1- Double integrals over rectangular

if $f(x,y)$ is Continous through the rectangular region $R: a \le x \le b$, $c \le y \le d$ then the double integrals over R are defined by

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,d$$

Example

$$\iint_R x y e^{xy^2}\,dA \quad . \quad R: 0 \le x \le 2, \; 0 \le y \le 1$$

Solution $\int_0^2 \int_0^1 x\, y e^{xy^2}\,dy\,dx$

$t = y^2$

$dt = 2y\,dy$

$$= \frac{1}{2} \int_0^2 \int_0^1 x\, e^{xt}\,dt\,dx$$

$$= \frac{1}{2} \int_0^2 \cancel{x} \frac{e^{xt}}{\cancel{x}} \Big|_0^1 dx$$

$$= \frac{1}{2} \int_0^2 (e^x - 1)\,dx = \frac{1}{2} \left[e^x - x \right]_0^2$$

$$= \frac{1}{2} \left[e^2 - 2 - 1 + 0 \right] = \frac{1}{2}(e^2 - 3)$$

2- Double integrals over bounded non Rectangular region

let $f(x,y)$ be Continuous on region R

if ① R is defined by $a \le x \le b$, $g_1(x) \le y \le g$

with g_1 and g_2 are Continuous on $[a,b]$ the

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

② if R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ with h_1 and h_2 are Continuous on $[c,d]$ Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Examples

:- Sketch the region of integration and evaluate the integral,

① $$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}}\,dy\,dx$$

$$= \frac{3}{2}\int_1^4 \left. \frac{e^{\frac{y}{\sqrt{x}}}}{\frac{1}{\sqrt{x}}} \right|_0^{\sqrt{x}} dx \Longrightarrow \frac{3}{2}\int_1^4 \sqrt{x}\,(e-1)\,dx$$

$$= \frac{3}{2}(e-1)\int_1^4 x^{1/2}\,dx \Rightarrow \frac{3}{2}(e-1)\left[\frac{x^{3/2}}{3/2}\right]_1^4$$

$$\Rightarrow = (e-1)(8-1) = 7(e-1)$$

② $$\int_0^{3/2}\int_1^{4-2u}\left(\frac{4-2u}{v^2}\right)dv\,du$$

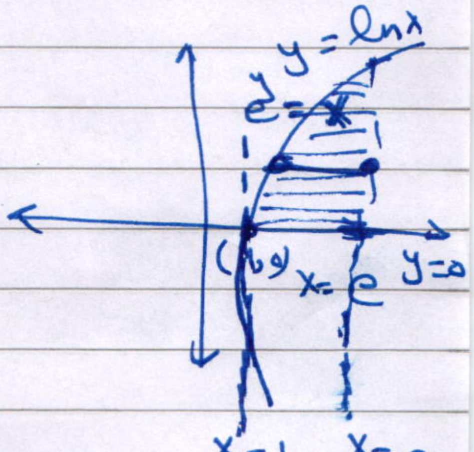
$$=\int_0^{3/2}(4-2u)\left[-\frac{1}{v}\right]_1^{4-2u}du$$

$$=\int_0^{3/2}(4-2u)\left[-\frac{1}{4-2u}+1\right]$$

$$=\int_0^{3/2}(-1+4-2u)\,du=\int_0^{3/2}(3-2u)\,du$$

$$=\left[3u-u^2\right]_0^{3/2}=\frac{9}{2}-\frac{9}{4}-0+0=\frac{9}{4}$$

③ $$\int_1^e\int_0^{\ln x} x\,y\,dy\,dx$$



$$=\int_0^1\int_{e^y}^{e} x\,y\,dx\,dy$$

$$=\int_0^1 y\,\frac{x^2}{2}\Big|_{e^y}^{e}dy=\frac{1}{2}\int_0^1 y\left(e^2-e^{2y}\right)dy$$

