Multiple Integrals

① Double integrals over rectangle

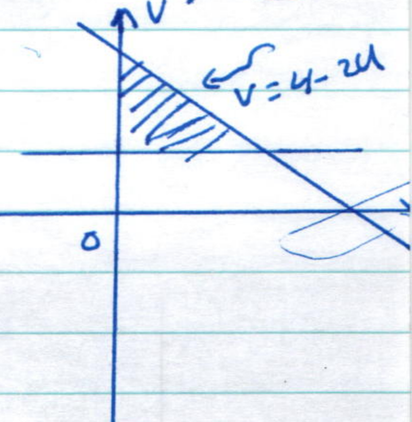
if $f(x,y)$ is continous through the rectangular region $R: a \le x \le b$, $c \le y \le d$ then the double integrals over R are defined by

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Example : $\displaystyle\int_0^{3/2}\int^{4-2u} \frac{4-2u}{v^2}\,dv\,du$

Sketch the region of integration and evaluate the integrals



$$= \int_0^{3/2} (4-2u)\left[-\frac{1}{v}\right]_1^{4-2u} du$$

$$= \int_0^{3/2} (4-2u)\left[-\frac{1}{4-2u}+1\right]$$

$$= \int_0^{3/2} (-1+4-2u)\,du = \int_0^{3/2} (3-2u)\,du$$

$$= \left[3u-u^2\right]_0^{3/2} = \frac{9}{2}-\frac{9}{4} = \frac{9}{4}$$

Example ②

Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 1 - 6x^2y$

and $R: 0 \leq x \leq 2, \quad -1 \leq y \leq 1$

Solution :-
$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{0}^{2}(1-6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1}\left(x - 2x^3y\right)\Big|_0^2\,dy$$

$$= \int_{-1}^{1}(2-16y)\,dy = \left[2y - 8y^2\right]_{-1}^{1} = \boxed{4}$$

Reversing the order of integration gives the same answer.

$$\int_0^2\int_{-1}^{1}(1-6x^2y)\,dy\,dx = \int_0^2\left(y - 3x^2y^2\right)\Big|_{-1}^{1}\,dx$$

$$= \int_0^2 (1-3x^2) - (-1-3x^2)\,dx$$

$$= \int_0^2 2\,dx = 2x\Big|_0^2 = \boxed{4}$$

Example ③

$$\iint_R x y e^{xy^2} \, dA \quad , R: 0 \le x \le 2, \; 0 \le y \le 1$$

Solution : $\displaystyle\int_0^2 \int_0^1 x y e^{xy^2} \, dy \, dx$

$t = y^2$
$dt = 2y \, dy$

$$= \frac{1}{2} \int_0^2 \int_0^1 x e^{xt} \, dt \, dx$$

$$= \frac{1}{2} \int_0^2 x \cdot \frac{e^{xt}}{x} \Big|_0^1 \, dx$$

$$= \frac{1}{2} \int_0^2 (e^x - 1) \, dx$$

$$= \frac{1}{2} \left[e^x - x \right]_0^2$$

$$= \frac{1}{2} \left[e^2 - 2 - 1 + 0 \right] = \frac{1}{2} (e^2 - 3)$$

Example (4) : calculate $\int_2^4 \int_1^3 (xy^2+y)\, dy\, dx$

Solution :- $\int_1^3 (xy^2+y)\, dy = \left[\frac{xy^3}{3} + \frac{y^2}{2}\right]_1^3$

$= \left(9x + \frac{9}{2}\right) - \left(\frac{x}{3} + \frac{1}{2}\right)$

$= \frac{26x}{3} + 4$

Therefore $\int_2^4 \int_1^3 (xy^2+y)\, dy\, dx$

$= \int_2^4 \left(\frac{26x}{3} + 4\right) dx$

$= \left[\frac{13x^2}{3} + 4x\right]_2^4$

$= \left[\frac{13 \times 16}{3} + 4x\right]_2^4$

$= \left(\frac{13 \times 16}{3} + 16\right) - \left(\frac{13 \times 4}{3} + 8\right)$

$= 13 \times 4 + 8 = 60$

Triple integrals in rectangular Coordinates

The volume V of a closed bounded region D in space is $V = \iiint_D dv$

How to evaluate the triple integrals?

