# الجامعة التكنولوجية

## قسم الهندسة الكيمياوية

## المرحلة الاولى

# الرياضيات

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# Preliminaries

## Real Numbers & the Real Line :

Much of calculus is based on properties of the real number system. Real numbers that can be expressed as decimals, such as

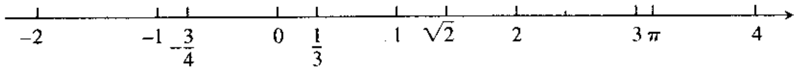
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever.

The real numbers can be represented geometrically as points on a number line called the real line.




**Rules for Inequalities**

If $a$, $b$, and $c$ are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If $a$ and $b$ are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

We distinguish three special subsets of real numbers.

1. The natural numbers, namely $1, 2, 3, 4, \ldots$
2. The integers, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The rational numbers, namely the numbers that can be expressed in the form of a function $m/n$, where $m$ and $n$ are integers and $n \neq 0$. Examples are

$\frac{1}{3}$, $-\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}$, $\frac{200}{13}$ and $57 = \frac{57}{1}$

The rational numbers are precisely the real numbers with decimal expansions that are either

a) terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000\ldots = 0.75 \quad \text{or}$$

b) eventually repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}$$

The bar indicates the block of repeating digits.

Real numbers that are not rational are called irrational number. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are $\pi$, $\sqrt{2}$, $\sqrt[3]{5}$ and $\log_{10} 3$

Set $\Rightarrow$ Elements

$\cup$ union

$\cap$ intersection

$\phi$ empty set

The set $A$ consisting of the natural numbers (or positive integers) less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\}$$

The entire set of integers is written as

$$\{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x \mid x \text{ is an integer and } 0 < x < 6\}$$

is the set of positive integers less than 6.

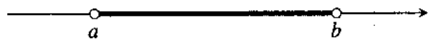
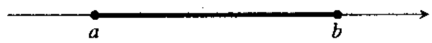
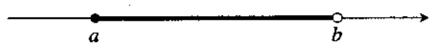
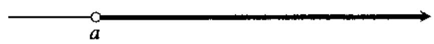
## Intervals

Finite intervals : intervals of numbers corresponding to line segments.

Infinite intervals : intervals corresponding to rays and the real line.

Finite intervals : 1. Closed
2. Half-open
3. Open

Types of intervals

| | Notation | Set description | Type | Picture |
|---|---|---|---|---|
| **Finite:** | $(a, b)$ | $\{x \mid a < x < b\}$ | Open |  |
| | $[a, b]$ | $\{x \mid a \le x \le b\}$ | Closed |  |
| | $[a, b)$ | $\{x \mid a \le x < b\}$ | Half-open |  |
| | $(a, b]$ | $\{x \mid a < x \le b\}$ | Half-open |  |
| **Infinite:** | $(a, \infty)$ | $\{x \mid x > a\}$ | Open |  |
| | $[a, \infty)$ | $\{x \mid x \ge a\}$ | Closed |  |
| | $(-\infty, b)$ | $\{x \mid x < b\}$ | Open |  |
| | $(-\infty, b]$ | $\{x \mid x \le b\}$ | Closed |  |
| | $(-\infty, \infty)$ | $\mathbb{R}$ (set of all real numbers) | Both open and closed |  |

Example : Solve the following inequalities and show their solution sets on the real line.

a) $2x - 1 < x + 3$

b) $-\frac{x}{3} < 2x + 1$

c) $\frac{6}{x-1} \geqslant 5$

Solution

a) $2x - 1 < x + 3 \quad (+1)$

$2x < x + 4 \quad (-x)$

$x < 4$

The solution set is the open interval $(-\infty, 4)$.

b) $-\frac{x}{3} < 2x + 1 \quad (\times 3)$

$-x < 6x + 3 \quad (+x)$

$0 < 7x + 3 \quad (-3)$

$-3 < 7x \quad (\div 7)$

$-\frac{3}{7} < x$

The solution set is the open interval $(-3/7, \infty)$.

c) The inequality $6/(x-1) \geqslant 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, $(x-1)$ is positive and the inequality will be preserved if we multiply both sides by $(x-1)$.

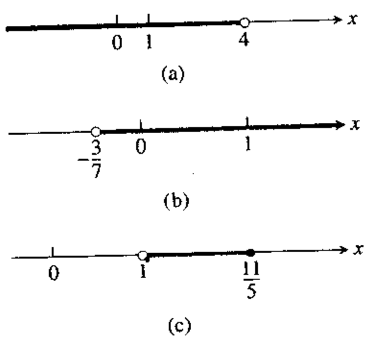
$$\frac{6}{x-1} \geqslant 5 \quad (\times (x-1))$$

$$6 \geqslant 5x - 5 \quad (+5)$$

$$11 \geqslant 5x$$

$$\frac{11}{5} \geqslant x \quad \text{or} \quad x \leqslant \frac{11}{5}$$

The solution set is the half-open interval $(1, 11/5]$.



## Absolute Value

The absolute value of a number $x$, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases}$$

Example: Finding absolute values

$|3| = 3$ , $|0| = 0$ , $|-5| = 5$ , $|-|a|| = |a|$

**Absolute Value Properties**

| | | |
|---|---|---|
| 1. | $\lvert -a \rvert = \lvert a \rvert$ | A number and its additive inverse or negative have the same absolute value. |
| 2. | $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$ | The absolute value of a product is the product of the absolute values. |
| 3. | $\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$ | The absolute value of a quotient is the quotient of the absolute values. |
| 4. | $\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$ | The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values. |

Note. $|-a| \neq -|a|$

$|-3| = 3 \qquad -|3| = -3$

* if $a$ & $b$ differ in sign, then $|a+b|$ is less than $|a|+|b|$

$|-3+5| = |2| = 2 < |-3| + |5| = 8$

* in all other cases $|a+b|$ equals $|a| + |b|$

$|3+5| = |8| = |3| + |5|$

$|-3-5| = |-8| = 8 = |-3| + |-5|$

**Absolute Values and Intervals**

If $a$ is any positive number, then

| | | | |
|---|---|---|---|
| 5. | $\lvert x \rvert = a$ | if and only if | $x = \pm a$ |
| 6. | $\lvert x \rvert < a$ | if and only if | $-a < x < a$ |
| 7. | $\lvert x \rvert > a$ | if and only if | $x > a$ or $x < -a$ |
| 8. | $\lvert x \rvert \leq a$ | if and only if | $-a \leq x \leq a$ |
| 9. | $\lvert x \rvert \geq a$ | if and only if | $x \geq a$ or $x \leq -a$ |

Example : Solve the equation with absolute values.

$$|2x-3| = 7$$

solution: By property 5, $2x-3 = \pm 7$

$$2x - 3 = 7 \qquad\qquad 2x - 3 = -7$$
$$2x = 10 \qquad\qquad 2x = -4$$
$$x = 5 \qquad\qquad x = -2$$

The solutions of $|2x-3| = 7$ are $x = 5$ & $x = -2$

Example: Solve the inequality with absolute values.

$$\left|5 - \frac{2}{x}\right| < 1$$

Solution: By property 6, $-1 < 5 - \frac{2}{x} < 1 \quad (-5)$

$$-6 < -\frac{2}{x} < -4 \qquad (x - \tfrac{1}{2})$$

$$3 > \frac{1}{x} > 2 \qquad \text{(reciprocals)}$$

$$\frac{1}{3} < x < \frac{1}{2}$$

The solution set is the open interval $(1/3, 1/2)$

Example : Solve the inequality and show the solution set on the real line:

a) $|2x-3| \leqslant 1$ b) $|2x-3| \geqslant 1$

Solution:

a) $|2x-3| \leqslant 1$

By property 8, $-1 \leqslant 2x-3 \leqslant 1 \quad (+3)$

$$2 \leqslant 2x \leqslant 4 \qquad (\div 2)$$

$$1 \leqslant x \leqslant 2$$

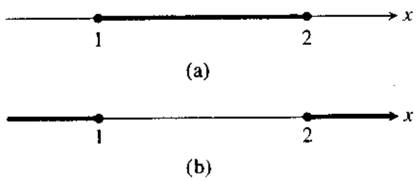
The solution set is the closed interval $[1, 2]$.

b) $|2x-3| \geqslant 1$

By property 9,

$$2x-3 \geq 1 \quad \text{or} \quad 2x-3 \leq -1 \qquad (\div 2)$$
$$x-\frac{3}{2} \geq \frac{1}{2} \quad \text{or} \quad x-\frac{3}{2} \leq -\frac{1}{2} \qquad (+\frac{3}{2})$$
$$x \geq 2 \quad \text{or} \quad x \leq 1$$

The solution set is $(-\infty, 1] \cup [2, \infty)$.



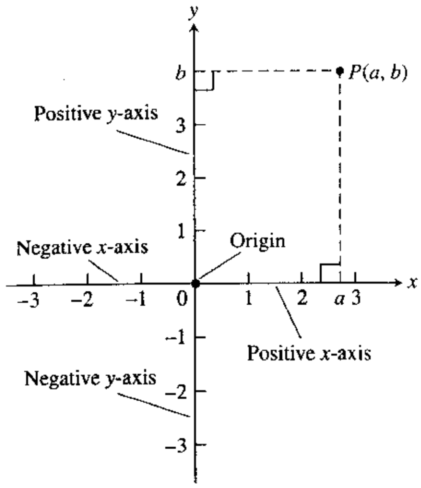

## Lines, Circles, and Parabolas

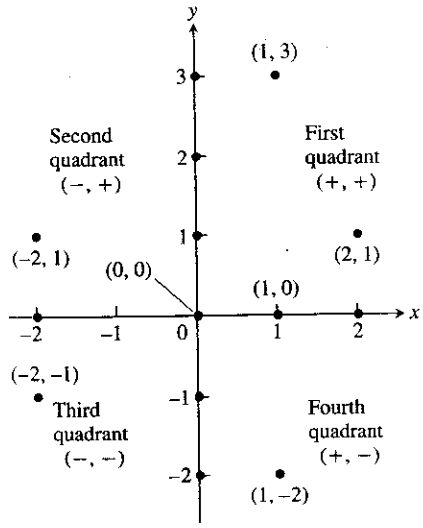
### Cartesian Coordinates in the Plane

Coordinate axes 1. $x$-coordinate (abscissa)
2. $y$-coordinate (ordinate)

Coordinate pair, Origin

Cartesian Coordinate = Rectangular Coordinate.



