

# الجامعة التكنولوجية

## قسم الهندسة الكيميائية

### المرحلة الثانية

### الرياضيات II

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# Multiple Integrals

## Integration of functions of one variable

We start by recalling the basics of integration with respect to a single variable. We have results such as:

$$\int_a^b f(x) \, dx = [F(x)]_a^b := F(b) - F(a)$$

where  $F$  is an “antiderivative” or indefinite integral of  $f$ . ( $f = dF/dx$ .)

$$\begin{aligned}\int \lambda f(x) \, dx &= \lambda \int f(x) \, dx, \\ \int (f(x) + g(x)) \, dx &= \int f(x) \, dx + \int g(x) \, dx.\end{aligned}$$

Useful techniques include:

- Integration by parts.
- Integration by substitution.
- Use of partial fractions.

Some specific integrals can be found on the formula sheet.

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**Question 1.** Calculate  $\int_0^1 (3x^4 + x^2 e^{x^3}) \, dx$ .

**Solution.** This integral is designed to use a few different techniques! Firstly we can split the integral as

$$\int_0^1 (3x^4 + x^2 e^{x^3}) \, dx = 3 \int_0^1 x^4 \, dx + \int_0^1 x^2 e^{x^3} \, dx.$$

The first integral on the right-hand side is straightforward:

$$\int_0^1 x^4 \, dx = \left[ \frac{x^5}{5} \right]_0^1 = \frac{1}{5}.$$

To calculate the second integral on the right hand side we use integration by substitution; taking  $u = x^3$  we have  $du/dx = 3x^2$  so  $x^2 dx = \frac{1}{3} du$ . Therefore

$$\int x^2 e^{x^3} dx = \int \frac{1}{3} e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C.$$

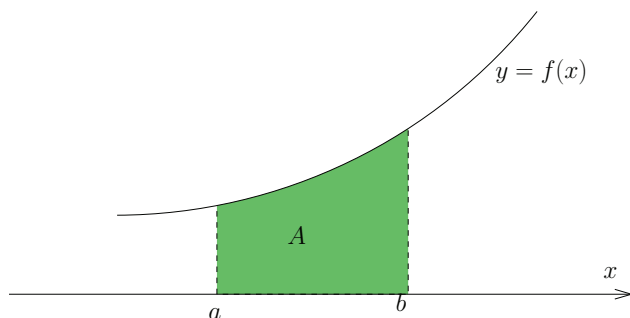
The definite integral can then be computed as

$$\int_0^1 x^2 e^{x^3} dx = \left[ \frac{1}{3} e^{x^3} \right]_0^1 = \frac{1}{3} e^1 - \frac{1}{3} e^0 = \frac{e-1}{3}.$$

Combining the above calculations we finally get

$$\int_0^1 (3x^4 + x^2 e^{x^3}) dx = 3 \times \frac{1}{5} + \frac{e-1}{3} = \frac{4+5e}{15}.$$

One interpretation of an integral like this is the area under a graph: assuming that  $f$  is positive, the area of a strip of width  $dx$  and length  $f$  is  $f dx$  and then  $A = \int_a^b f(x) dx$  is the total area between the  $x$  axis and the curve  $y = f(x)$  for  $a < x < b$ .

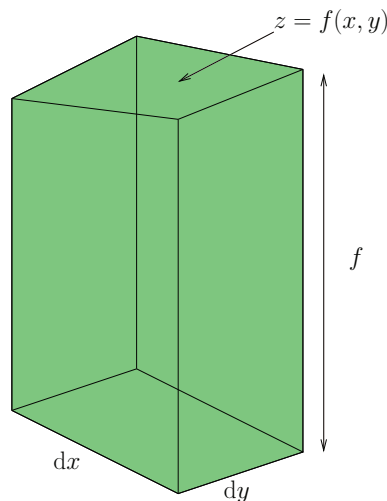


More generally, if  $f$  represents the density of some quantity, *i.e.* it is the amount per unit length,  $f dx$  is the amount in a short length  $dx$  and  $A = \int_a^b f(x) dx$  is the total amount between  $x = a$  and  $x = b$ .

## 2 Integration of functions of two variables

Thinking of a single integral as giving an area, we might ask: what is the volume under a surface  $z = f(x, y)$  lying above a rectangle in the  $x - y$  plane,  $a < x < b$ ,  $c < y < d$ ? (We again take  $f > 0$  for simplicity.)