1. integrate firstly with respect to z.
2. Sketch the Projection R of the solid on the xy-plane. from this, determine the limits of integration for the double integrals over R.
3. Integrate the double integrals as in the previous Sections $\iiint_D f(x,y,z)\, dv$

$$= \iint_R \left[\int f(x,y,z)\, dz \right] dA$$

* Defintion

The average value of a function $f(x,y,z)$ over a region D in space is defined by

$$av(f) = \frac{1}{\text{volume of } D} \iiint_D f(x,y,z)\, dv$$

②

Example ① :- Evaluate $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz \, dy \, dx$

$$= \int_0^1 \int_0^{1-x^2} x z \Big|_3^{4-x^2-y} dy \, dx = \int_0^1 \int_0^{1-x^2} x(4-x^2-y-3) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} (x - x^3 - xy) \, dy \, dx = \int_0^1 \left[xy - x^3 y - x\frac{y^2}{2} \right]_0^{1-x^2}$$

$$= \int_0^1 x(1-x^2) - x^3(1-x^2) - \frac{1}{2}x(1-x^2)^2 \Big] dx$$

$$= \int_0^1 \left[x - \cancel{x^3} - x^3 + x^5 - \frac{1}{2}x + \cancel{x^3} - \frac{1}{2}x^5 \right] dx$$

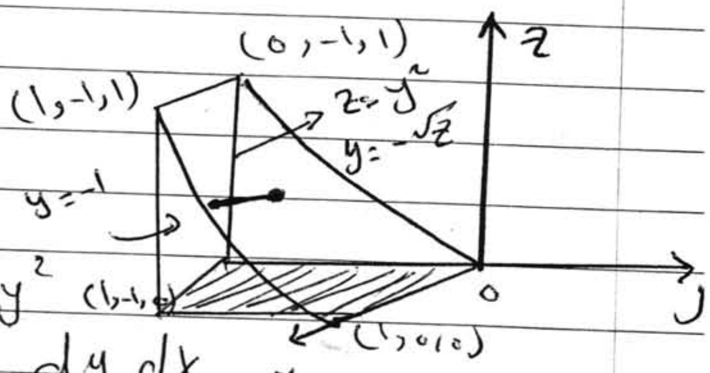
$$= \int_0^1 \left[\frac{1}{2}x - x^3 + \frac{1}{2}x^5 \right] dx$$

$$= \frac{1}{4}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6 = \frac{1}{4} - \frac{1}{4} + \frac{1}{12} - 0 = \boxed{\frac{1}{12}}$$

Class

Example 2 : (a) Evaluate $\int_0^1\int_{-1}^0\int_0^{y^2} dz\,dy\,dx$

(b) Rewrite the integral as an equivalent integral in the order $dy\,dz\,dx$



(a) Solution $\int_0^1\int_{-1}^0 z\Big|_0^{y^2} dy\,dx$

$$=\int_0^1\int_{-1}^0 y^2\,dy\,dx=\int_0^1 \frac{y^3}{3}\Big|_{-1}^0 dx=\frac{1}{3}\int_0^1 0-(-1)^3\,dx$$