# Increments and Straight Lines

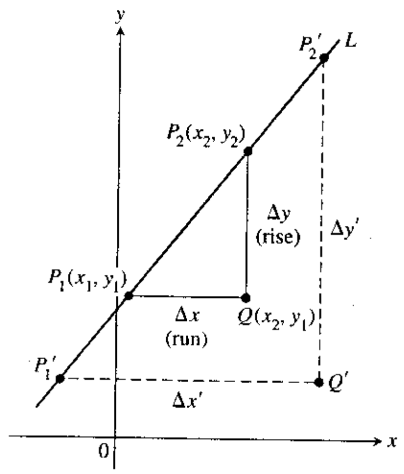
When a particle moves from one point in the plane to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

$$\Delta x = x_2 - x_1$$
$$\Delta y = y_2 - y_1$$

Any nonvertical line in the plane has the property that the ratio

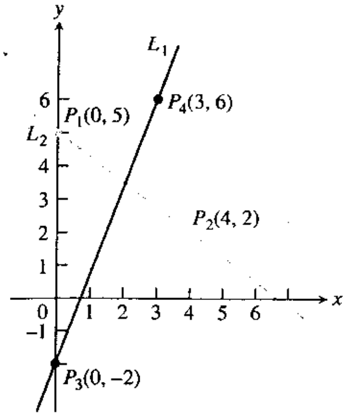
$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope}$$

has the same value for every choice of the two points $P_1(x_1, y_1)$ & $P_2(x_2, y_2)$ on the line.




Triangles $P_1QP_2$ and $P_1'Q'P_2'$ are similar, so the ratio of their sides has the same value for any two points on the line. This common value is the line's slope.

The slope tells up the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right.




The slope of $L_1$ is

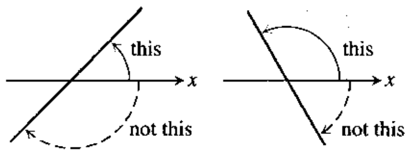
$$m = \frac{\Delta y}{\Delta x} = \frac{6 - (-2)}{3 - 0} = \frac{8}{3}.$$

That is, $y$ increases 8 units every time $x$ increases 3 units. The slope of $L_2$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 5}{4 - 0} = \frac{-3}{4}.$$

That is, $y$ decreases 3 units every time $x$ increases 4 units.

The angle of inclination of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line

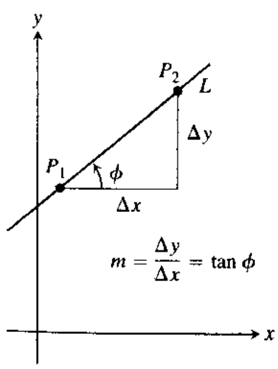



Angles of inclination are measured counterclockwise from the $x$-axis.

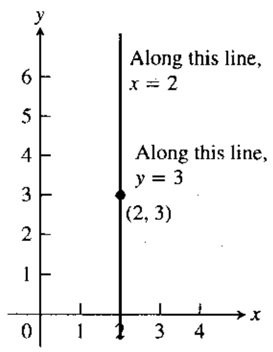
The inclination of a horizontal line is $0°$. The inclination of a vertical line is $90°$.

$$0 \leqslant \phi < 180$$

$$m = \tan \phi$$




The slope of a nonvertical line is the tangent of its angle of inclination.




The standard equations for the vertical and horizontal lines through (2, 3) are $x = 2$ and $y = 3$.

$$m = \frac{y - y_1}{x - x_1} \Rightarrow y - y_1 = m(x - x_1)$$

$$\therefore \boxed{y = y_1 + m(x - x_1)}$$ *the point-slope equation*

*Example*: Write an equation for the line through the point $(2,3)$ with slope $-3/2$.

Solution: $x_1 = 2$, $y_1 = 3$, & $m = -3/2$

$$y = y_1 + m(x - x_1) \Rightarrow y = 3 - \frac{3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + 6$$

when $x = 0$, $y = 6$ so the line intersects the y-axis at $y = 6$

Example: Write an equation for the line through $(-2,-1)$ & $(3,4)$.

Solution:

$$m = \frac{-1-4}{-2-3} = \frac{-5}{-5} = 1$$

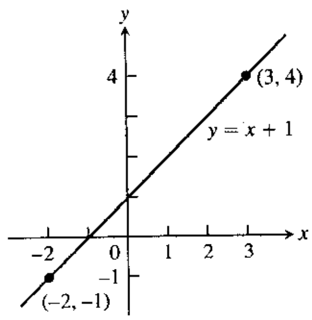
with $(-2,-1)$

$$y = y_1 + m(x - x_1)$$
$$y = -1 + 1 \cdot (x - (-2))$$
$$y = -1 + x + 2$$
$$y = x + 1$$

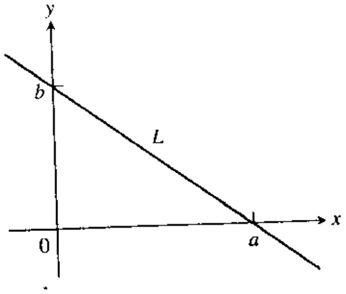
with $(3,4)$

$$y = y_1 + m(x - x_1)$$
$$y = 4 + 1 \cdot (x - 3)$$
$$y = 4 + x - 3$$
$$y = x + 1$$

Same result




The y-coordinate of the point where a nonvertical line intersects the y-axis is called the y-intercept of the line. Similarly, the x-intercept of a nonhorizontal line is the x-coordinate of the point where it crosses the x-axis




Line $L$ has $x$-intercept $a$ and $y$-intercept $b$.

$$y = b + m(x - 0)$$

$\boxed{y = mx + b}$ the slope-intercept equation.

if intercept $(b) =$ zero $\Rightarrow$ $y = mx$ pass through the origion ( Linear equation ).

$$Ax + By = C \qquad (A \; \& \; B \text{ not both } 0)$$

the general linear equation.

Example : Find the slope and y-intercept of the line

$$8x + 5y = 20$$

Solution :

$$8x + 5y = 20$$

$$5y = -8x + 20$$

$$y = -\frac{8}{5}x + 4$$

The slope is $m = -8/5$ . The y-intercept is $b = 4$

## Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination, so they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines $L_1$ and $L_2$ are perpendicular, their slopes $m_1$ and $m_2$ satisfy $m_1 m_2 = -1$ , so each slope is the negative reciprocal of the other :

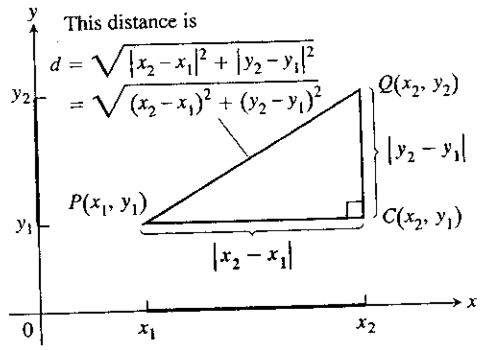
$$m_1 = -\frac{1}{m_2} \quad , \quad m_2 = -\frac{1}{m_1}$$

# Distance and Circles in the Plane

The distance between points in the plane is calculated with a formula

$$P(x_1, y_1) \quad \& \quad Q(x_2, y_2)$$

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$




The distance from the origin to $P(x, y)$ is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

By definition, a circle of radius $a$ is the set of all points $P(x, y)$ whose distance from some center $C(h, k)$ equals $a$. From the distance formula, $P$ lies on the circle if and only if

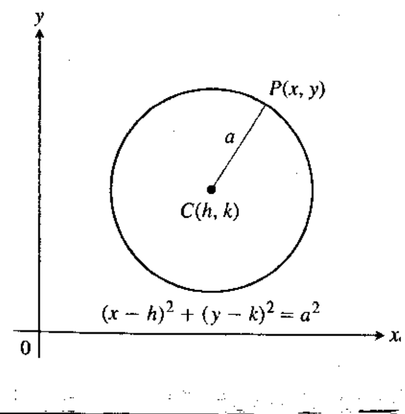
$$\sqrt{(x-h)^2 + (y-k)^2} = a$$

so

$$\boxed{(x-h)^2 + (y-k)^2 = a^2}$$

This is the standard equation of a circle with center $(h, k)$ and radius $a$. The circle of radius $a = 1$ and centered at the origin is the unit circle with equation

$$x^2 + y^2 = 1$$



Example : Find the center & radius of the circle

$$(x-1)^2 + (y+5)^2 = 3$$

Solution : $h = 1$ , $k = -5$ & $a^2 = 3$

$\therefore$ The center is $(1, -5)$ & the radius is $a = \sqrt{3}$

Example : Find an equation for the circle of radius 2 centered at $(3,4)$ ?

Solution : $(x-h)^2 + (y-k)^2 = a^2$

$h = 3$ , $k = 4$ & $a = 2$

$$(x-3)^2 + (y-4)^2 = 4$$

Example : Find the center and radius of the circle

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

Solution : We convert the equation to standard form by completing the squares in $x$ & $y$ :

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

$$(x^2 + 4x \quad) + (y^2 - 6y \quad) = 3$$

$$\left(x^2 + 4x + \left(\tfrac{4}{2}\right)^2\right) + \left(y^2 - 6y + \left(\tfrac{-6}{2}\right)^2\right) = 3 + \left(\tfrac{4}{2}\right)^2 + \left(\tfrac{-6}{2}\right)^2$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$$

$$(x+2)^2 + (y-3)^2 = 16$$

The center is $(-2, 3)$ & the radius is $a = 4$

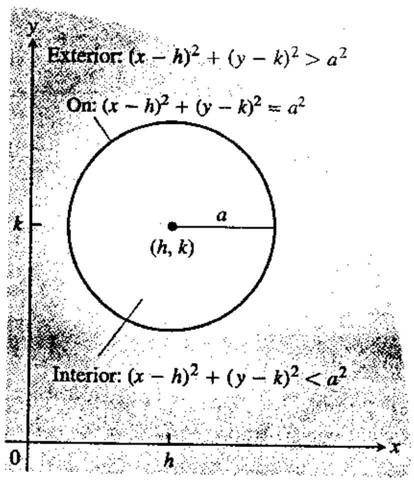
Note :

$(x-h)^2 + (y-k)^2 < a^2$

the interior region of the circle

$(x-h)^2 + (y-k)^2 > a^2$

the exterior region of the circle



## Parabolas :

The general equation is

$$y = ax^2 + bx + c \qquad a \neq 0$$

if $a > 0 \Rightarrow$ the parabola open upward,
if $a < 0 \Rightarrow$ " " " downward.

