bar{y}) = (6\,, 9.23846)$

$$S_{xx} = \sum x_i^2 - \frac{1}{n}[\sum x_i]^2 = 650 - \frac{1}{13}(78)^2$$

$$S_{xx} = 182$$

$$S_{yy} = \sum y_i^2 - \frac{1}{n}\left[\sum y_i\right]^2 = 1483.08 - \frac{1}{13}(120.1)^2$$

$$S_{yy} = 373.5436$$

$$S_{x,y} = \sum x_i y_i - \frac{1}{n}\left[\sum x_i\right]\left[\sum y_i\right]$$

$$= 968.95 - \frac{1}{3}(78)(120.1)$$

$$S_{x,y} = 248.35$$

$$b = \frac{S_{xy}}{S_{xx}} = \frac{284.35}{182} = 1.36456$$

$$a = \bar{y} - b\bar{x} = 9.23846 - (1.36456)(6) = 1.0511$$

$\therefore$ the best – fit regression eqn. Is

$$y = 1.0511 + 1.36456x$$

Variance of experimentel pt^s. around the line :

This must be found from the residuals,

$$e_i = y_i - \hat{y} = y_i - (a + bx_i) = y_i - a - bx_i$$

$$SSE = \sum (y_i - a - bx_i)^2$$

Since $a = \bar{y} - b\bar{x}$ — (9)

$$\bar{y} = a + b\bar{x} \rightarrow a = \bar{y} - b\bar{x}$$

$$\therefore \quad SSE = \sum [(y_i - \bar{y}) - b(x_i - \bar{x})]^2$$

$$= \sum (y_i - \bar{y})^2 - 2b \sum (x_i - \bar{x})(y_i - \bar{y}) + b^2 \sum (x_i - \bar{x})^2$$

$$SSE = S_{yy} - 2bS_{xy} + b^2 S_{xx}$$

$$\because \quad b = \frac{S_{xy}}{S_{xx}}$$

$$\therefore \quad SSE = S_{yy} - 2bS_{xy} + \frac{(S_{xy})(S_{xy})}{(S_{xx})(S_{xx})} \cdot S_{xx}$$

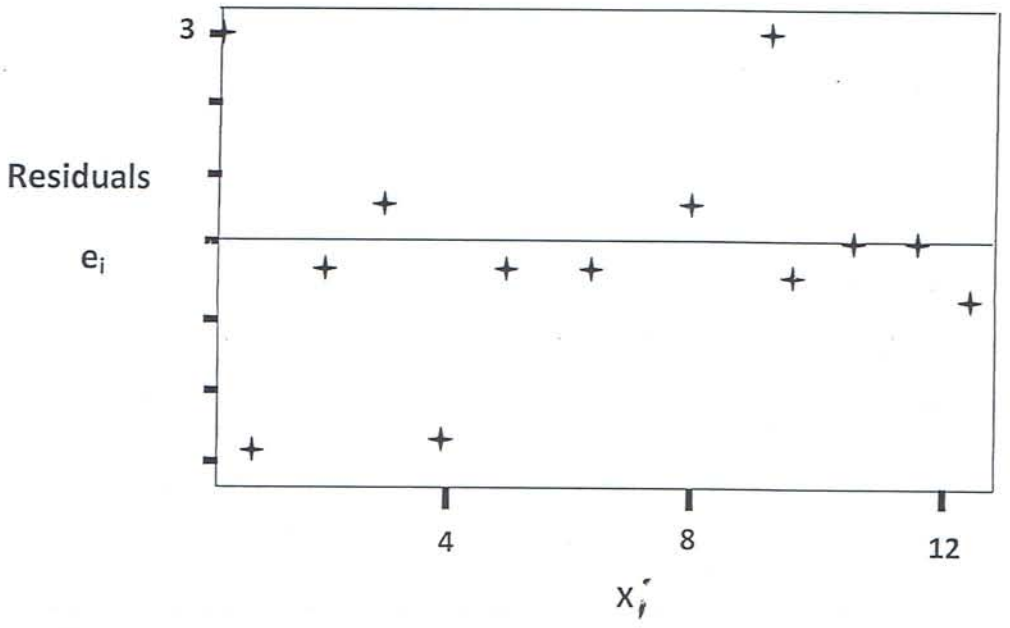
$$= S_{yy} - 2bS_{xy} + bS_{xy}$$

$$\therefore \quad SSE = S_{yy} - bS_{xy}$$

The estimate of the variance of the pt^s. about the line is :

$$S^2_{y\backslash x} = \frac{SSE}{n-2} = \frac{S_{yy} - bS_{xy}}{n-2}$$

This quantity is a measure of the scatter of experimental pt^s. around the line.