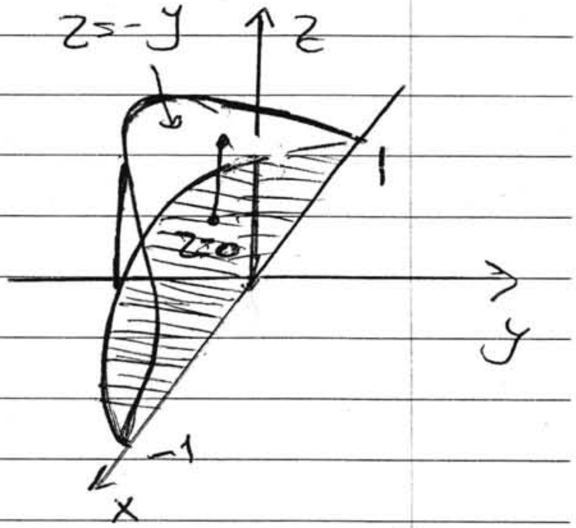
$$=\frac{1}{3}\int_0^1 dx=\frac{1}{3}x\Big|_0^1=\boxed{\frac{1}{3}}$$

(b) $$\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\,dz\,dx=\int_0^1\int_0^1 y\Big|_{-1}^{-\sqrt{z}} dz\,dx$$

$$=\int_0^1\int_0^1(-\sqrt{z}+1)\,dz\,dx=\int_0^1\Big[-\frac{2}{3}z^{3/2}+z\Big]_0^1 dx$$

$$=\int_0^1\left(\frac{-2}{3}+1\right)dx=\int_0^1\frac{1}{3}dx=\frac{1}{3}x\Big|_0^1=\boxed{\frac{1}{3}}$$

2 Find the volume of the region cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$



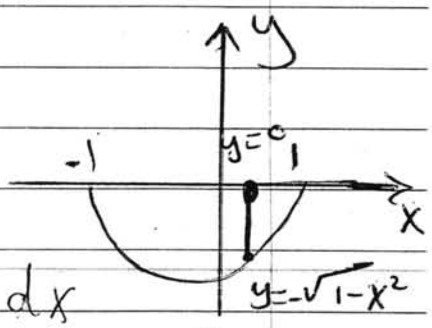
$$V=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx$$

$$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} -y\,dy\,dx=\int_{-1}^{1}-\frac{y^2}{2}\Big|_{-\sqrt{1-x^2}}^{0}dx$$

$$=-\frac{1}{2}\int\int_{-1}^{1} 0-\left(-\sqrt{1-x^2}\right)dx$$

$$=\frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx=\frac{1}{2}*2\int_{0}^{1}(1-x^2)\,dx$$

$$=\left[x-\frac{x^3}{3}\right]_0^1=1-\frac{1}{3}=\boxed{\frac{2}{3}}$$



$y^2=1-x^2$

$y=-\sqrt{1-x^2}$

Class

(4)

(3) Find the average value of $F(x,y,z) = x+y-z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x=1$, $y=1$, $z=2$

Solution :-

$$V(D) = \int_0^1\int_0^1\int_0^2 dz\,dy\,dx$$

$$= \int_0^1\int_0^1 z\Big|_0^2 dy\,dx$$

$$= \int_0^1\int_0^1 2\,dy\,dx = \int_0^1 2y\Big|_0^1 dx = \int_0^1 2\,dx = 2(x)\Big|_0^1 = 2$$

$$av(f) = \frac{1}{2}\int_0^1\int_0^1\int_0^2 (x+y-z)\,dz\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^1 \left(xz+yz-\frac{z^2}{2}\right]_0^2 dy\,dx = \frac{1}{2}\int_0^1\int_0^1 [2x+2y-2]\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[2xy+y^2-2y\right]_0^1 dx = \frac{1}{2}\int_0^1 (2x+1-2)\,dx$$

$$= \frac{1}{2}\int_0^1 (2x-1)\,dx = \frac{1}{2}\left[x^2-x\right]_0^1 = \frac{1}{2}[1-1] = 0$$

(5)