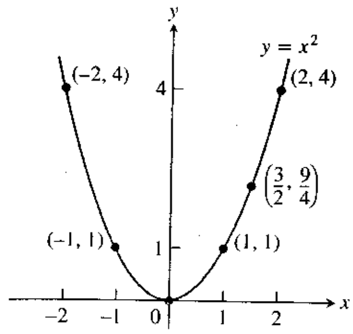
The graph of an equation of the form

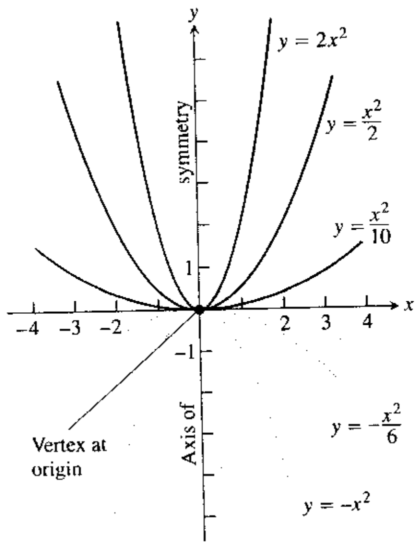
$$y = ax^2$$

is a parabola whose axis (axis of symmetry) is the $y$-axis. The parabola's vertex (point where the parabola and axis cross) lies at the origin.

Notes :

1. The larger the value of $|a|$, the narrower the parabola.
2. If $a = 0$, then we have $y = bx + c$ which is an equation for a line.
3. The axis is the line $x = -b/2a$. Its $x$-coordinate is $x = -b/2a$; its $y$-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.




Example : Graph the equation

$$y = -\frac{1}{2}x^2 - x + 4$$

Solution : $y = ax^2 + bx + c$

$$\therefore \quad a = -\frac{1}{2}, \quad b = -1, \quad c = 4$$

Since $a < 0$, the parabola opens downward.

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1 \Rightarrow x = -1 \text{ (Vertical line)}$$

$$\therefore\ y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2} \Rightarrow y = \frac{9}{2}$$

The vertex is $(-1, 9/2)$

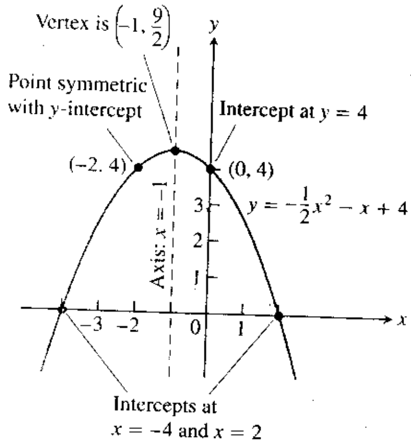
The $x$-intercepts are where $y = 0$

$$-\frac{1}{2}x^2 - x + 4 = 0$$

$$x^2 + 2x - 8 = 0$$

$$(x-2)(x+4) = 0$$

$$x = 2 \quad , \quad x = -4$$




## Functions; Domain and Range :

In calculus we may want to refer to an unspecified function without having any particular formula in mind. A symbolic way to say "$y$ is a function of $x$" is by writing

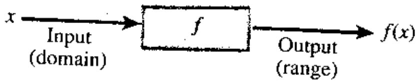
$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"})$$

In this notation, the symbol $f$ represents the function. The letter $x$, called the independent variable, represents the input value of $f$, and $y$, the dependent variable, represents the corresponding output value of $f$ at $x$.

- DEFINITION Function

A function from a set $D$ to a set $Y$ is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

The set $D$ of all possible input values is called the domain of the function. The set of all values of $f(x)$ as $x$ varies throughout $D$ is called the range of the function. The range may not include every element in the set $Y$.




Example : Find the domain & range of the following

$y = x^2$ , $y = 1/x$ , $y = \sqrt{x}$ , $y = \sqrt{4-x}$

& $y = \sqrt{1-x^2}$

Solution :

$y = x^2$

domain $\Rightarrow$ $\{x \mid x \text{ is real}\}$ or $(-\infty, \infty)$

range $\Rightarrow$ $\{y \mid y \geqslant 0\}$ or

$[0, \infty)$

| $x$ | $y = x^2$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| $-1$ | 1 |
| $-2$ | 4 |

$y = \dfrac{1}{x}$

domain $\Rightarrow$ $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

range $\Rightarrow$ $\{y \mid y \neq 0\}$ or

$(-\infty, 0) \cup (0, \infty)$

| $x$ | $y = 1/x$ |
|---|---|
| 0 | ? |
| 1 | 1 |
| 2 | 1/2 |
| $-1$ | $-1$ |
| $-2$ | $-1/2$ |

$y = \sqrt{x}$

domain $\Rightarrow$ $\{x \mid x \geqslant 0\}$ or $[0, \infty)$

range $\Rightarrow$ $\{y \mid y \geqslant 0\}$ or $[0, \infty)$

$y = \sqrt{4-x}$ $\Rightarrow$ $4 - x \geqslant 0 \Rightarrow x \leqslant 4$

domain $\Rightarrow$ $\{x \mid x \leqslant 4\}$ or $(-\infty, 4]$

range $\Rightarrow$ $\{y \mid y \geqslant 0\}$ or $[0, \infty)$

$y = \sqrt{1-x^2}$ $\quad$ $1 - x^2 \geqslant 0 \Rightarrow x^2 \leqslant 1$

$x \leqslant 1$

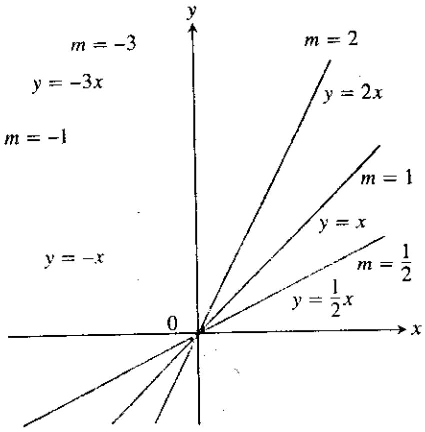
$-x \leqslant 1 \Rightarrow x \geqslant -1$

domain $\Rightarrow$ $\{x \mid x \leqslant 1 \text{ or } x \geqslant -1\}$ or $[-1, 1]$

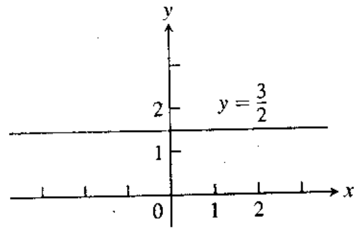
range $\Rightarrow$ $[0, 1]$

# Identifying Functions:

- Linear Functions: A function of the form $f(x) = mx + b$, for constants $m$ and $b$, is called a linear function.




The collection of lines $y = mx$ has slope $m$ and all lines pass through the origin.

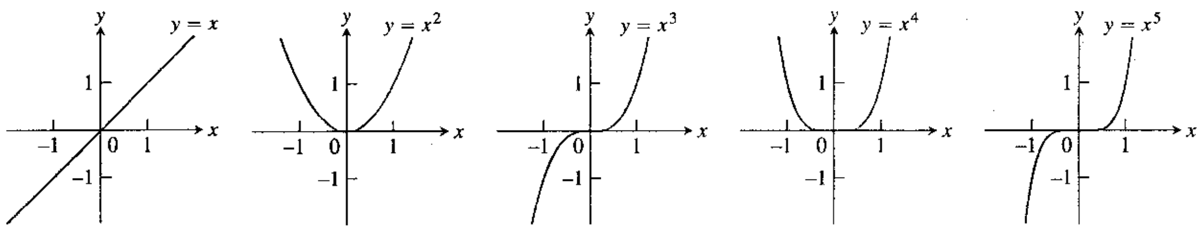



A constant function has slope $m = 0$.

- Power Functions: A function $f(x) = x^a$, where $a$ is a constant, is called a power function. There are several important cases to consider.

a) $a = n$, a positive integer.

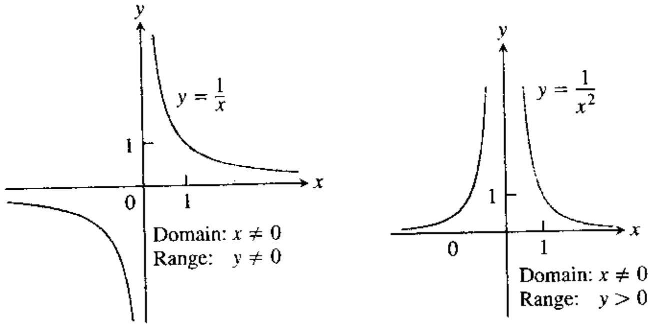
$f(x) = x^n$ for $n = 1, 2, 3, 4, 5$




b) $a = -1$ or $a = -2$

$f(x) = x^{-1} = 1/x$ & $g(x) = x^{-2} = 1/x^2$

Both functions are defined for all $x \neq 0$ (You can never divide by zero).




(a) (b)

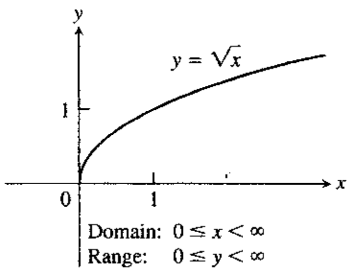
c) $a = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \& \frac{3}{2}$

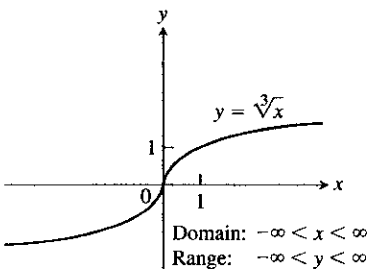
$f(x) = x^{1/2} = \sqrt{x}$ square root function.

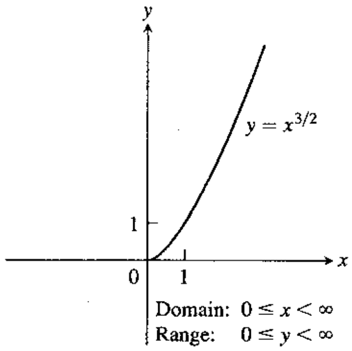
$g(x) = x^{1/3} = \sqrt[3]{x}$ cube root function.