Example 2)

For the data of example (1) calculate the standard deviation of pt^s. about the regression line, then plot residuals against x.

$S^2_{y/x} = \frac{SSE}{n-2} = \frac{S_{yy} - b\,S_{xy}}{n-2}$

$S^2_{y/x}$ = variance

$\sqrt{S^2_{y/x}}$ = standard div

$S_{y/x} = \sqrt{\frac{SSE}{n-2}}$

Soln. :

$\hat{y} = a + bx$

$\hat{y} = 1.0511 + 1.36456\,x$ (from ex. (1))

Residual $e_i = y_i - \hat{y}$

$$S_{y\backslash x} = \sqrt{\frac{SSE}{n-2}}$$

x_i	y_i	$\hat{y}$	e_i
0	3.85	1.05	+ 2.8
1	0.03	2.41	- 2.38
2	3.5	3.77	- 0.27
3	6.13	5.13	+ 1.0
4	4.07	6.49	- 2.42

$$b = \frac{S_{xy}}{S_{xx}} = \frac{284.35}{182} = 1.36456$$

$$a = \bar{y} - b\bar{x} = 9.23846 - (1.36456)(6) = 1.0511$$

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$$SSE = \sum (y_i - a - bx_i)^2$$

Since $a = \bar{y} - b\bar{x}$ — (9)

$$\bar{y} = a + b\bar{x} \rightarrow a = \bar{y} - b\bar{x}$$

$$\therefore \quad SSE = \sum [(y_i - \bar{y}) - b(x_i - \bar{x})]^2$$

$$= \sum (y_i - \bar{y})^2 - 2b \sum (x_i - \bar{x})(y_i - \bar{y}) + b^2 \sum (x_i - \bar{x})^2$$

$$SSE = S_{yy} - 2bS_{xy} + b^2 S_{xx}$$

$$\because \quad b = \frac{S_{xy}}{S_{xx}}$$

$$\therefore \quad SSE = S_{yy} - 2bS_{xy} + \frac{(S_{xy})(S_{xy})}{(S_{xx})(S_{xx})} \cdot S_{xx}$$

$$= S_{yy} - 2bS_{xy} + bS_{xy}$$

$$\therefore \quad SSE = S_{yy} - bS_{xy}$$

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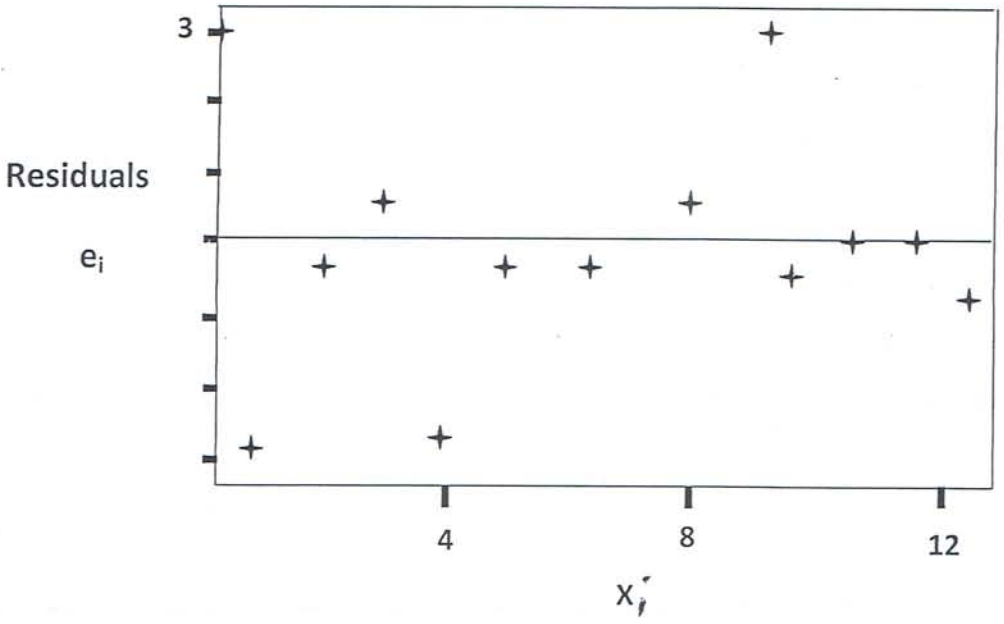