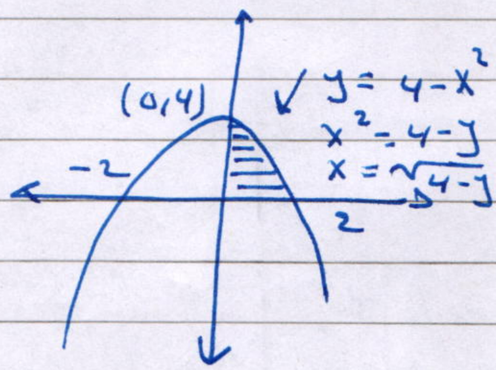
$$= \frac{1}{2}\int_0^1 e^2 y \, dy - \frac{1}{2}\int_0^1 y e^{2y} \, dy$$

$$= \frac{1}{2} e^2 \frac{y^2}{2}\Big|_0^1 - \frac{1}{2}\left[\frac{1}{2} y e^{2y} - \frac{1}{4} e^{2y}\right]_0^1$$

$$= \cancel{\frac{1}{4} e^2} - \cancel{\frac{1}{4} e^2} + \frac{1}{8} e^2 + 0 - \frac{1}{8}$$

$$= \frac{1}{8} e^2 - \frac{1}{8} \Longrightarrow \frac{1}{8}\left(e^2 - 1\right)$$

(4) $$\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} \, dy \, dx$$



$$= \int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} \, dx \, dy$$

$$= \int_0^4 \frac{x^2}{2}\Big|_0^{\sqrt{4-y}} \frac{e^{2y}}{4-y} \, dy \Longrightarrow \frac{1}{2}\int_0^4 (\cancel{4-y}) \cdot \frac{e^{2y}}{\cancel{4-y}} \, dy$$

$$= \frac{1}{2}\int_0^4 e^{2y} \, dy \Longrightarrow \frac{1}{2} \frac{e^{2y}}{2}\Big|_0^4 = \frac{1}{4}\left(e^8 - 1\right)$$

Double integrals as volume

if $f(x,y)$ is positive, we interpret the double integrals of f over a region R as the volume of the solid bounded below by R and above by the surface $z = f(x,y)$ that is $V = \iint_R f(x,y)\, dA$

$$Z = \iint_R z\, dA$$

Example ① Find the volume of the region bounded above by the surface $z = 16 - x^2 - y^2$ and below by the square $R: 0 \le x \le 2 \,;\, 0 \le y \le 2$

solution :-

$$V = \iint_R (16 - x^2 - y^2)\, dA$$

$$= \int_0^2 \int_0^2 (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - y^2 x\right]_0^2 dy$$

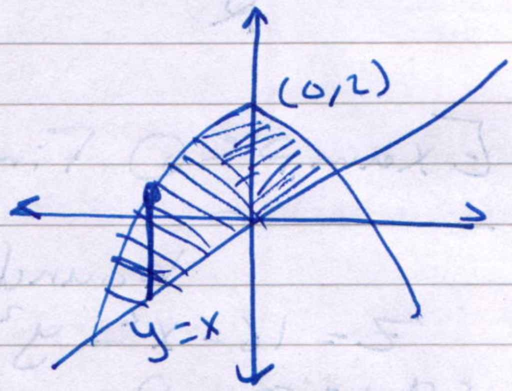
$$= \int_0^2 \left(32 - \frac{8}{3} - 2y^2\right) dy = \int_0^2 \left[\frac{80}{3} - 2y^2\right] dy$$

$$= \left[\frac{80}{3}y - \frac{2y^3}{3}\right]_0^2 = \frac{176}{3} - \frac{16}{3} = \frac{160}{3}$$

Example 2

Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabala $y = 2 - x^2$ and the line $y = x$ in the xy plane

Solution



$$V = \iint_R x^2 \, dA$$

$$= \int_{-2}^{1} \int_{x}^{2-x^2} x^2 \, dy \, dx$$

$$= \int_{-2}^{1} x^2 y \Big|_{x}^{2-x^2} dx$$

$$= \int_{-2}^{1} x^2(2-x^2) - (x) \, dx$$

$$= \int_{-2}^{1} (2x^2 - x^4 - x^3) \, dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{4}x^4\right]_{-2}^{1} = \frac{63}{20}$$

$$x = 2 - x^2$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$\boxed{\therefore x = 1, \quad x = -2}$$