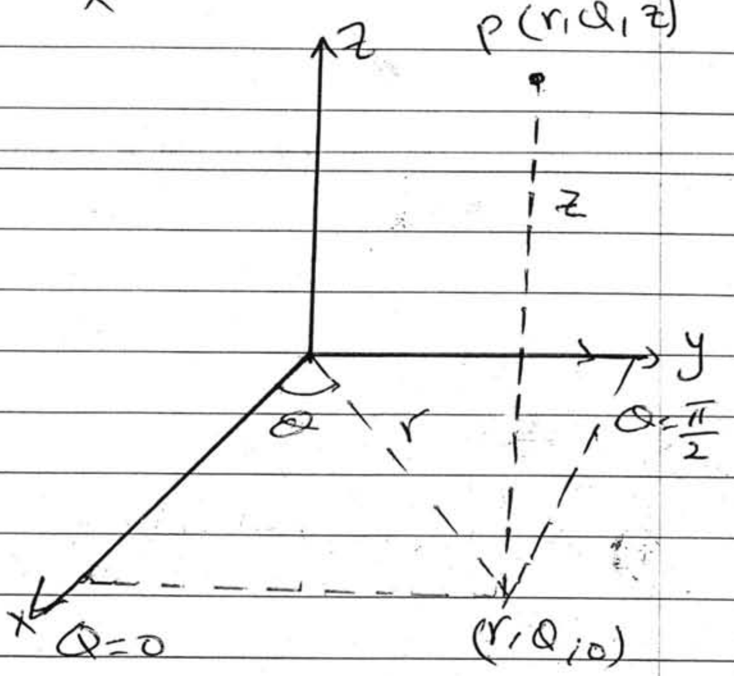
(4) Triple integrals in cylindrical and spherical Coordinates.

(1) Integration in cylindrical Coordinates

cylindrical Coordinates of a Point P in space is
P in space is $P(r, \theta, z)$ in which (i) r and θ are
the Polar Coordinates of the vertical Projection of
P on the xy_plane.
(ii) Z is the rectangular vertical Coordinates
Projection

$x = r\cos\theta \quad , y = r\sin\theta$

$x^2 + y^2 = r^2$ and $\tan\theta = \frac{y}{x}$



6

Example ① : Evaluate : $\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta+1)\,r\,d\theta\,dz\,dr$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r^2\sin\theta+r)\,d\theta\,dz\,dr$$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\Big[-r^2\cos\theta+r\theta\Big]_0^{2\pi}\,dz\,dr$$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\Big[-r^2+2\pi r+r^2-0\Big]\,dz\,dr$$

$$=\int_0^2\int_{r-2}^{\sqrt{4-r^2}}2\pi r\,dz\,dr=2\pi\int_0^2 rz\Big|_{r-2}^{\sqrt{4-r^2}}dr$$

$$=2\pi\int_0^2\Big[r\sqrt{4-r^2}-r(r-2)\Big]dr$$

$$=2\pi\int_4^0\left(-\frac{1}{2}\right)\sqrt{w}\,dw-2\pi\int_0^2(r^2-2r)\,dr$$

$w=4-r^2$
$dw=-2r\,dr$
$-\frac{1}{2}dw=r\,dr$

$$=\pi\int_0^4 w^{1/2}\,dw-2\pi\left[\frac{r^3}{3}-r^2\right]_0^2$$

$$=\pi\,\frac{w^{3/2}}{3/2}\Big|_0^4-2\pi\left[\frac{8}{3}-4\right]$$

$$=\frac{2}{3}\pi\left[(4)^{3/2}-0\right]-2\pi\left(\frac{-4}{3}\right)$$

$$=\frac{2}{3}\pi(8)+\frac{8\pi}{3}+=\frac{16\pi}{3}+\frac{8\pi}{3}=\frac{24\pi}{3}=\boxed{8\pi}$$

(5) Convert the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$ to an equivalent integral in cylinder Coordinate and evaluate the resulting integral.