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$S^2_{y/x}$ = variance

$\sqrt{S^2_{y/x}}$ = standard div

$S_{y/x} = \sqrt{\frac{SSE}{n-2}}$

Soln. :

$\hat{y} = a + bx$

$\hat{y} = 1.0511 + 1.36456\,x$ (from ex. (1))

Residual $e_i = y_i - \hat{y}$

$$S_{y\backslash x} = \sqrt{\frac{SSE}{n-2}}$$

x_i	y_i	$\hat{y}$	e_i
0	3.85	1.05	+ 2.8
1	0.03	2.41	- 2.38
2	3.5	3.77	- 0.27
3	6.13	5.13	+ 1.0
4	4.07	6.49	- 2.42

5	7.07	7.85	- 0.78
6	8.66	9.21	- 0.55
7	11.65	10.57	+ 1.08
8	15.23	11.93	+ 3.3
9	12.29	13.56	- 1.27
10	14.74	14.65	+ 0.09
11	16.02	16.01	+ 0.01
12	16.86	17.37	- 0.51

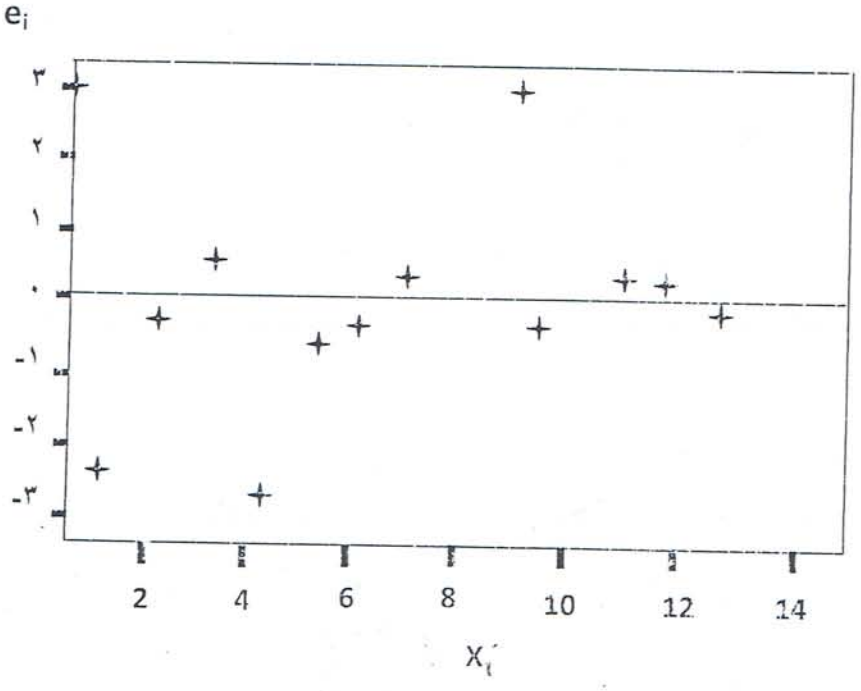
$$SSE = S_{yy} - bS_{xy}$$

$$= 373.5436 - 1.36456(248.35)$$

$$= 34.655$$

$$S_{y\backslash x} = \sqrt{\frac{SSE}{n-2}} = \sqrt{\frac{34.655}{13-2}}$$

$$S_{y\backslash x} = 1.775$$



Relation forms :

1. Straight line through origin $\rightarrow y = m\,x$

2. Other single constant forms are all transformable to straight line through origin.

$y = me^x$ define $Y = y$, $X = e^x$ $\rightarrow Y = mX$

3. straight line $\rightarrow y = a_0 + a_1 x$

4. straight line forms :

- Two – constant relations may be transformed to straight line :

$y = ae^{bx}$ exponential $\rightarrow \ln y = \ln a + bx$

$y = ax^b$ power $\rightarrow \ln y = \ln a + b \ln x$

$y = \frac{1}{a_0 + a_1 x}$ hyporbola $\rightarrow \frac{1}{y} = a_0 + a_1 x$

5. Higher constant relations :

These may be polynomials or other forms that contain more than two constants. It is not usually possible to transform them in to st. line forms.

e.g. /

$y = a_0 + a_1 x + a_2 x^2$ 2^{nd} degree polynomial

$y = a + be^{cx}$ modified exponential

$\frac{a}{y} = b + cx$ $\rightarrow \frac{1}{y} = \frac{b}{a} + \frac{c}{a} x$ (st. line form).