Areas, moments and Centers of mass

Areas of bounded regions in the Plane

$$\text{Area} = \iint_R dA$$

Examples

① Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

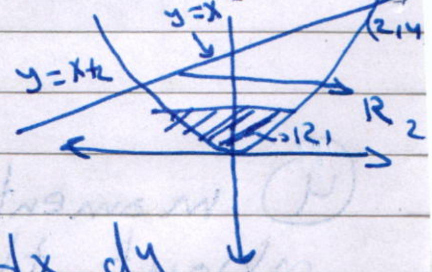
Solution :-

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

② Find the area of the region R enclosed by the Parabala $y = x^2$ and the line $y = x + 2$

Solution :- if we integrate with respect to x then to y, we will divided the region R intwo regions R_1 and R_2



$$A = \iint_{R_1} dA + \iint_{R_2} dA$$

$$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$$

on the other hand, reversing the order of integration gives:-

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx$$, clearly this results is

is Simpler one

$$A = \int_{-1}^{2} y \Big|_{x^2}^{x+2} d\cancel{x} = \int_{-1}^{2} (x+2+x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x + \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

First and Second moments and Centers of mass

mass and moment formulas for thin plates covering region in xy_plane

(1) Density: $\rho(x,y)$: mass : $M = \iint \rho(x,y)\, dA$

(2) First moments : $M_x = \iint y\, \rho(x,y)\, dA$

$M_y = \iint x\, \rho(x,y)\, dA$

(3) Center of mass:

$$\bar{x} = \frac{M_x}{M}, \quad \bar{y} = \frac{M_y}{M}$$

(4) moments of inertia (Second moments).

about the x_axis : $I_x = \iint y^2 \rho(x,y)\, dA$

" " y_axis : $I_y = \iint x^2 \rho(x,y)\, dA$

" " origin: $I_o = \iint (x^2+y^2)\, \rho(x,y)\, dA$

(5) Radii of gyration about the x_axis $R_{\cancel{x}} = \sqrt{\frac{I_x}{M}}$, about y_axis $R_y = \sqrt{\frac{I_y}{M}}$

about the origin: $R_o = \sqrt{\frac{I_o}{M}}$

Example : A thin plate covers the triangular region bounded by the x-axis and the lines $x=1$ and $y=2x$ in the first quadrant. The plate's density at the point (x,y) is $\rho(x,y)=6x+6y+6$. Find the plate's mass, first moment, center of mass, moments of intertia and radii of gyration about the coordinate axes.

Solution :- $M=\int_0^1\int_0^{2x}\rho(x,y)\,dy\,dx$

$$=\int_0^1\int_0^{2x}(6x+6y+6)\,dy\,dx \Rightarrow \int_0^1 (6xy+3y^2+6y)\Big|_0^{2x} dx$$

$$=\int_0^1(12x^2+12x^2+12x)\,dx=(8x^3+6x^2)\Big|_0^1=14$$

First moment: $M_x=\int_0^1\int_0^{2x} y\,\rho(x,y)\,dy\,dx$

$$=\int_0^1\int_0^{2x} y(6x+6y+6)\,dy\,dx$$

$$=\int_0^1\left[3xy^2+2y^3+3y^2\right]_0^{2x}dx=\int_0^1(12x^3+16x^3+12x^2)\,dx$$

$$=(3x^4+4x^4+4x^3)\Big|_0^1=11$$

$$M_y=\int_0^1\int_0^{2x} x\,\rho(x,y)\,dy\,dx \Rightarrow \int_0^1\int_0^{2x} x(6x+6y+6)\,dy\,dx$$

$$=\int_0^1(6x^2y+3xy^2+6xy)\Big|_0^{2x}dx=\int_0^1(12x^3+12x^3+12x^2)\,dx$$

$$=(6x^4+4x^3)\Big|_0^1=10$$

Center of mass $\bar{x}=\dfrac{M_x}{M}=\dfrac{10}{14}$, $\bar{y}=\dfrac{M_y}{M}=\dfrac{11}{14}$

$$I_x = \iint y^2 \rho(x,y)\, dA = \int_0^1 \int_0^{2x} y^2 (6x+6y+6)\, dy\, dx$$

$$= \int_0^1 \left(2xy^3 + \frac{3}{2}y^4 + 2y^3\right)\Big|_0^{2x} dx$$

$$= \int_0^1 (16x^4 + 24x^4 + 16x^3)\, dx = (8x^5 + 4x^4)\Big|_0^1 = 12$$

Similarly, the moment of inertia about the y-axis is $I_y = \iint x^3 \rho(x,y)\, dA$