First look at a small rectangle of length  $dx$  and width  $dy$ . Its area is  $dA = dx dy$ . Then the volume between that small rectangle and the surface is the height of the enclosed (approximately) cuboidal region times the area:  $f dA = f dx dy$ .



The volume between the  $x - y$  plane and the surface in the slab of width  $dx$  between  $y = c$  and  $y = d$  is then given by integrating with respect to  $y$ :  $(\int_c^d f(x, y) dy) dx$ . (In doing this integral,  $x$  is held fixed.) Finally, to get the total volume, we must integrate with respect to  $x$  from  $a$  to  $b$ :

$$\text{volume} = V = \int_a^b \left( \int_c^d f(x, y) dy \right) dx. \quad (7.1)$$

We usually abbreviate this double integral as

$$V = \int_a^b \int_c^d f(x, y) dy dx. \quad (7.2)$$

Noting that, for cases we'll be considering at least, we could have integrated with respect to  $x$  first and then  $y$ , we also have

$$V = \int_c^d \int_a^b f(x, y) dx dy = \int \int_D f(x, y) dx dy, \quad (7.3)$$

where  $D$  is our region of integration, here  $D$  is the rectangle  $a < x < b$ ,  $c < y < d$ .

More generally if  $f$  once again represents some sort of density, *e.g.* mass per unit area in a sheet of metal, the double integral  $\int \int_D f(x, y) dx dy$  is the total amount in the region  $D$ .

**Question 2.** Calculate  $\int_2^4 \int_1^3 (xy^2 + y) dy dx$ .

**Solution.** Calculating the inner integral first gives

$$\int_1^3 (xy^2 + y) dy = \left[ \frac{xy^3}{3} + \frac{y^2}{2} \right]_1^3 = \left( 9x + \frac{9}{2} \right) - \left( \frac{x}{3} + \frac{1}{2} \right) = \frac{26x}{3} + 4.$$

Therefore

$$\begin{aligned} \int_2^4 \int_1^3 (xy^2 + y) \, dy \, dx &= \int_2^4 \left( \frac{26x}{3} + 4 \right) \, dx \\ &= \left[ \frac{13x^2}{3} + 4x \right]_2^4 = \left( \frac{13 \times 16}{3} + 16 \right) - \left( \frac{13 \times 4}{3} + 8 \right) = 13 \times 4 + 8 = 60. \end{aligned}$$

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**Question 3.** Calculate  $\int_0^1 \int_2^3 2xy \, dx \, dy$ .

**Solution.** Calculating the inner integral first gives

$$\int_2^3 2xy \, dx = [x^2 y]_2^3 = 9y - 4y = 5y.$$

Therefore

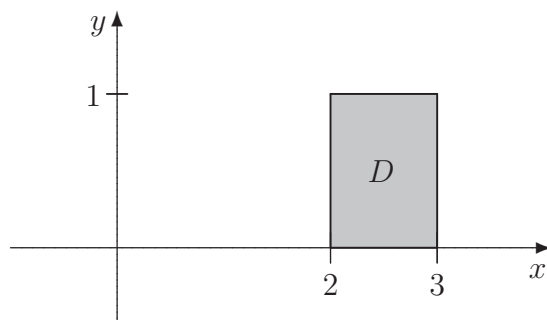
$$\int_0^1 \int_2^3 2xy \, dx \, dy = \int_0^1 5y \, dy = \left[ \frac{5y^2}{2} \right]_0^1 = \frac{5}{2}.$$

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**question 4 .** For the double integral  $\int_2^3 \int_0^1 2xye^{xy^2} \, dy \, dx$  the region of integration  $D$  is the rectangle given by

$$0 \leq y \leq 1, \quad 2 \leq x \leq 3.$$



**Question 5 .** Calculate  $\int_2^3 \int_0^1 2xye^{xy^2} \, dy \, dx$ .

**Solution.** Firstly consider the inner integral

$$\int_0^1 2xye^{xy^2} \, dy.$$

We will use integration by substitution for this integral; taking  $u = xy^2$  we have  $\frac{\partial u}{\partial y} = 2xy$  so  $du = 2xy \, dy$  and hence

$$\int 2xye^{xy^2} \, dy = \int e^u \, du = e^u + C = e^{xy^2} + C.$$

Thus

$$\int_0^1 2xye^{xy^2} dy = [e^{xy^2}]_0^1 = e^x - 1.$$

It follows that

$$\int_2^3 \int_0^1 2xye^{xy^2} dy dx = \int_2^3 (e^x - 1) dx = [e^x - x]_2^3 = (e^3 - 3) - (e^2 - 2) = e^3 - e^2 - 1.$$