Example)

Transform $p = exp.(a + \frac{1}{bx})$ and define parameters :

$$\ln p = a + \frac{1}{bx} \qquad \therefore \; Y = \ln p \quad , X = \frac{1}{x}$$

$$A_0 = a \quad , A_1 = \frac{1}{b}$$

Example 1)

Fit the following data to a straight line :

Time :	0	3	5	8	10	12
Speed :	0.28	11.2	18.3	29.1	36.2	43.4

Solu. :

$$N = 6 \quad , \sum x_i = 38 \quad , \sum y_i = 138.48$$

$$\sum x_i^2 = 342 \quad , \quad \sum y_i^2 = 4501.2 \quad , \quad \sum x_i y_i = 1240.7$$

$$\bar{y} = 23.08 \quad , \quad \bar{x} = 6.33$$

$$b = \frac{S_{xy}}{S_{xx}} \quad , \quad a = \bar{y} - b\bar{x}$$

$$S_{xy} = \sum xy - \frac{1}{n}\left(\sum x\right)\left(\sum y\right)$$

$$= 1240.7 - \tfrac{1}{6}(38)(138.48) = 363.7$$

$$S_{xx} = \sum x_i^2 - \frac{1}{n}\left(\sum x\right)^2 = 342 - \frac{1}{6}(38)^2 = 101.3$$

$$b = \frac{363.7}{101.3} = 3.59 \quad , \qquad a = 23.08 - 3.59 * 6.33 = 0.34$$

$b = 3.59 \quad , \quad a = 0.34$

The relation :

$S = 0.34 + 3.59t$

Example 2)

Fit the following data to $p = exp.\left[a + \frac{1}{bx}\right]$

x :	22	23	24	25	26
p :	0.368	0.223	0.134	0.082	0.05

Solu. :

Transform relation to st. line form :

$$\ln p = a + \frac{1}{bx} \quad , Y = \ln p \qquad , \; X = \frac{1}{x}$$

$$A_0 = a \quad , A_1 = \frac{1}{b}$$

$$N = 5 \quad , \sum x_i = 0.2091 \quad , \sum y_i = -10.007$$

$$\sum x_i^2 = 8.77X10^{-3} \qquad , \; \sum xy = -0.4097$$

$$S_{xy} = \sum xy - \frac{1}{n}\left(\sum x\right)\left(\sum y\right)$$

$$= -0.4097 - \tfrac{1}{5}(-0.2091)(-10.007)$$

$$= 8.79X10^{-3}$$

$$S_{xx} = \sum x_i^2 - \frac{1}{n}\left(\sum x\right)^2$$

$$= 8.77X10^{-3} - \frac{1}{5}(0.2091)^2$$

$$= 2.54X10^{-5}$$

$$A_1 = \frac{S_{xy}}{S_{xx}} = \frac{8.79X10^{-3}}{2.54X10^{-5}} = 3.46X10^2 = 346$$

$$\bar{x} = 0.0418 \quad , \quad \bar{y} = -2.0014$$

$$A_0 = -2.0014 - 346 * 0.0418 = -16.5$$

$$A_1 = 346 \quad , \quad A_0 = -16.5$$

$$\therefore \quad a = A_0 = -16.5$$

$$A_1 = \frac{1}{b} \quad \rightarrow \quad b = 2.89X10^{-3}$$

$$\therefore \quad p = exp.\left[-16.5 + \frac{1}{2.89X10^{-3}x}\right]$$

Example 3)

Fit the data in (1) above to a st. line that passes through the origin.

$$y = m\,x$$

$$\frac{\partial \sum(y_i - \hat{y})^2}{\partial m} = 0 \quad \rightarrow \quad \frac{\partial \sum(y_i - m\,x_i)^2}{\partial\, m} = 0$$

$$-2\sum(y_i - m\,x_i)\,(x_i) = 0 \quad \rightarrow \quad \sum y_i x_i = m\sum x_i^2$$

$$\therefore \quad m = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{1240.7}{342} = 3.628$$

$$\therefore \quad Relation\ is \quad y = 3.628\,x$$

The Least square parabola :

The least square parabola approximating the set of pt^s. (X_1,Y_1) , (X_2,Y_2), … , (X_n,Y_n) has the eqn.

$$Y = a_0 + a_1X + a_2X^2$$

Where the constants a_0 , a_1 and a_2 are determined by solving simultaneously the eqn^s.

$$\left\{ \begin{array}{c} \sum Y = a_0N + a_1\sum X + a_2\sum X^2 \\ \sum XY = a_0\sum X + a_1\sum X^2 + a_2\sum X^3 \\ \sum X^2Y = a_0\sum X^2 + a_1\sum X^3 + a_2\sum X^4 \end{array} \right\}$$

Called the normal eqn^s. for the least square parabola.