$$= \int_0^1 \int_0^{2x} (6x+6y+6)\, dy\, dx = \frac{39}{5}$$

$$I_0 = I_x + I_y = 12 + \frac{39}{5} = \frac{99}{5}$$

$$R_x = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{12}{14}} = \sqrt{\frac{6}{7}}, \quad R_y = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{39}{70}}$$

$$R_0 = \sqrt{\frac{I_0}{m}} = \sqrt{\frac{99}{70}}$$

Example (2): Find the Centroid of the region in the First quadrant that is bounded above by the line $y=x$ and below by the Parabala $y=x^2$.

Solution :- Set $\rho = 1$ $\quad m = \int_0^1 \int_0^x 1\, dy\, dx = \int_0^1 [y]_{x^2}^{x} dx$

$$= \int_0^1 (x - x^2)\, dx = \left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right) dx = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 (xy)_{x^2}^{x} dx = \int_0^1 (x^2 - x^3)\, dx = \frac{1}{12}$$

From these values of M_x, M, M_y we find

$\bar{x} = \frac{M_y}{m} = \frac{1}{2}$, $\bar{y} = \frac{M_x}{m} = \frac{2}{5}$ $\therefore$ Centeroid is the $\left(\frac{1}{2}, \frac{2}{5}\right)$ Points.

Double integrals in Polar Coordinates

if $f(r,\theta)$ is defined function over region R bounded by the rays $\theta = \alpha$, $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$, $r = g_2(\theta)$ with $0 \leq g_1(\theta) \leq g_2(\theta)$, we define the integration of $f(r,\theta)$ over R by

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

The area of a closed bounded region R in the Polar Coordinates, Plane is

$$A = \iint_R dA = \iint_R r\, dr\, d\theta$$

So Area of $R = \int_\alpha^\beta \frac{r^2}{2}\, d\theta = A$

Area between two curves $A = \int_\alpha^\beta \frac{r_1^2 - r_2^2}{2}\, d\theta$

Changing Cartesian integrals into Polar integrals

Sometimes a double integrals eaiser to evaluate using Polar Coordinates.

This is especially true if the region of integration can be easily defined using a Polar equation.

The following formula for converted between rectangula and Polar Coordinates are needed.

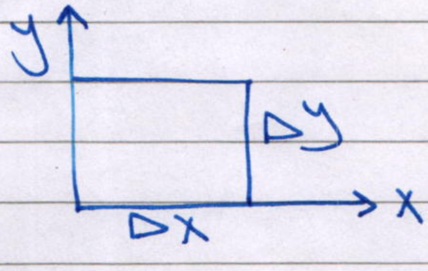
$x = r\cos\theta \quad , \quad y = r\sin\theta$

$r^2 = x^2 + y^2 \quad , \quad \tan\theta = \frac{y}{x}$

Rectangule Coordinates

$\iint f(x,y)\, dA$

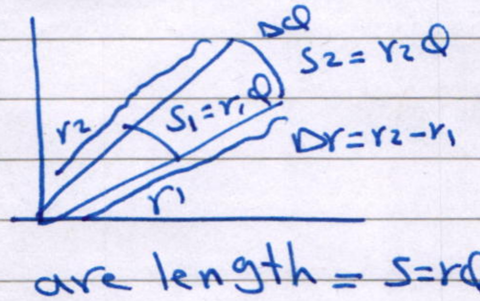
$dA = dx\, dy$



Polar coordinates

$\iint f(r,\theta)\, r\, dr\, d\theta$

$dA = r\, dr\, d\theta$



arc length $= S = r\theta$

let $r_1 \equiv r_2 \equiv r$

$\Delta A = r\,\Delta r\,\Delta\theta$

$\Delta\theta = d\theta$

$\Delta r = dr$

$dA = r\,dr\,d\theta$

an in general

$x = r\cos\theta \quad , y = r\sin\theta$

so $x = g(r,\theta)$, $y = h(r,\theta)$

Examples :-

Change the Cartesian integral in an equivalent Polar integral, then evaluate the Polar integral