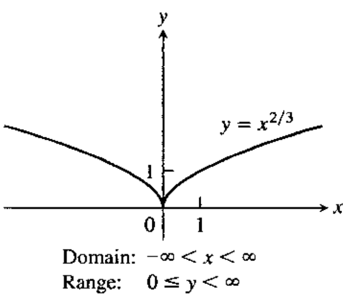
<u>Note:</u> $y = x^{2/3} = (x^{1/3})^2$

$y = x^{3/2} = (x^{1/2})^3$



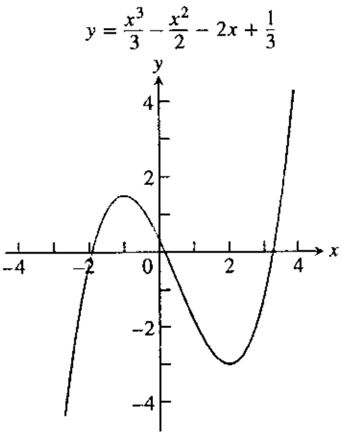

Polynomials : A function p is a polynomial if

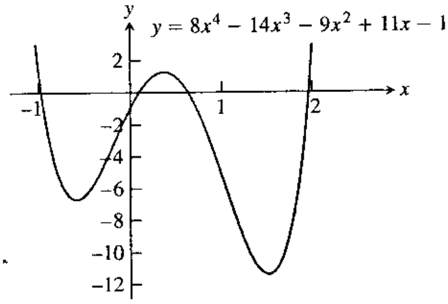
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

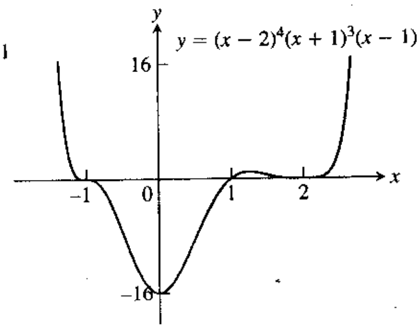
where $n$ is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the coefficients of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then $n$ is called the degree of the polynomial.

polynomials of degree 2 $\Rightarrow$ $p(x) = ax^2 + bx + c$ $\Rightarrow$ quadratic function.

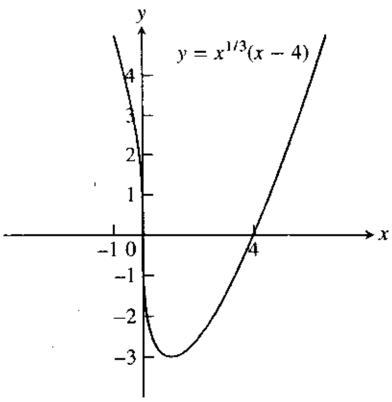
polynomials of degree 3 $\Rightarrow$ $p(x) = ax^3 + bx^2 + cx + d$ $\Rightarrow$ cubic function.

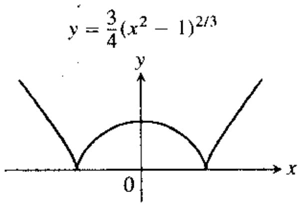


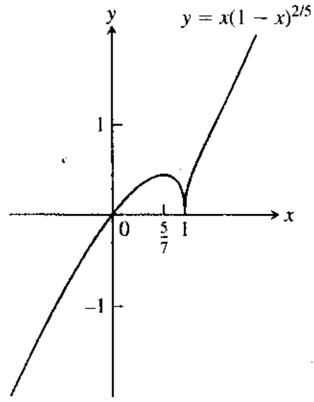



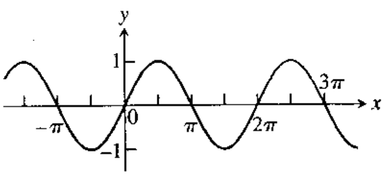

Algebraic Functions : Any function constructed from polynomials using algebraic operation (addition, subtraction, multiplication, division, and taking roots) lies within the class of algebraic functions.



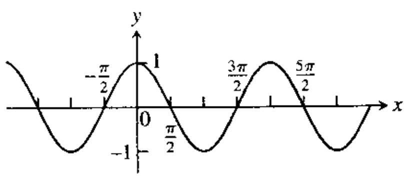




Trigonometric Functions : Sine , Cosine functions , . . .

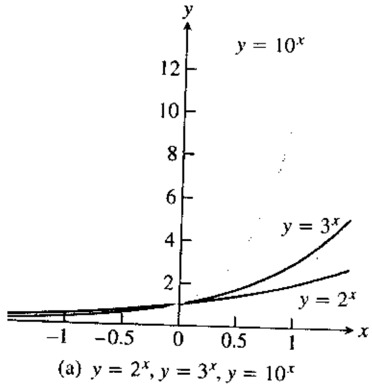


$f(x) = \sin x$

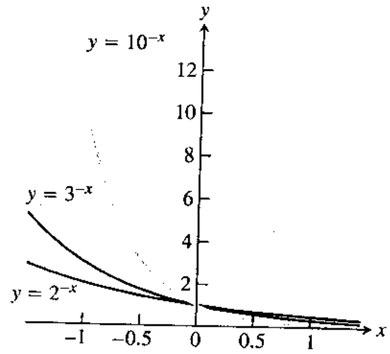


$f(x) = \cos x$

Exponential Functions : Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called exponential functions. All exponential functions have domain $(-\infty, \infty)$ and rang $(0, \infty)$.

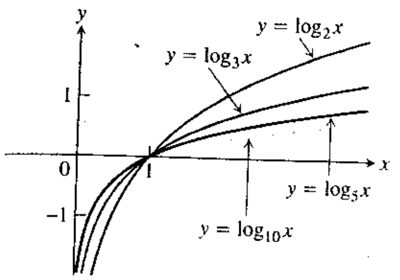



(a) $y = 2^x, y = 3^x, y = 10^x$




(b) $y = 2^{-x}, y = 3^{-x}, y = 10^{-x}$

Logarithmic Functions : These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the inverse functions of the exponential functions. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.




Transcendental Functions : These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well (such as the hyperbolic functions).

## Increasing Versus Decreasing Functions :

If the graph of a function climbs or rises as you move from left to right, we say that the function is increasing. If the graph descends or falls as you move from left to right, the function is decreasing.

| Function | Where increasing | Where decreasing |
|---|---|---|
| $y = x^2$ | $0 \le x < \infty$ | $-\infty < x \le 0$ |
| $y = x^3$ | $-\infty < x < \infty$ | Nowhere |
| $y = 1/x$ | Nowhere | $-\infty < x < 0$ and $0 < x < \infty$ |
| $y = 1/x^2$ | $-\infty < x < 0$ | $0 < x < \infty$ |
| $y = \sqrt{x}$ | $0 \le x < \infty$ | Nowhere |
| $y = x^{2/3}$ | $0 \le x < \infty$ | $-\infty < x \le 0$ |

## Even and Odd Functions :

A function $y = f(x)$ is an
even function of $x$ if $f(-x) = f(x)$
odd function of $x$ if $f(-x) = -f(x)$
for every $x$ in the function's domain.

Example : Recognize even & odd functions of the following:
$f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^2 + 1$, & $f(x) = x + 1$

Solution : $f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all $x$; Symmetry about $y$-axis.

$f(x) = x^3$ Odd function: $(-x)^3 = -x^3$ for all $x$; symmetry about the origin.

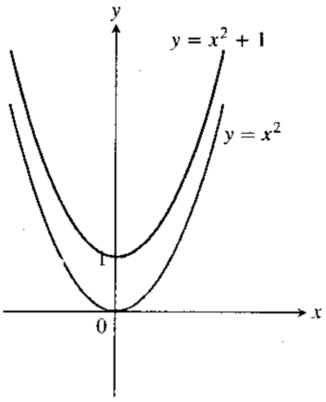
$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all $x$; symmetry about $y$-axis

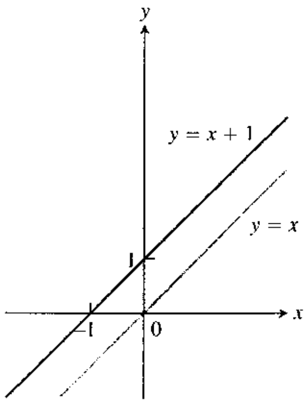
$f(x) = x + 1 \Rightarrow$ $-x + 1 \ne x + 1 \Rightarrow$ not even

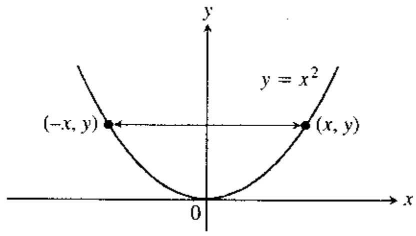
$-x + 1 \ne -(x+1) \Rightarrow$ not odd

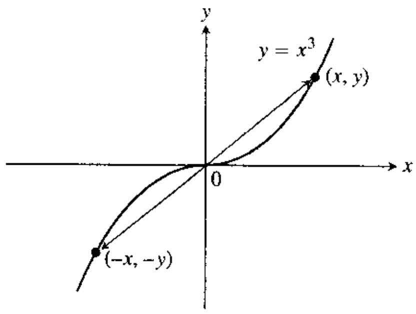
The symmetry is lost.

$f(x) = x$ Odd function : $(-x) = -x$ for all $x$ ; symmetry about the origin.




## Sums, Differences, Products, and Quotients :

If $f$ and $g$ are functions, then for every $x$ that belongs to the domains of both $f$ and $g$ (that is, for $x \in D(f) \cap D(g)$), we define functions by the formulas

$$(f+g)(x) = f(x) + g(x).$$
$$(f-g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)\,g(x).$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If $c$ is a real number, then the function $cf$ is defined for all $x$ in the domain of $f$ by

$$(cf)(x) = cf(x).$$

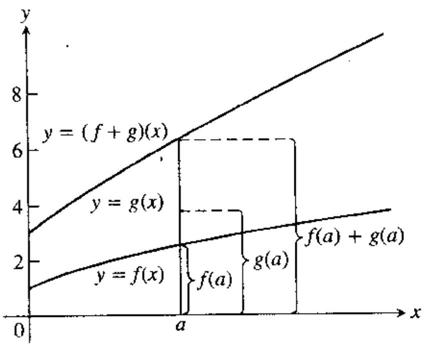
Example : The functions defined by the formulas

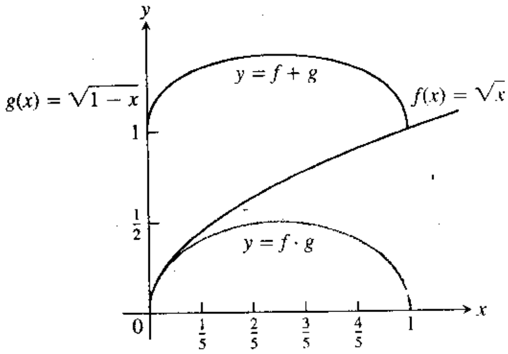
$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{1-x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$.
The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1]$$

| Function | Formula | Domain |
|---|---|---|
| $f + g$ | $(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$ | $[0, 1] = D(f) \cap D(g)$ |
| $f - g$ | $(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$ | $[0, 1]$ |
| $g - f$ | $(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$ | $[0, 1]$ |
| $f \cdot g$ | $(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$ | $[0, 1]$ |
| $f/g$ | $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$ | $[0, 1)$ ($x = 1$ excluded) |
| $g/f$ | $\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$ | $(0, 1]$ ($x = 0$ excluded) |



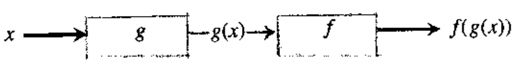




## Composite Functions :

If $f$ and $g$ are functions, the composite function $f \circ g$ ("$f$ composed with $g$") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers $x$ in the domain of $g$ for which $g(x)$ lies in the domain of $f$.