* this technique can be extended to obtain normal eqn^s. for **cubic** and **quartic** curves.

Example 4)

Fit the following data to an eqn. of the form

$y = a_0 + a_1x + a_2x^2$, by the method of least squares.

X	Y	X_{new}	X^2	X^3	X^4	XY	X^2Y
10	157	-5	25	.	625	-785	3925
20	179	-3	9	.	81	-537	1611
30	210	-1	1	.	1	-210	210
40	252	1	1	.	1	252	252
50	302	3	9	.	81	906	2718
60	361	5	25	.	625	1805	9025
	1461	$\sum X = 0$	70	0	1414	1431	17741

Using least square method to obtain the normal eqn^{s}. for 2nd. Order polynomial (parabola).

$$\sum Y = a_0 N + a_1 \sum X + a_2 \sum X^2$$

$$\sum XY = a_0 \sum X + a_1 \sum X^2 + a_2 \sum X^3$$

$$\sum X^2 Y = a_0 \sum X^2 + a_1 \sum X^3 + a_2 \sum X^4$$

Subst to obtain :

$1461 = 6\ a_0 + 70\ a_2$ .....(1)

$1431 = 70\ a_1$ .....(2) $\rightarrow$ $a_1 = 20.44$

$17741 = 70\ a_0 + 1414\ a_2$ .....(3)

Eqn. (1) * 70 – eqn. (3) * 6

$12270 = 420\ a_0 + 4900\ a_2$

$-$

$106446 = 420\ a_0 + 8484\ a_2$

$-4176 = -3584\ a_2$ $\rightarrow$ $a_2 = 1.165$

$\therefore$ $a_0 = 229.9$

$y = 229.9 + 20.44\,x + 1.165\,x^2$

Correlation :

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التقارب - التوابع

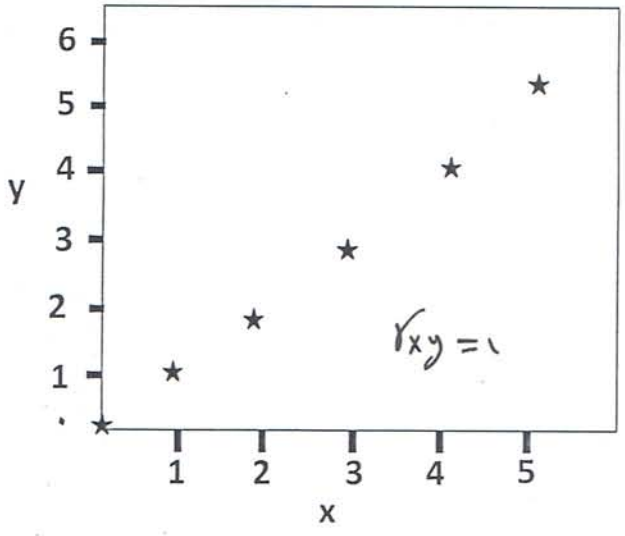
Is a measure of the association between two random variables, both variables are assumed to be varying randomly. We do assume for this analysis that X and Y are related linearly. So the usual correlation coefficient gives a measure of the linear association between X and Y.

$$r_{xy} = \frac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}} = \frac{\sum xy - \frac{1}{n}\sum x \sum y}{\sqrt{\sum x^2 - \frac{1}{n}(\sum x)^2 \cdot \sum y^2 - \frac{1}{n}(\sum y)^2}}$$