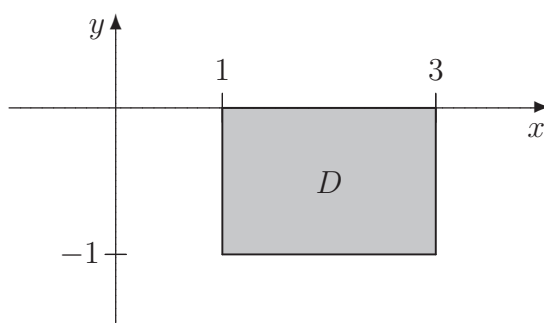
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**Question 6 .** Let  $D$  be the rectangle given by

$$-1 \leq y \leq 0, \quad 1 \leq x \leq 3.$$

Calculate  $\iint_D (x^2 + y) dy dx$ .

**Solution.** The region of integration is the following:



Now

$$\begin{aligned} \iint_D (x^2 + y) dy dx &= \int_1^3 \int_{-1}^0 (x^2 + y) dy dx = \int_1^3 \left[ x^2 y + \frac{y^2}{2} \right]_{-1}^0 dx = \int_1^3 \left( x^2 - \frac{1}{2} \right) dx \\ &= \left[ \frac{x^3}{3} - \frac{x}{2} \right]_1^3 = \left( 9 - \frac{3}{2} \right) - \left( \frac{1}{3} - \frac{1}{2} \right) = \frac{23}{3}. \end{aligned}$$

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Of course a region of integration need not be rectangular. If  $D$  can be described by  $g(x) < y < h(x)$  for  $a < x < b$ , the (double) integral of  $f(x, y)$  over  $D$  will be

$$\int \int_D f(x, y) dy dx = \int_a^b \left( \int_{g(x)}^{h(x)} f(x, y) dy \right) dx.$$

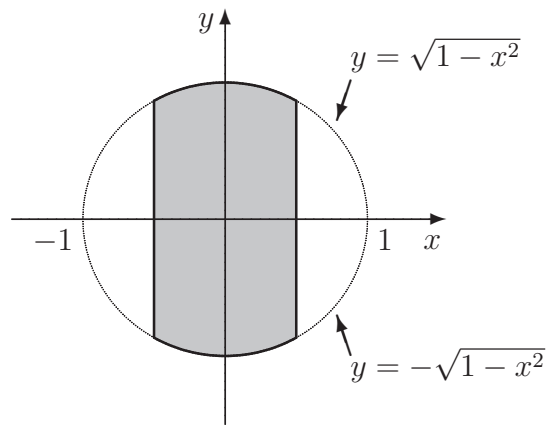
We can drop the brackets and simply write this as  $\int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx$ .

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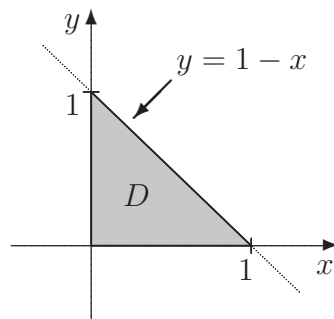
**question 7 .** Consider the region given by

$$-\frac{1}{2} \leq x \leq \frac{1}{2}, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}.$$

This region is part of a disc.



**question 8.** Consider the region  $D$ :



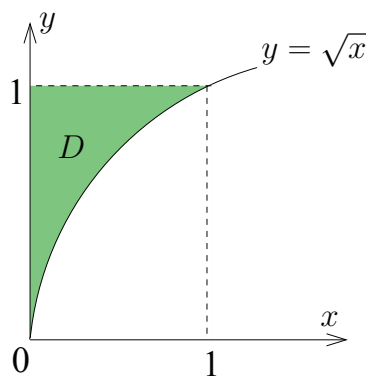
In this region we have  $0 \leq x \leq 1$  whilst, for a given  $x$ ,  $0 \leq y \leq 1 - x$ . Thus the region  $D$  is described by

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x.$$

**Question 9 .** Calculate  $\int \int_D y \, dA$  over the region  $D$  below:

**Solution.** In this region we have  $0 \leq x \leq 1$ , whilst, for a given  $x$ , we have

$$\sqrt{x} \leq y \leq 1.$$



Therefore

$$\int \int_D y \, dA = \int_0^1 \int_{\sqrt{x}}^1 y \, dy \, dx.$$

Calculating the inner integral first gives

$$\int_{\sqrt{x}}^1 y \, dy = \left[ \frac{y^2}{2} \right]_{\sqrt{x}}^1 = \frac{1}{2} - \frac{x}{2}.$$