Solution :-

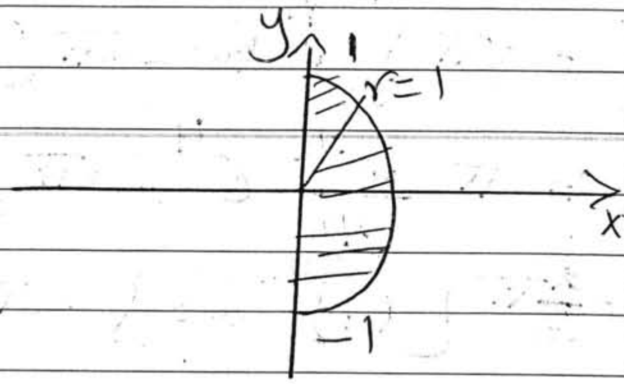
$$\int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{0}^{r\cos\theta} r^2\,dz\,r\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\int_{0}^{1} z\Big|_{0}^{r\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2} \frac{r^5}{5}\Big|_{0}^{1}\cos\theta\,d\theta = \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta$$

$$=\frac{2}{5}\int_{0}^{\pi/2}\cos\theta\,d\theta$$

$$=\frac{2}{5}\sin\theta\Big|_{0}^{\pi/2} = \frac{2}{5}[1-0] = \frac{2}{5}$$



(6) Find the volume of the region bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$ and $x+y+z=4$

Solution

$$\int_0^{2\pi}\int_0^{2}\int_0^{4-x-y} dz\, r\, dr\, d\theta$$

$$=\int_0^{2\pi}\int_0^{2} z\Big|_0^{4-x-y} r\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} (4-x-y)\, r\, dr\, d\theta$$

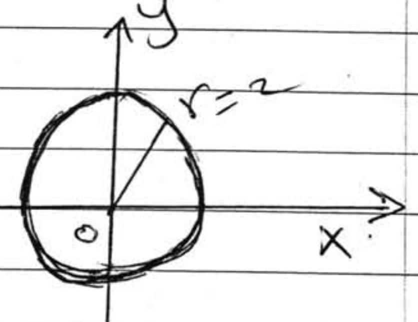
$$=\int_0^{2\pi}\int_0^{2} \left[4r-r^2\cos\theta-r^2\sin\theta\right] dr\, d\theta$$

$$=\int_0^{2\pi} \left[2r^2-\frac{1}{3}r^3\cos\theta-\frac{1}{3}r^3\sin\theta\right]_0^2 d\theta$$

$$=\int_0^{2\pi} \left[8-\frac{8}{3}\cos\theta-\frac{8}{3}\sin\theta\right] d\theta$$

$$=\left[8\theta-\frac{8}{3}\sin\theta+\frac{8}{3}\cos\theta\right]_0^{2\pi}$$

$$=16\pi-0+\frac{8}{3}-\frac{8}{3}=16\boxed{\pi}$$



Function and Definite integrals

1. The Error Function
2. The Gamma Function
3. The Beta Function
4. The Factorial function.

1 The Error Function

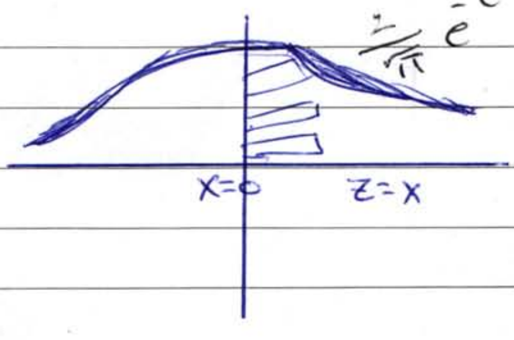
- the error function denoted erf, is defined by the integral

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz .$$

and clearly represents the area under the Curve e^{-z^2}, from $Z=0$ to $Z=X$

Z is a dummy variable because it only enables the curve to be described and any variable would do this.

- the variable Z is eliminated by the limits of integration thus leaving x as the only variable.