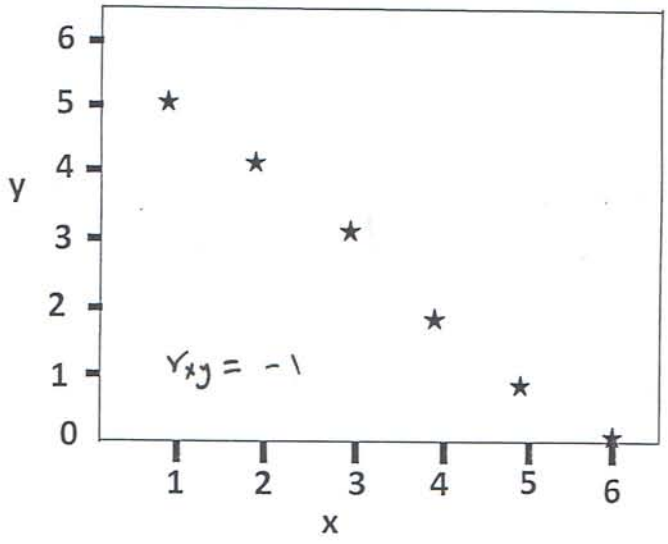
For perfect correlation $\rightarrow r = \pm 1$

If there is no systematic relation between X and Y at all ,

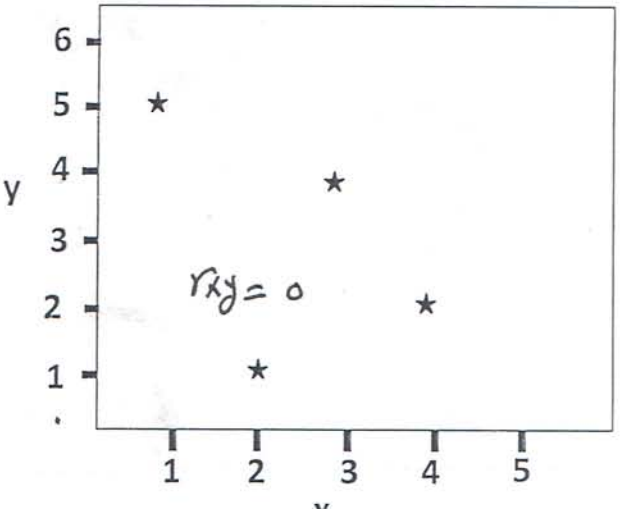
$r_{xy} \simeq 0$

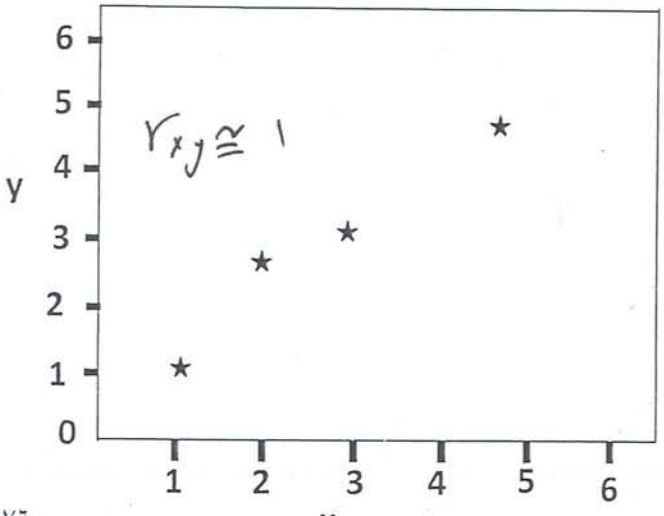


(a)



(b)





This fig. illustrate various correlation coefficients.

A) $r_{xy} = 1$

B) $r_{xy} = -1$

C) $r_{xy} = 0$

D) $r_{xy} \simeq 1$

Tutorial sheet (7)

q. 1) It is required to fit the following eqn^{s}. to a straight line; so determine the constants, then calculate the correlation coefficient. (r_{xy}).

A.

$$y = axe^{x} + bx^{2.2}$$

x :	1	2	3	4	5
y :	37.5	32.0	25.8	28.6	37.6

B.

$$y = ae^{x} + be^{-x}$$

x :	0	0.2	0.4	0.6	0.8	1.0
y :	-1.12	0.026	1.15	2.32	3.59	5.0

C.

$$\ln y = axe^{x} + bx$$

x :	0.21	0.27	0.35	0.38	0.43
y :	10	22	70	100	240

D.

$$y = \frac{x}{a + bx}$$

y :	3.5	7.2	12.6	16.4	20.2
x :	100	200	300	400	500

E.

$$C^2 = \frac{C_i^2}{2C_iKt + 1}$$

C :	2.5	1.65	1.18	0.95	0.88
t :	10	15	20	25	30

F.

$$K = A\, e^{-E/RT}$$

K :	1.22	2.72	4.95	7.39	11.0
T :	316.46	322.58	331.16	336.7	340.14

Multiple and Partial Correlation

- Multiple Correlation :

The degree of relationship existing between three or more variables is called multiple correlation. The fundamental principles involved in problems of multiple correlation are analogous to those of simple correlation.