Two functions can be composed at $x$ whenever the value of one function at $x$ lies in the domain of the other. The composite is denoted by $f \circ g$.

Example : Finding Formulas for Composites

If $f(x) = \sqrt{x}$ and $g(x) = x+1$ , find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ f)(x)$ d) $(g \circ g)(x)$.

Solution :

a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$

Domain $\Rightarrow [-1, \infty)$ $\qquad x+1 \geqslant 0 \Rightarrow x \geqslant -1$

b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$

Domain $\Rightarrow [0, \infty)$

c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$

Domain $\Rightarrow [0, \infty)$

d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x+2$

Domain $\Rightarrow (-\infty, \infty)$

Example : If $f(x) = x^2$ & $g(x) = \sqrt{x}$ , find $(f \circ g)(x)$ & the domain?

Solution : $(f \circ g)(x) = (\sqrt{x})^2 = x$

Domain $\Rightarrow [0, \infty)$ , not $(-\infty, \infty)$

## Shifting a Graph of a Function :

Shift Formulas :

1. Vertical Shifts $\qquad y = f(x) + k$

a) Shifts the graph of $f$ up $k$ units if $k > 0$

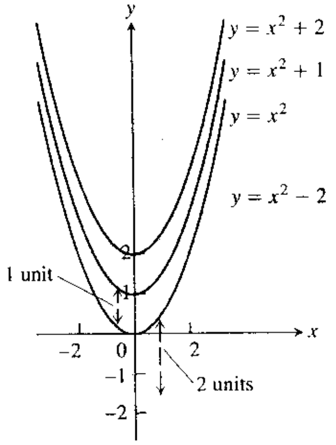
b) Shifts the graph of $f$ down $|k|$ units if $k < 0$

2. Horizontal Shifts $\qquad y = f(x+h)$

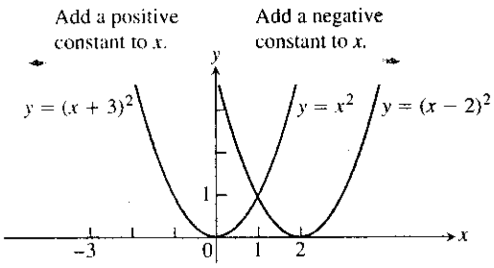
a) Shifts the graph of $f$ left $h$ units if $h > 0$

b) Shifts the graph of $f$ right $|h|$ units if $h < 0$

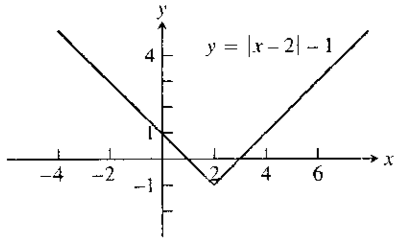
* $k$ & $h$ are constants.




To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for $f$ (Example 4a and b).




To shift the graph of $y = x^2$ to the left, we add a positive constant to $x$. To shift the graph to the right, we add a negative constant to $x$ (Example 4c).




Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

## Scaling and Reflecting a Graph of a Function :

To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function $f$, or the independent variable $x$, by an appropriate constant $c$. Reflections across the coordinate axes are special cases where $c = -1$.

1. For $c > 1$

a) $y = c f(x)$ Stretches the graph of $f$ vertically by a factor of $c$.

b) $y = \frac{1}{c} f(x)$ Compresses the graph of $f$ vertically by a factor of $c$.

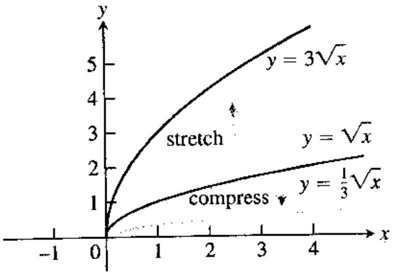
c) $y = f(cx)$ Compresses the graph of $f$ horizontally by a factor of $c$.

d) $y = f(x/c)$ Stretches the graph of $f$ horizontally by a factor of $c$.

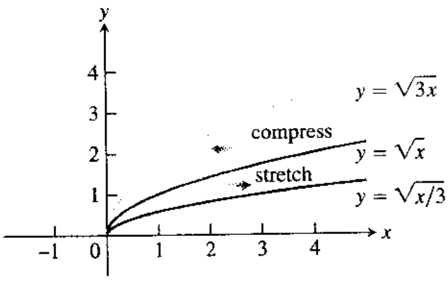
2. For $c = -1$

a) $y = -f(x)$ Reflects the graph of $f$ across the $x$-axis.

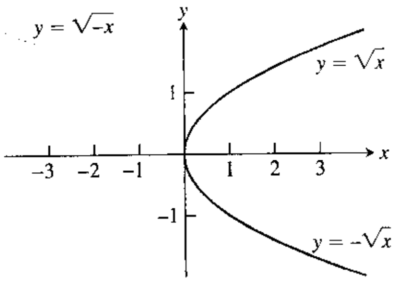
b) $y = f(-x)$ Reflects the graph of $f$ across the $y$-axis.




Vertically stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3




Horizontally stretching and compressing the graph $y = \sqrt{x}$ by a factor of 3




Reflections of the graph $y = \sqrt{x}$ across the coordinate axes ■

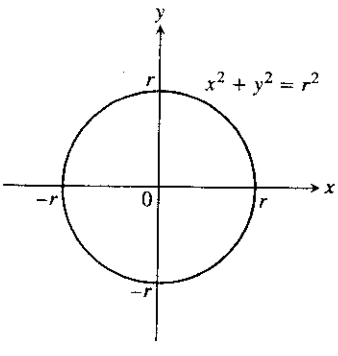
## Ellipses :

$x^2 + y^2 = r^2$ the standard equation for a circle of radius $r$ centered at the origin.

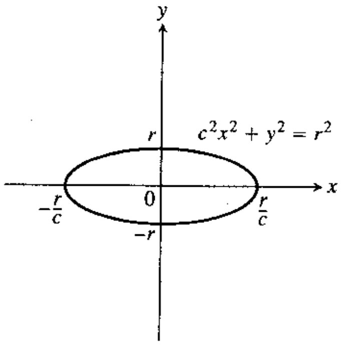
substituting $cx$ for $x$ gives

$$c^2x^2 + y^2 = r^2 \quad \ldots (1)$$

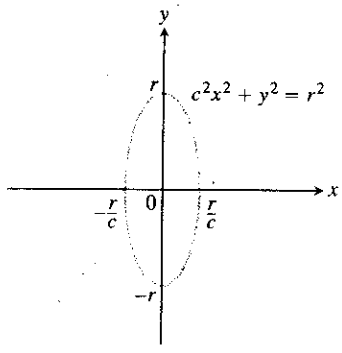
1. If $0 < c < 1$, the graph horizontally stretches the circle.
2. If $c > 1$, the circle is compressed horizontally.




(a) circle




(b) ellipse, $0 < c < 1$




(c) ellipse, $c > 1$

Notes :

1. In figure (b), the line segment joining the points $(\pm r/c, 0)$ is called the major axis of the ellipse; the minor axis is the line segment joining $(0, \pm r)$.
2. In figure (c), the major axis is the line segment joining the points $(0, \pm r)$ and the minor axis is the line segment joining the points $(\pm r/c, 0)$.

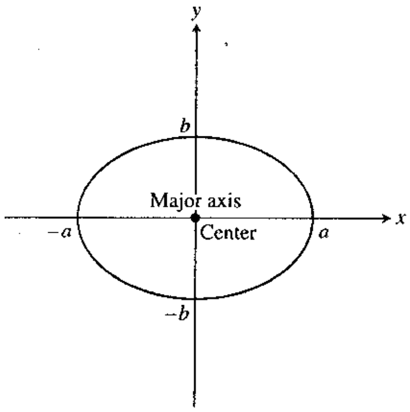
3. In both cases, the major axis is the line segment having the longer length.

If we divide both sides of equation (1) by $r^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots (2)$$

where $a = r/c$ & $b = r$

1. If $a > b$, the major axis is horizontal.
2. If $a < b$, the major axis is vertical.




Graph of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, where the major axis is horizontal.

Substituting $x - h$ for $x$, & $y - k$ for $y$, in equation (2)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \ldots (3)$$

the standard equation of an ellipse with center at $(h, k)$.

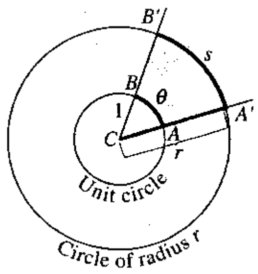
# Trigonometric Functions :

## Radian Measure :

The radian measure of the angle $ACB$ at the center of the unit circle equals the length of the arc that $ACB$ cuts from the unit circle.

$$s = r\theta$$

where $s$: the length of arc cut from a circle.
$r$: the radius of a circle.
$\theta$: the angle producing the arc, radian.




$$\pi \text{ radians} = 180^\circ$$

Conversion Formulas

$$1 \text{ degree} = \frac{\pi}{180} \; (\approx 0.02) \text{ radians}$$

Degrees to radians : multiply by $\frac{\pi}{180}$

For example, $45^\circ$ in radian measure

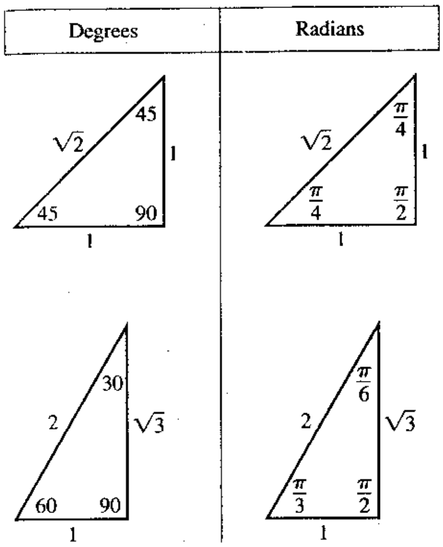
$$45 \times \frac{\pi}{180} = \frac{\pi}{4} \text{ rad.}$$

$$1 \text{ radian} = \frac{180}{\pi} \; (\approx 57) \text{ degrees}$$

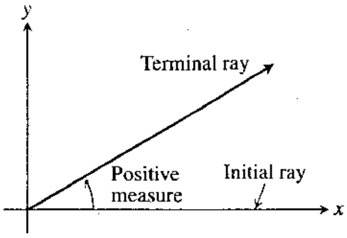
Radians to degrees : multiply by $\frac{180}{\pi}$