The factor $\frac{2}{\sqrt{\pi}}$ is introduced for convenience so that

$$\boxed{\operatorname{erf}\infty = 1}$$

$$\operatorname{erf}\infty = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-z^2}\,dz$$

or $$\operatorname{erf}\infty = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx.$$

To prove that $\operatorname{erf}\infty = 1$

Take $I = \int_0^\infty e^{-x^2}\,dx$ and solve

$$I = \int_0^\infty e^{-x^2}\,dx$$

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy.$$

$$I^2 = \int_0^\infty dx \int_0^\infty e^{-(x^2+y^2)}\,dy = \boxed{\iint_{R_{xy}} e^{-(x^2+y^2)}\,dy}$$

where the region of integration R_{xy} is the whole of the positive quadrant of the xy-plane.

Now transforming I^2 to Polar Coordinates θ & r using the equations

$x = r\cos\theta \quad , \quad y = r\sin\theta$

then the element of area ($dx\,dy$) has to be replaced by ($r\,dr\,d\theta$), and x^2+y^2 by r^2

$$I^2 = \iint_{R_{r,\theta}} e^{-r^2} r\,d\theta\,dr = \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{r=0}^{r=\infty} e^{-r^2} r\,dr$$

$$I^2 = \int_0^{\pi/2} \left(-1/2\, e^{-r^2}\right)\Big]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

$$\text{So } I^2 = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2} = \int_0^{\infty} e^{-x^2}\,dx$$

$$\therefore \operatorname{erf}\infty = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1$$

Properties of the error Function

① Differentiating error function equation gives directly

$$\frac{d}{dx}\operatorname{erf}x = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

using this result to integrate $\operatorname{erf}x$, by parts give

② Integration of $\operatorname{erf}x$

$$\int \operatorname{erf}x\,dx = x.\operatorname{erf}x - \int x \frac{2}{\sqrt{\pi}} e^{-x^2}\,dx + C$$
$$= x.\operatorname{erf}x + \frac{1}{\sqrt{\pi}} e^{-x^2} + C$$

where c is the constant of integration

$\int \operatorname{erf} x \, dx$, is sometimes tabulated under the symbol ($i \operatorname{erf} x$) with

$c = -\frac{1}{\sqrt{\pi}}$ so that $i \operatorname{erf} 0 = 0$

Another related function which is sometimes tabulated is the complementary error function ($\operatorname{erfc} x$), this is defined by the equation

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-x^2} \, dx$$

2 The gamma function $\Gamma(x)$

This is defined by the integral,

$$\Gamma(x) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

set $x = x+1$, then integrate by parts gives,

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\, dt = -t^{x} e^{-t} \Big]_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\Gamma(x+1) = x\,\Gamma(x), \quad (x>0)$$

when $x=n$, and n being a positive integer >1 we have,

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = \cdots = n!\,\Gamma(1)$$

and since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, give,

$$\Gamma(n+1) = n!$$

if $n=1 = \Gamma(2) = 1\,\Gamma(1) = 1!$

if $n=2 = \Gamma(3) = 2\,\Gamma(2) = 2\,\Gamma * 1\,\Gamma(1) = 2!$

if $n=3 = \Gamma(4) = 3\,\Gamma(3) = 3*2\,\Gamma(2) = 3\times 2\times 1\,\Gamma(1) = 3!$

in which $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t}\, dt$

and solved by putting $t = u^2$

and integrating, $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}$

$\because t = u^2 \Rightarrow dt = 2u\,du$ & $e^{-t} = e^{-u^2}$

$t^{-1/2} = (u^2)^{-1/2} = u^{-1} = \frac{1}{u}$

Now substinte for those in main integral you get

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\, du = 2 \cdot \frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\pi}}$$

Since we had prove $\int_0^\infty e^{-u^2}\, du = \boxed{\frac{\sqrt{\pi}}{2}}$

Previously so $\Gamma(1.5) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

The relation $\Gamma(x+1) = x\,\Gamma(x)$ is also useful in defining the Γ-function for negative values of x

if were-writting this relation

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

- Note that $\Gamma(x)$ becomes in finite at $x=0$

Class

Note :- All negative integral values of x becomes infinite, and it is important to emphasise that the value of $\Gamma(x)$ for negative values of x are not given by the integral form but by the recurrence relation

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

For tables of the Γ - function see Jahnke and Emde ("Tables of Functions" Dover Publications, 1945.)