- Subscript Notation :

- Regression equation. Regression plane :

A regression eqn. Is eqn. For estimating a dependent variable, X_1 , from the independent variables X_2 , X_3 , … and is called a regression eqn. Of X_1 on X_2 , X_3 ,… and can be written as $X_1=F(X_2 , X_3 , \ldots)$.

- The simplest reg. eqn. Of X_1 on X_2 and X_3 :

$X_1 = b_{1.23} + b_{12.3} X_2 + b_{13.2} X_3$ .....(1)

If we keep X_3 constant in eqn. (1) , the graph of X_1 vs. X_2 is a straight line with slop $b_{12.3}$. if we keep X_2 constant, the graph of X_1 vs. X_3 is a straight line with slop $b_{13.2}$.

The subscript after the dot (.) indicate the variables held constant in each case.

- X_1 varies partially because of variation in X_2 and partially because of variation in X_3 , so $b_{12.3}$ and $b_{13.2}$ called the partial regression coefficients of X_1 on X_2 keeping X_3 constant and of X_1 on X_3 keeping X_2 constant respectively eqn. (1) is called a **linear regression** of X_1 on X_2 and X_3 . In a three dimensional

rectangular co-ordinate system it represents a plane called a **regression plane**.

- Normal eqn^s. for the least square reg. plane :

The least square reg. plane of X_1 on X_2 and X_3 has the eqn. (1) where $b_{1.23}$, $b_{12.3}$ and $b_{13.2}$ are determined by solving simultaneously the normal eqn^s.

$$\left\{\begin{array}{l} \sum X_1 = b_{1.23}N + b_{12.3}\sum X_2 + b_{13.2}\sum X_3 \\ \sum X_1X_2 = b_{1.23}\sum X_2 + b_{12.3}\sum X_2^2 + b_{13.2}\sum X_2X_3 \\ \sum X_1X_3 = b_{1.23}\sum X_3 + b_{12.3}\sum X_2X_3 + b_{13.2}\sum X_3^2 \end{array}\right\} \ldots eqn.\,(2)$$

These can be obtained by multiplying both sides of eqn. (1) by 1 , X_2 and X_3 and summing on both sides.

If $x_1 = X_1 - \bar{X}_1$, $x_2 = X_2 - \bar{X}_2$, $x_3 = X_3 - \bar{X}_3$

The lst. Eqn. of (2) divided by N , to get :

$$\bar{X}_1 = b_{1.23} + b_{12.3}\bar{X}_2 + b_{13.2}\bar{X}_3 \longrightarrow$$

Subtracting this eqn. from eqn. (1) $X_1 = b_{1.23} + b_{12.3}X_2 + b_{13.2}X_3$

$$X_1 - \bar{X}_1 = b_{12.3}(X_2 - \bar{X}_2) + b_{13.2}(X_3 - \bar{X}_3)$$

Or :

$$x_1 = b_{12.3}\,x_2 + b_{13.2}\,x_3 \qquad \ldots\ldots (3)$$

Where $b_{12.3}$ and $b_{13.2}$ are obtained by solving simultaneously the eqn^s.

$$\left\{ \begin{array}{l} \sum x_1 x_2 = b_{12.3} \sum x_2^2 + b_{13.2} \sum x_2 x_3 \\ \sum x_1 x_3 = b_{12.3} \sum x_2 x_3 + b_{13.2} \sum x_3^2 \end{array} \right\} \quad \ldots\ldots (4)$$

These eqn^s. which are equivalent to the normal equ^s. (2) and can be obtained by multiplying both sides of (3) by x_2 and x_3 and summing both sides.

Example 1)

The following table shows the corresponding values of three variables X_1 , X_2 and X_3 . find the least square reg. eqn. of X_3 on X_1 and X_2 .