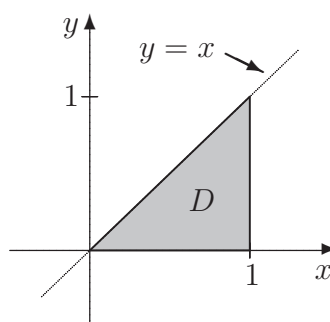
Hence

$$\int_D y \, dA = \int_0^1 \left( \frac{1}{2} - \frac{x}{2} \right) dx = \left[ \frac{x}{2} - \frac{x^2}{4} \right]_0^1 = 0 - \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{1}{4}.$$

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**question 10.** Consider the region  $D$ :



In this region we have  $0 \leq y \leq 1$ , whilst, for a given  $y$ ,  $y \leq x \leq 1$ .

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**Question 1 1** Calculate  $\iint_D (3 - x - y) \, dA$  where  $D$  is the region described in the previous question.

**Solution.** Recall that  $D$  is described by

$$0 \leq y \leq 1, \quad y \leq x \leq 1.$$

Therefore

$$\iint_D (3 - x - y) \, dA = \int_0^1 \int_y^1 (3 - x - y) \, dx \, dy.$$

Now

$$\int_y^1 (3 - x - y) \, dx = \left[ 3x - \frac{x^2}{2} - yx \right]_y^1 = \left( 3 - \frac{1}{2} - y \right) - \left( 3y - \frac{y^2}{2} - y^2 \right) = \frac{5}{2} - 4y + \frac{3y^2}{2}.$$

It follows that

$$\iint_D (3 - x - y) \, dA = \int_0^1 \left( \frac{5}{2} - 4y + \frac{3y^2}{2} \right) dy = \left[ \frac{5y}{2} - 2y^2 + \frac{y^3}{2} \right]_0^1 = \left( \frac{5}{2} - 2 + \frac{1}{2} \right) - 0 = 1.$$

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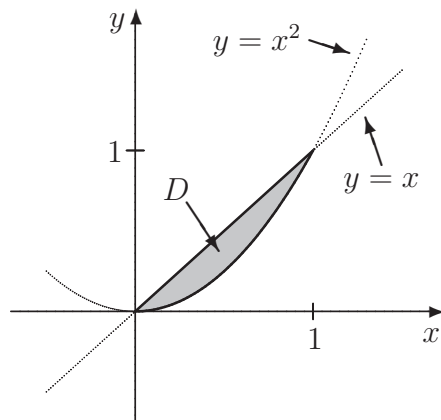


### 3 Interchanging the order of integration

As noted before, we can swap the order in which the integrals are carried out :  $\int \int_D f \, dy \, dx = \int \int_D f \, dA = \int \int_D f \, dx \, dy$ . It is sometimes easier to calculate the value of a double integral doing the integrations in one order than the other.

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**Example 7.6.** Consider the region  $D$  which lies between the line  $y = x$  and the parabola  $y = x^2$ :



This region can be described by

$$0 \leq x \leq 1, \quad x^2 \leq y \leq x.$$

On the other hand, the parabola is also given by the equation  $x = \sqrt{y}$  so the region  $D$  can also be described by

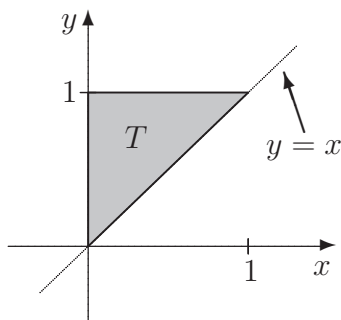
$$0 \leq y \leq 1, \quad y \leq x \leq \sqrt{y}.$$

It follows that we have

$$\int_0^1 \int_{x^2}^x f(x, y) \, dy \, dx = \iint_D f(x, y) \, dA = \int_0^1 \int_y^{\sqrt{y}} f(x, y) \, dx \, dy.$$

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**Question 1 .** Calculate  $\iint_T e^{y^2} \, dA$  where  $T$  is the triangular region with vertices  $(0, 0)$ ,  $(0, 1)$  and  $(1, 1)$ :



**Solution.** As a first attempt we may describe  $T$  by

$$0 \leq x \leq 1, \quad x \leq y \leq 1.$$

It follows that

$$\iint_T e^{y^2} dA = \int_0^1 \int_x^1 e^{y^2} dy dx.$$

The inner integral is then

$$\int_x^1 e^{y^2} dy$$

which can't be evaluated very easily! We've got stuck!