We now show how certain integrals may be evaluated in terms of the Γ - function.

(3) The Beta function (β)

The beta-function $\beta(m,n)$ is defined by the integral:-

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

for $m > 0$ & $n > 0$

It is necessarily symmetric in m and n

$$\beta(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

Since, by putting $1-x = u$ and

$du = -dx$ in above integral will give the formula of beta-function $\beta(n,m)$

$$\beta(n,m) = -\int_1^0 u^{m-1}(1-u)^{n-1}\,du = \beta(m,n).$$

An alternative form of the beta-function obtained by putting

$x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta$

$1-x = \cos^2\theta$ and substitute in main eq of $\beta(m,n)$ to get

$$\beta(m,n) = \int_0^{\pi/2} \sin\theta^{2(m-1)} \cos\theta^{2(n-1)} \cdot 2\sin\theta\cos\theta\,d\theta$$

Class

where the limits of integration changed from $0 \longrightarrow 1$, to $0 \rightarrow \frac{\pi}{2}$, by

Putting $x=1$, & $x=0$ in eq. $x=\sin^2\theta$

to find $\theta = 0 \longrightarrow \frac{\pi}{2}$

So $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

From which we may derive a reduction formula relating $B(m,n)$ as

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} B(1,1),$$

where $B(1,1) = 1$ and $B(\frac{1}{2},\frac{1}{2}) = \pi$

and how to prove that $B(1,1) = 1$ and $B(\frac{1}{2},\frac{1}{2}) = \pi$ is by using any one of the above two

relations and it leave it for you to prove it. the relation between Gamma and Beta function is

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

(4) The Factorial Function

This is defined by the integral.

$$n! = \int_0^\infty e^{-t} t^n \, dt = \Gamma(n+1)$$

also $n! = n(n-1)(n-2) - - - - \times 3 \times 2 \times 1 \times 0!$

where $0! = 1$ and to Prove that

$$0! = \int_0^\infty e^{-t} \, dt = -e^{t} \Big|_0^\infty = 1$$

For example $3! = 3 \times 2 \times 1 \times 0! = 3 \times 2!$

$4! = 4 \times 3 \times 2 \times 1 \times 0! = 4 \times 3!$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0! = 7 \times 6! = 7 \times 6 \times 5!$

$= 7 \times 6 \times 5 \times 4!$

① $\boxed{\text{Prove that} \quad \Gamma(n+1) = n!}$

This is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Set $x = x+1$, then integrate by parts

gives

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\, dt = -t^{x} e^{-t}\Big]_0^\infty + x\int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\Gamma(x+1) = x\,\Gamma(x) \quad , \quad (x > 0)$$

when $x = n$, and n being a positive integer ≥ 1
we have, $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\Gamma(1)$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\,\Gamma(1)$$

and since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, gives

$$\Gamma(n+1) = n!$$

② Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Solution:- $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} dt$, and solved by

Putting $t = u^2$ and integrating, $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} dt$

$\therefore t = u^2 \Rightarrow dt = 2u\,du$ and $e^{-t} = e^{-u^2}$

and $t^{-1/2} = (u^2)^{-1/2} = u^{-1} = \frac{1}{u}$

Now substitute for those in main integral you get $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2} du = \cancel{2} \cdot \frac{\sqrt{\pi}}{\cancel{2}} = \boxed{\sqrt{\pi}}$