ⓐ $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)\, dx\, dy$

$x = \sqrt{4-y^2}$

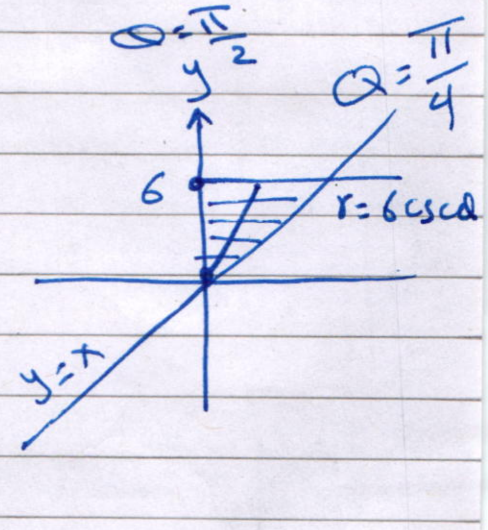
$x^2 + y^2 = 4$

$= \displaystyle\int_0^{\pi/2} \int_0^2 r^2 . r . dr\, d\theta$

$= \displaystyle\int_0^{\pi/2} \frac{r^4}{4} \Big|_0^2 d\theta \Rightarrow = 4\int_0^{\pi/2} d\theta = 4\theta \Big|_0^{\pi/2}$

$= 4\left(\frac{\pi}{2}\right) = 2\pi$

ⓑ $\displaystyle\int_0^6 \int_0^y x\, dx\, dy$



$y = 6$

$r\sin\theta = 6 \Rightarrow r = 6\csc\theta$

$= \displaystyle\int_{\pi/4}^{\pi/2} \int_0^{6\csc\theta} r\cos\theta\, r\, dr\, d\theta$

$= \displaystyle\int_{\pi/4}^{\pi/2} \frac{r^3}{3} \Big|_0^{6\csc\theta} \cos\theta\, d\theta$

$$= 72\int_{\pi/4}^{\pi/2} \csc^3\theta \cos\theta \, d\theta = 72\int_{\pi/4}^{\pi/2} \frac{\csc^2}{\sin\theta} \cos\theta \, d\theta$$

$$= 72\int_{\pi/4}^{\pi/2} \csc^2\theta \cot\theta \, d\theta = -72\int_{\pi/4}^{\pi/2} \cot\theta \, d(\cot\theta)$$

$$= -72 \frac{\cot^2\theta}{2}\Big|_{\pi/4}^{\pi/2} = -36\left[\cot^2\frac{\pi}{2} - \cot^2\frac{\pi}{4}\right]$$

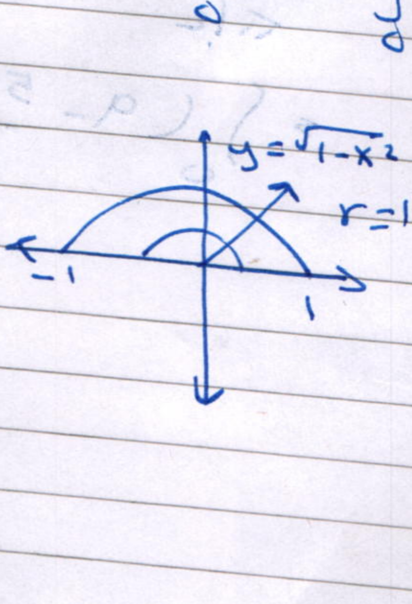
$$= -36[0-1] = 36$$

Example2 : Evaluate $\iint_R e^{x^2+y^2}\, dy\, dx$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi} \frac{1}{2} e^{r^2}\Big]_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e-1)\, d\theta$$



Finding limits of integration in Polar Coordinates.

Example ①: Determine the volume of $z=\sqrt{9-x^2-y^2}$ over the region $x^2+y^2\leq 4$ in the first octant.