As a second attempt let us describe  $T$  the other way; that is

$$0 \leq y \leq 1, \quad 0 \leq x \leq y.$$

Then

$$\iint_T e^{y^2} dA = \int_0^1 \int_0^y e^{y^2} dx dy.$$

Evaluating the inner integral first gives

$$\int_0^y e^{y^2} dx = [xe^{y^2}]_0^y = ye^{y^2}.$$

It follows that

$$\iint_T e^{y^2} dA = \int_0^1 ye^{y^2} dy = \left[ \frac{e^{y^2}}{2} \right]_0^1 = \frac{e}{2} - \frac{1}{2} = \frac{1}{2}(e - 1),$$

where we used the substitution  $u = y^2$  to evaluate the integral.

**Note.** On evaluating a double integral  $I = \int \int_D f(x, y) dA$  by doing the  $y$  integral first, *i.e.* taking  $I = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$ , we integrate along vertical strips between the lower boundary, say  $y = g_1(x)$ , and the upper boundary, say  $y = g_2(x)$  (this gives a result which generally depends upon  $x$ , but definitely does not depend on  $y$ ):  $I_y(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy$ . We then total up the contributions of strips by integrating  $I_y$  from the lowest value of  $x$  taken in  $D$ , say  $a$ , to the largest, say  $b$  (see Fig. 7.1(i)):  $I = \int_a^b I_y(x) dx$ . On the other hand, doing the  $y$  integral first, we take  $I = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$ , we integrate along horizontal strips between the left-hand boundary, say  $x = h_1(y)$ , and the right-hand boundary, say  $x = h_2(y)$  (this gives a result which generally depends upon  $y$ , but definitely does not depend on  $x$ ):  $I_x(y) = \int_{h_1(y)}^{h_2(y)} f(x, y) dx$ . We then total up the contributions of strips by integrating  $I_x$  from the lowest value of  $y$  taken in  $D$ , say  $c$ , to the largest, say  $d$  (see Fig. 7.1(ii)). (Either way, the final result does not depend on either  $x$  or  $y$ !)

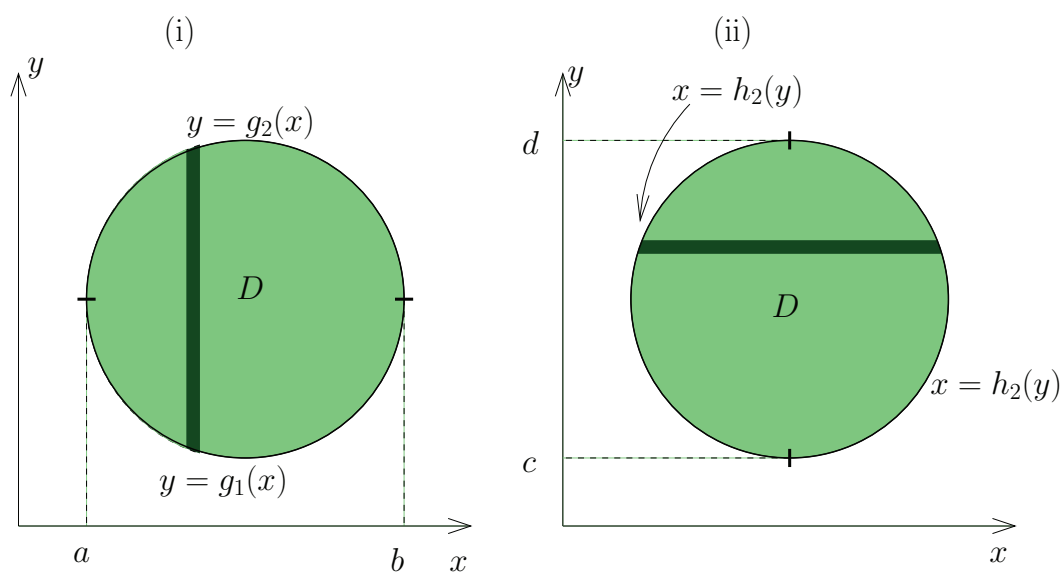


Figure 1: Different orders of integration over  $D$ .