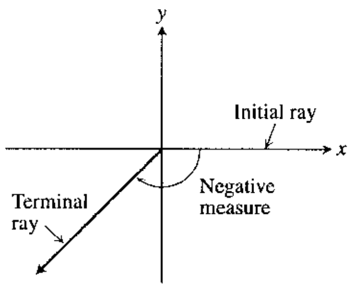
For example, $\frac{\pi}{6}$ in degree

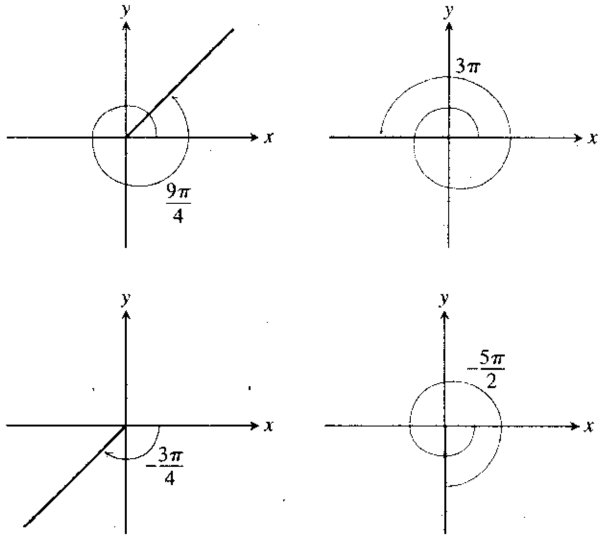
$$\frac{\pi}{6} \times \frac{180}{\pi} = 30^\circ \text{ degree.}$$



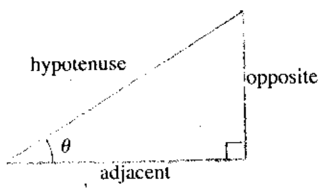

Angles measured counterclockwise $\Rightarrow$ positive measures.
Angles measured clockwise $\Rightarrow$ negative measures.



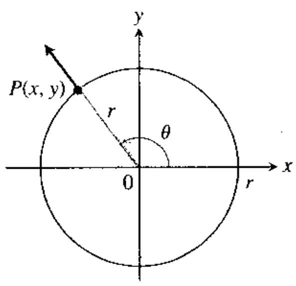

The Six Basic Trigonometric Functions :

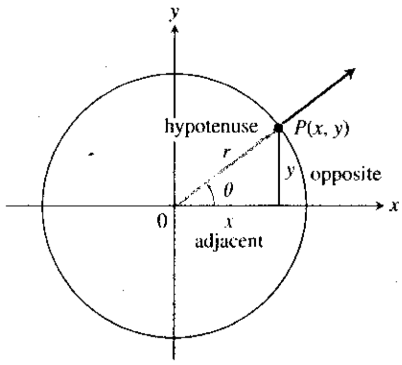



$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\quad \csc\theta = \frac{\text{hyp}}{\text{opp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\quad \sec\theta = \frac{\text{hyp}}{\text{adj}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\quad \cot\theta = \frac{\text{adj}}{\text{opp}}$




Sine : $\sin\theta = \frac{y}{r}$ $\qquad$ Cosecant : $\csc\theta = \frac{r}{y}$

Cosine : $\cos\theta = \frac{x}{r}$ $\qquad$ secant : $\sec\theta = \frac{r}{x}$

tangent : $\tan\theta = \frac{y}{x}$ $\qquad$ cotangent : $\cot\theta = \frac{x}{y}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

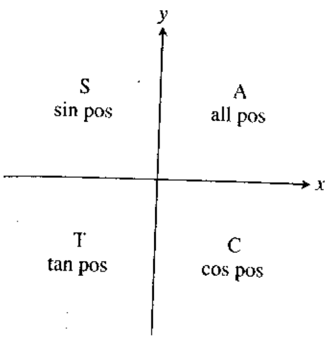
- $\tan\theta$ & $\sec\theta$ are not defined if $x=0$ or not defined if $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

- $\cot\theta$ & $\csc\theta$ are not defined if $y=0$ or not defined if $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

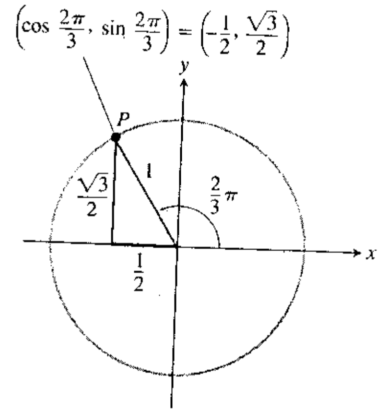
$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$




The CAST rule, remembered by the statement "All Students Take Calculus," tells which trigonometric functions are positive in each quadrant.




The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

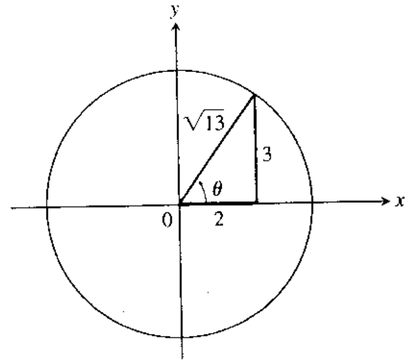
Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of $\theta$

| **Degrees** | −180 | −135 | −90 | −45 | 0 | 30 | 45 | 60 | 90 | 120 | 135 | 150 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **$\theta$ (radians)** | $-\pi$ | $\frac{-3\pi}{4}$ | $\frac{-\pi}{2}$ | $\frac{-\pi}{4}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| $\sin\theta$ | 0 | $\frac{-\sqrt{2}}{2}$ | −1 | $\frac{-\sqrt{2}}{2}$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | −1 | 0 |
| $\cos\theta$ | −1 | $\frac{-\sqrt{2}}{2}$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{-\sqrt{2}}{2}$ | $\frac{-\sqrt{3}}{2}$ | −1 | 0 | 1 |
| $\tan\theta$ | 0 | 1 | | −1 | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | | $-\sqrt{3}$ | −1 | $\frac{-\sqrt{3}}{3}$ | 0 | | 0 |

Example : If $\tan\theta = \frac{3}{2}$ & $0 < \theta < \frac{\pi}{2}$ , find the five other trigonometric functions of $\theta$.

solution :

$$\tan\theta = \frac{3}{2}$$




the right triangle of height 3 (opposite) and base 2 (adjacent).

hypotenuse $= \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

$\cos\theta = \frac{2}{\sqrt{13}}$ , $\sin\theta = \frac{3}{\sqrt{13}}$ , $\sec\theta = \frac{\sqrt{13}}{2}$

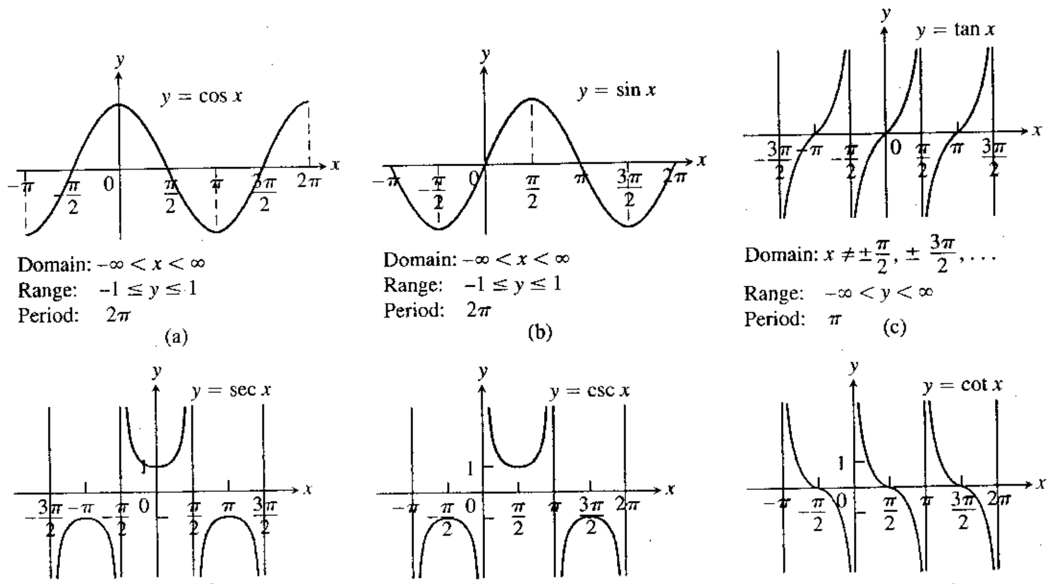
$\csc\theta = \frac{\sqrt{13}}{3}$ , $\cot\theta = \frac{2}{3}$

Periodicity and Graphs of the Trigonometric Functions :

$\cos(\theta+2\pi) = \cos\theta$ $\quad$ $\sin(\theta+2\pi) = \sin\theta$ $\quad$ $\tan(\theta+2\pi) = \tan\theta$

$\sec(\theta+2\pi) = \sec\theta$ $\quad$ $\csc(\theta+2\pi) = \csc\theta$ $\quad$ $\cot(\theta+2\pi) = \cot\theta$

$\cos(\theta-2\pi) = \cos\theta$ $\quad$ $\sin(\theta-2\pi) = \sin\theta$




Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: $2\pi$
(a)

Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: $2\pi$
(b)

Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: $\pi$
(c)

Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: $2\pi$
(d)

Domain: $x \ne 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: $2\pi$
(e)

Domain: $x \ne 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: $\pi$
(f)

Period $\pi$ : $\quad \tan(x+\pi) = \tan x$

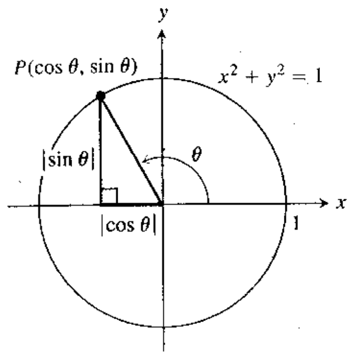
$\quad \cot(x+\pi) = \cot x$

Period $2\pi$ : $\sin(x+2\pi) = \sin x$
$\cos(x+2\pi) = \cos x$
$\sec(x+2\pi) = \sec x$
$\csc(x+2\pi) = \csc x$