$f(x,y)=\sqrt{9-x^2-y^2}$, $f(r,\theta)=\sqrt{9-r^2}$

$x^2+y^2\leq 4$, $r^2\leq 4$, $r\leq 2$

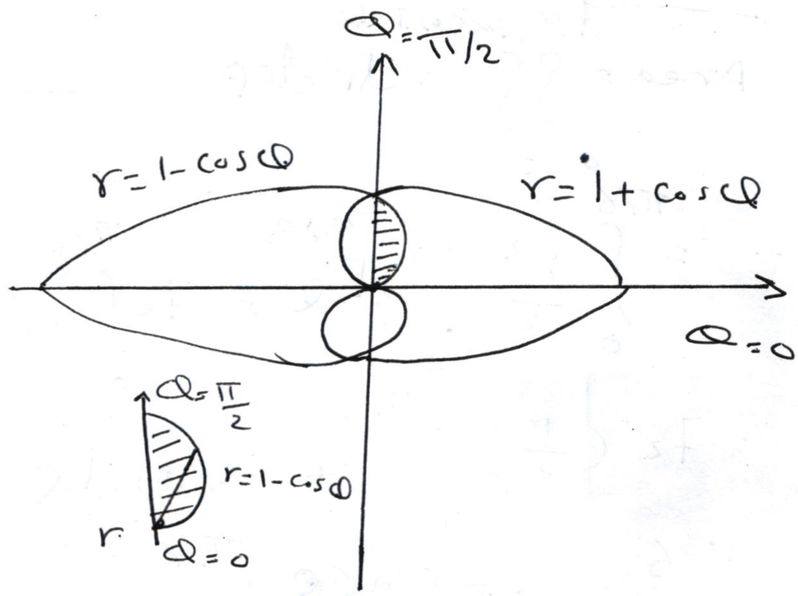
$$\int_0^{\pi/2}\int_0^2\sqrt{9-r^2}\; r\,dr\,d\theta = \int_0^{\pi/2}-\frac{1}{2}\int_0^2(9-r^2)^{1/2}\,(-2r)\,dr\,d\theta$$

$$\int_0^{\pi/2}\left(-\frac{1}{2}\,\frac{2}{3}\,(9-r^2)^{3/2}\right]_0^2 d\theta$$

$$=\int_0^{\pi/2}\left(9-\frac{5\sqrt{5}}{3}\right)d\theta = \left(\frac{27-5\sqrt{5}}{6}\right)\pi$$

Examples 2 :-

① Find the area of the region on Common to the interior of the Cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$



Solution :-

$$\text{area} = 4\int_0^{\pi/2}\int_{0}^{1-\cos\theta} r \, dr \, d\theta$$

$$= 4\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{(1-\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta$$

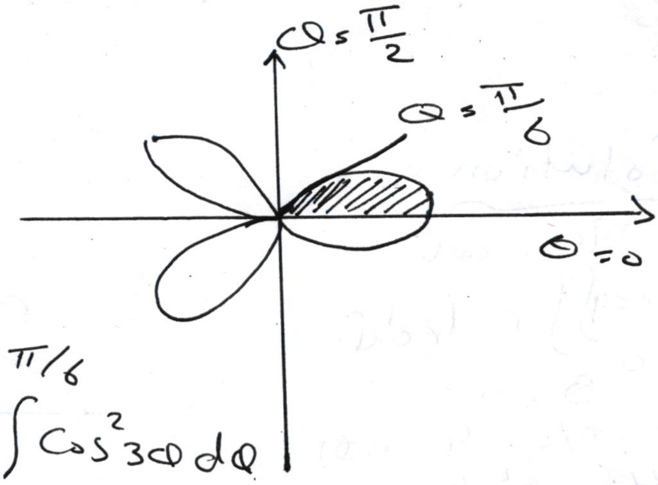
$$= 2\int_0^{\pi/2} \left[1 - 2\cos\theta + \cos^2\theta\right] d\theta$$

$$= 2\int_0^{\pi/2} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \Big] d\theta$$

$$= 2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{3\pi}{2} - 4 = \frac{3\pi - 8}{2}$$

② Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.



Solution

$$\text{Area} = \int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6} \frac{r^2}{2}\Big|_0^{12\cos 3\theta} d\theta = 72\int_0^{\pi/6} \cos^2 3\theta\, d\theta$$

$$= 72\left(\frac{1}{2}\right)\int_0^{\pi/6} (1+\cos 6\theta)\,d\theta$$

$$= 36\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/6}$$

$$= 12\pi$$