X_1	X_2	X_3
3	16	90
5	10	72
6	7	54
8	4	42
12	3	30
14	2	12
$\sum = 48$	$\sum = 42$	$\sum = 300$

$X_3 = f(X_1 , X_2)$

$X_3 = b_{3.12} + b_{31.2} X_1 + b_{32.1} X_2$

The normal eqn^s. of the least square reg. line :

$$\left\{ \begin{array}{c} \sum X_3 = b_{3.12} N + b_{31.2} \sum X_1 + b_{32.1} \sum X_2 \\ \sum X_3 X_1 = b_{3.12} \sum X_1 + b_{31.2} \sum X_1^2 + b_{32.1} \sum X_2 X_1 \\ \sum X_3 X_2 = b_{3.12} \sum X_2 + b_{31.2} \sum X_1 X_1 + b_{32.1} \sum X_2^2 \end{array} \right\}$$

Or by using eqn. (4)

$x_3 = b_{31.2}\, x_1 + b_{32.1}\, x_2$

$$\sum x_3 x_1 = b_{31.2} \sum x_1^2 + b_{32.1} \sum x_1 x_2$$

$$\sum x_3 x_2 = b_{31.2} \sum x_1 x_2 + b_{32.1} \sum x_2^2$$

$\bar{X}_1 = 8$, $\bar{X}_2 = 7$, $\bar{X}_3 = 50$

x_1	x_2	x_3	x_3x_1	x_3x_2	x_2x_1	x_1^2	x_2^2
-5	9	40	-200	360	-45	25	81
-3	3	22	-66	66	-9	9	9
–2	0	4	-8	0	0	4	0
0	-3	-8	0	24	0	0	9
4	-4	-20	-80	80	-16	16	16
6	-5	-38	-228	190	-30	36	25
$\sum = 0$	$\sum = 0$	$\sum = 0$	$\sum = -582$	$\sum = 720$	$\sum = -10($	$\sum = 90$	$\sum = 140$

-582 = $b_{31.2}$ (90) + $b_{32.1}$ (-100) .....(1)

720 = $b_{31.2}$ (-100) + $b_{32.1}$ (140) .....(2)

Eqn. (1) * (100) + eqn. (2) * (90) yield :

6600 = $b_{32.1}$ (2600)

∴ $b_{32.1}$ = 2.54

-582 = $b_{31.2}$ (90) + 2.54 (-100)

$b_{31.2}$ = -3.64

subst. the above constant in following eqn.

$\bar{X}_3 = b_{3.12} + b_{31.2}\bar{X}_1 + b_{32.1}\bar{X}_2$

$50 = b_{3.12} + (-3.64)\,(8) + (2.54)\,(7)$

$b_{3.12} = 61.34$

- Standard error of estimate : Can be defined as :

$$S_{1.23} = \sqrt{\frac{\sum(X_1 - X_{1.est})^2}{N}}$$

$X_{1.est.}$ calculated from reg. eqn. :

$X_1 = b_{1.23} + b_{12.3}\, X_2 + b_{13.2}\, X_3$

- Coefficient of Multiple correlation :

$$R_{1.23} = \sqrt{1 - \frac{S_{1.23}^2}{S_1^2}}$$

$$S_1 = \sqrt{\frac{\sum(X_1 - \bar{X}_1)^2}{N}} = \sqrt{\frac{\sum x_1^2}{N}}$$

OR :

$$R_{.123} = \sqrt{1 - \frac{\sum(X_1 - X_{1.est})^2}{\sum(X_1 - \bar{X}_1)^2}}$$

Example 2)

Find the standard error of estimate of X_3 on X_1 and X_2 of example (1) :

Soln. :

- The reg. eqn. of X_3 on X_1 and X_2 :

$X_3 = 61.34 - 3.64\ X_1 + 2.54\ X_2$

$$S_{3.12} = \sqrt{\frac{\sum(X_3 - X_{3.est})^2}{N}}$$

X_3	$X_{3.est.}$	$(X_3 - X_{3.est})^2$
90	91.06	1.124
72	68.54	11.97
54	57.3	10.89
42	42.4	0.16
30	25.3	22.1
12	15.43	11.76
		$\sum$ =58.0

$$\therefore\ S_{3.12} = \sqrt{\frac{58}{6}}$$

$S_{3.12} = 3.11$

- The correlation coefficient of X_3 on X_2 and X_1

$$R_{3.12} = \sqrt{1 - \frac{S_{3.12}^2}{S_3^2}}$$

$$S_3 = \sqrt{\frac{\sum(X_3 - \bar{X}_3)^2}{N}} = \sqrt{\frac{\sum x_3^2}{N}}$$

x_3	x_3^2
40	1600
22	484
4	16
-8	64
-20	400
-38	1444
$\sum x_3 = 0$	$\sum = 4008$

$$S_3 = \sqrt{\frac{4008}{6}} = 25.85$$

$$\therefore \quad R_{3.12} = \sqrt{1 - \frac{(3.11)^2}{(25.85)^2}}$$

$R_{3.21} = 0.9927$