Even & Odd :

| Even | Odd |
|---|---|
| $\cos(-x) = \cos x$ | $\sin(-x) = -\sin x$ |
| $\sec(-x) = \sec x$ | $\tan(-x) = -\tan x$ |
| | $\csc(-x) = -\csc x$ |
| | $\cot(-x) = -\cot x$ |

Identities :




$x = r\cos\theta$
$y = r\sin\theta$
if $r = 1$ & $x^2 + y^2 = 1$

$\therefore$ $\boxed{\cos^2\theta + \sin^2\theta = 1}$

& $$\boxed{\begin{aligned} 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}}$$

Sum & Difference Formulas :

$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$

Double-Angle Formulas :

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\sin 2\theta = 2\sin\theta\cos\theta$

$$\begin{array}{r} \cos^2\theta + \sin^2\theta = 1 \\ \cos^2\theta - \sin^2\theta = \cos 2\theta \\ \hline 2\cos^2\theta = 1 + \cos 2\theta \end{array} \qquad \begin{array}{r} \cos^2\theta + \sin^2\theta = 1 \\ \mp\cos^2\theta \pm \sin^2\theta = \mp\cos 2\theta \\ \hline 2\sin^2\theta = 1 - \cos 2\theta \end{array}$$

Half-Angle Formulas:

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

## Limits of Function Values:

Let $f(x)$ be defined on an open interval about $x_0$, except possibly at $x_0$ itself. If $f(x)$ gets arbitrarily close to $L$ (as close to $L$ as we like) for all $x$ sufficiently close to $x_0$, we say that $f$ approaches the limit $L$ as $x$ approaches $x_0$, and we write

$$\lim_{x \to x_0} f(x) = L$$

which is read "the limit of $f(x)$ as $x$ approaches $x_0$ is $L$"

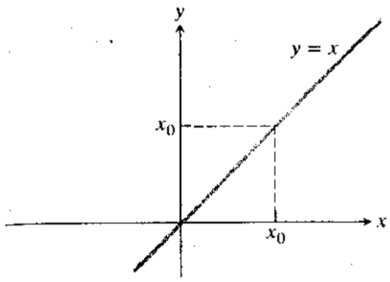
$\lim_{x \to 2} (4) = 4$ , $\lim_{x \to 2} (5x - 3) = 10 - 3 = 7$

$\lim_{x \to 13} (4) = 4$

$\lim_{x \to 3} x = 3$ , $\lim_{x \to -2} \frac{3x+4}{x+5} = \frac{-6+4}{-2+5} = -\frac{2}{3}$

a) If $f$ is the identity function $f(x) = x$

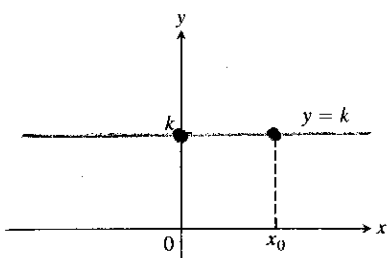
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0 .$$



(a) Identity function

b) If $f$ is the constant function $f(x) = k$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k .$$



(b) Constant function

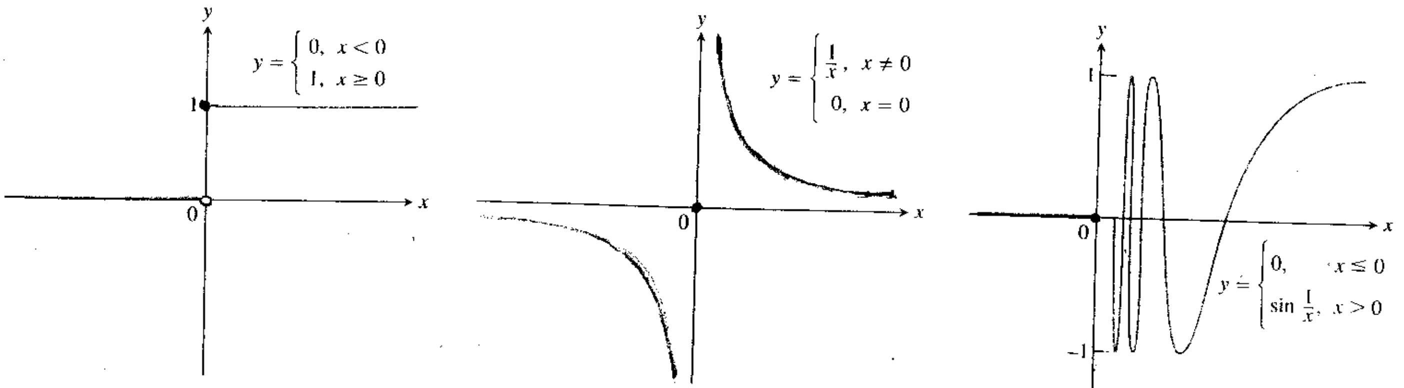
Example: Discuss the behavior of the following functions as $x \to 0$

a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0 \end{cases}$

b) $g(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c) $f(x) = \begin{cases} 0, & x \leqslant 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$

Solution:




(a) Unit step function $U(x)$

(b) $g(x)$

(c) $f(x)$

a) It jumps: The unit step function $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of $x$ arbitrarily close to zero, $U(x) = 0$. For positive values of $x$ arbitrarily close to zero $U(x) = 1$. There is no single value $L$ approached by $U(x)$ as $x \to 0$.

b) It grows too large to have a limit: $g(x)$ has no limit as $x \to 0$ because the values of $g$ grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to any real number.

c) It oscillates too much to have a limit: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and $-1$ in every open interval containing $0$. The values do not stay close to any one number as $x \to 0$.

# Calculating Limits Using the Limit Laws :

**THEOREM 1 Limit Laws**
If $L$, $M$, $c$ and $k$ are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $$\lim_{x \to c}(f(x) + g(x)) = L + M$$
The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $$\lim_{x \to c}(f(x) - g(x)) = L - M$$
The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$$
The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c}(k \cdot f(x)) = k \cdot L$$
The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$
The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If $r$ and $s$ are integers with no common factor and $s \neq 0$, then
$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$
provided that $L^{r/s}$ is a real number. (If $s$ is even, we assume that $L > 0$.)
The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

Examples :

1. $$\lim_{x \to c}(x^3 + 4x - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = c^3 + 4c^2 - 3$$

2. $$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$$
$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

3. $$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)}$$
$$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} = \sqrt{4(-2)^2 - 3} = \sqrt{13}$$

**THEOREM 2 Limits of Polynomials Can Be Found by Substitution**

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

**THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero**

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example :

$$\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = 0$$

Eliminating Zero Denominators Algebraically :

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c.

Examples :

1. $$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1} \frac{x+2}{x}$$

$$= \frac{1+2}{1} = 3$$

2. $$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} \times \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$\lim_{x\to 0} \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+100}+10)}$$

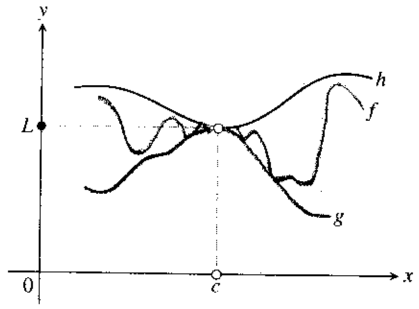
$$\lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{\sqrt{0+100}+10} = \frac{1}{20} = 0.05$$

**THEOREM 4** **The Sandwich Theorem**

Suppose that $g(x) \le f(x) \le h(x)$ for all $x$ in some open interval containing $c$, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.




The graph of $f$ is sandwiched between the graphs of $g$ and $h$.

The Sandwich Theorem is sometimes called the Squeeze Theorem or the Pinching Theorem.

Example :

$$1 - \frac{x^2}{4} \leqslant u(x) \leqslant 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0$$

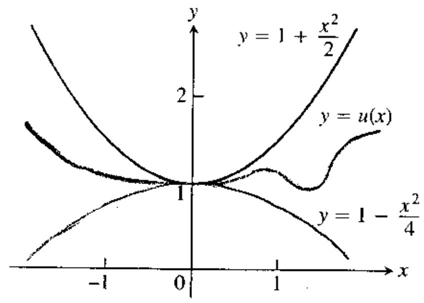
find $\lim_{x\to 0} u(x)$.

Solution :

$$\lim_{x\to 0} \left(1 - \frac{x^2}{4}\right) = \left(1 - \frac{0}{4}\right) = 1$$

$$\lim_{x\to 0} \left(1 + \frac{x^2}{2}\right) = \left(1 + \frac{0}{2}\right) = 1$$

$$\lim_{x\to 0} u(x) = 1$$




**THEOREM 5** If $f(x) \le g(x)$ for all $x$ in some open interval containing $c$, except possibly at $x = c$ itself, and the limits of $f$ and $g$ both exist as $x$ approaches $c$, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

# Finite Limits as $x \to \pm\infty$

The symbol for infinity ($\infty$) does not represent a real number. We use $\infty$ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$. When $x$ is positive and becomes increasingly large, $1/x$ becomes increasingly small. When $x$ is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit $0$ as $x \to \pm\infty$ or that $0$ is a limit of $f(x) = 1/x$ at infinity and negative infinity.

**DEFINITIONS** **Limit as $x$ approaches $\infty$ or $-\infty$**

1. We say that $f(x)$ has the **limit $L$ as $x$ approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $M$ such that for all $x$

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit $L$ as $x$ approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $N$ such that for all $x$

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

The basic facts to be verified by applying the formal definition are:

$$\lim_{x \to \pm\infty} k = k \qquad \text{and} \qquad \lim_{x \to \pm\infty} \frac{1}{x} = 0$$

**THEOREM** **Limit Laws as $x \to \pm\infty$**

If $L$, $M$, and $k$, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If $r$ and $s$ are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If $s$ is even, we assume that $L > 0$.)

Examples :

1. $\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = 5+0 = 5$

2. $\lim_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$

$= \lim_{x\to-\infty}\pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x}$

$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$

## Limits at Infinity of Rational Functions :

Example : ( Numerator & Denominator of Same Degree ):

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}$$

( Divide numerator and denominator by $x^2$)

$$= \frac{5+0-0}{3+0} = \frac{5}{3}$$

Example : ( Degree of Numerator Less Than Degree of Denominator)

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$$

( Divide numerator and denominator by $x^3$)

$$= \frac{0+0}{2-0} = 0$$

Example : ( Degree of Numerator Greater than Degree of Denominator)

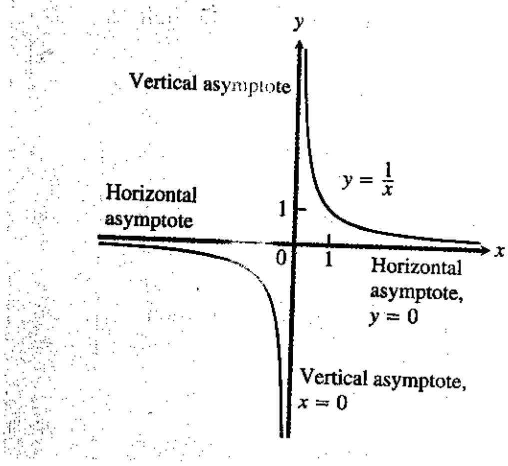
$$\lim_{x\to-\infty}\frac{2x^2-3}{7x+4} = \lim_{x\to-\infty}\frac{(2x)-(3/x)}{7+(4/x)}$$

( Divide numerator and denominator by $x$ )

$$= \frac{2(-\infty)-0}{7+0} = -\infty$$

## Horizontal Asymptotes :

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an asymptote of the graph.




In this figure, we observe that the $x$-axis is an asymptote of the curve on the right because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

We say that the $x$-axis is a horizontal asymptote of the graph of $f(x) = 1/x$.

**DEFINITION** **Horizontal Asymptote**
A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

## Vertical Asymptotes :

The distance between a point on the graph of $y = 1/x$ and the $y$-axis approaches zero as the point moves vertically along the graph and away from the origin (same figure above).

This behavior occurs because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

We say that the line $x=0$ (the $y$-axis) is a vertical asymptote of the graph of $y = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

**DEFINITION** **Vertical Asymptote**

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Example: Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x+3}{x+2}$$

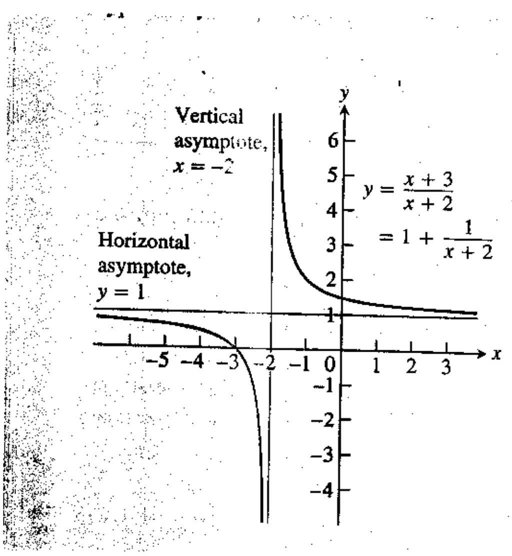
Solution:

$$\lim_{x \to \infty} \frac{x+3}{x+2} = \lim_{x \to \infty} \frac{1 + (3/x)}{1 + (2/x)} = \frac{1+0}{1+0} = 1$$

$\therefore$ $y = 1$ horizontal asymptote

$$\lim_{x \to -2} \frac{x+3}{x+2} = \infty \implies x + 2 = 0$$

$\therefore$ $x = -2$ vertical asymptote



Examples :

1. $\lim\limits_{x\to\infty} \left( x - \sqrt{x^2+1} \right)$

$$\lim_{x\to\infty} \left( x - \sqrt{x^2+1} \right) \cdot \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}$$

$$\lim_{x\to\infty} \frac{x^2-(x^2+1)}{x+\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{-1}{x+\sqrt{x^2+1}}$$

$$-\lim_{x\to\infty} \frac{(1/x)}{(x/x)+\left(\sqrt{(x^2/x^2)+(1/x^2)}\right)} = \frac{0}{1+\sqrt{1+0}} = 0$$

2. $$\lim_{x\to-\infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x\to-\infty} \frac{x^{1/3}-x^{1/5}}{x^{1/3}+x^{1/5}}$$

$$= \lim_{x\to-\infty} \frac{1-(x^{1/5}/x^{1/3})}{1+(x^{1/5}/x^{1/3})} = \lim_{x\to-\infty} \frac{1-(1/x^{2/15})}{1+(1/x^{2/15})}$$

$$= \frac{1-0}{1+0} = 1$$

3. $$\lim_{x\to\infty} \sqrt{\frac{2+3x}{1+5x}} = \sqrt{\lim_{x\to\infty} \frac{2+3x}{1+5x}}$$

$$= \sqrt{\lim_{x\to\infty} \frac{(2/x)+(3x/x)}{(1/x)+(5x/x)}} = \sqrt{\lim_{x\to\infty} \frac{(2/x)+3}{(1/x)+5}}$$

$$= \sqrt{\frac{0+3}{0+5}} = \sqrt{\frac{3}{5}}$$

4. $$\lim_{x\to\infty} \frac{x+3}{x-1} = \lim_{x\to\infty} \frac{1+(3/x)}{1-(1/x)} = \frac{1+0}{1-0} = 1$$

$\therefore\ y = 1$ horizontal asymptote.

$x - 1 = 0 \Rightarrow x = 1$ vertical asymptote.

5. $\lim\limits_{x \to 2} \dfrac{x^6 - 64}{x - 2}$

$$
\begin{array}{r|l}
\multicolumn{1}{r}{x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32} & \\
\cline{1-1}
x^6 - 64 & x - 2 \\
\cline{2-2}
\mp x^6 \pm 2x^5 & \\
\cline{1-1}
2x^5 - 64 & \\
\mp 2x^5 \pm 4x^4 & \\
\cline{1-1}
4x^4 - 64 & \\
\mp 4x^4 \pm 8x^3 & \\
\cline{1-1}
8x^3 - 64 & \\
\mp 8x^3 \pm 16x^2 & \\
\cline{1-1}
16x^2 - 64 & \\
\mp 16x^2 \pm 32x & \\
\cline{1-1}
32x - 64 & \\
\mp 32x \pm 64 & \\
\cline{1-1}
0 &
\end{array}
$$

$$\therefore \lim_{x \to 2} \frac{x^6 - 64}{x - 2} = \lim_{x \to 2} x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32$$

$$= (2)^5 + 2(2)^4 + 4(2)^3 + 8(2)^2 + 16(2) + 32 =$$

Note :

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

$$
\begin{array}{r|l}
\multicolumn{1}{r}{\frac{x}{2} + 1} & \\
\cline{1-1}
x^2 - 3 & 2x - 4 \\
\cline{2-2}
\mp x^2 \pm 2x & \\
\cline{1-1}
2x - 3 & \\
\mp 2x \pm 4 & \\
\cline{1-1}
1 &
\end{array}
$$

$$f(x) = \frac{x^2 - 3}{2x - 4} = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

# Continuity

**DEFINITION** **Continuous at a Point**

*Interior point*: A function $y = f(x)$ is **continuous at an interior point $c$** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

*Endpoint*: A function $y = f(x)$ is **continuous at a left endpoint $a$** or is **continuous at a right endpoint $b$** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

If a function $f$ is not continuous at a point $c$, we say that $f$ is discontinuous at $c$ and $c$ is a point of discontinuity of $f$. Note that $c$ need not be in the domain of $f$.

**Continuity Test**

A function $f(x)$ is continuous at an interior point of its domain $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists ($c$ lies in the domain of $f$)
2. $\lim_{x \to c} f(x)$ exists ($f$ has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Example: The function $y = \frac{1}{x}$ is continuous at every value of $x$ except $x = 0$. It has a point of discontinuity at $x = 0$.

**THEOREM** **Properties of Continuous Functions**

If the functions $f$ and $g$ are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number $k$
5. *Quotients*: $f/g$ provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing $c$, where $r$ and $s$ are integers

$$\lim_{x \to c} (f+g)(x) = \lim_{x \to c} (f(x) + g(x))$$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c) = (f+g)(c).$$

This shows that $f + g$ is continuous.

**THEOREM** **Composite of Continuous Functions**
If $f$ is continuous at $c$ and $g$ is continuous at $f(c)$, then the composite $g \circ f$ is continuous at $c$.

The following types of functions are continuous at every point in their domains:

1. Polynomials.
2. Rational functions: They have points of discontinuity at the zero of their denominators.
3. Root functions: ( $y = \sqrt[n]{x}$ , n a positive integer greater than 1 ).
4. Trigonometric functions.
5. Inverse trigonometric functions.
6. Exponential functions.
7. Logarithmic functions.

Note: The inverse function of any continuous function is continuous.

Examples:

1. $$f(x) = \begin{cases} \dfrac{2x^2 + x - 3}{x - 1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

① $f(1) = 2$

② $\lim\limits_{x \to 1} \dfrac{2x^2 + x - 3}{x - 1} = \lim\limits_{x \to 1} \dfrac{(2x+3)(x-1)}{(x-1)} = \lim\limits_{x \to 1} (2x+3)$

$= 2(1) + 3 = 5$

③ $\lim\limits_{x \to 1} f(x) \neq f(1)$ , $5 \neq 2$

$\therefore$ $f(x)$ discontinuous at $x = 1$

2. $$f(x) = \begin{cases} 3 + x & x \leqslant 1 \\ 3 - x & x > 1 \end{cases}$$

① $f(1) = 3 + 1 = 4$

② $\lim\limits_{x \to 1^-} 3 + x = 3 + 1 = 4$

$$\lim_{x \to 1^+} 3 - x = 3 - 1 = 2$$

$$\lim_{x \to 1^-} 3 + x \neq \lim_{x \to 1^+} 3 - x$$

$\therefore$ f(x) dicontinuous at $x = 1$

3. $$f(x) \begin{cases} \frac{1}{x-2} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

① $f(2) = 3$

② $$\lim_{x \to 2} \frac{1}{x-2} = \frac{1}{0} = \infty$$

No limit , $\therefore$ f(x) discontinuous at $x = 2$.

4. $$f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & x \neq 4 \\ 9 & x = 4 \end{cases}$$

① $f(4) = 9$

② $$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)}$$

$$= \lim_{x \to 4} (x+4) = 4 + 4 = 8$$

③ $$f(4) \neq \lim_{x \to 4} f(x)$$

$\therefore$ f(x) discontinuos at $x = 4$.

5. $$f(x) = \begin{cases} |x - 3| & x \neq 3 \\ 2 & x = 3 \end{cases}$$

① $f(3) = 2$

② $$\lim_{x \to 3^+} (x - 3) = 0$$

$$\lim_{x \to 3^-} -(x - 3) = 0$$

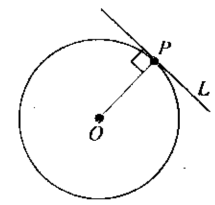
$$\lim_{x \to 3^+} (x - 3) = \lim_{x \to 3^-} -(x - 3) = 0$$

$\lim_{x \to 3} f(x) \neq f(3)$ , $0 \neq 2$

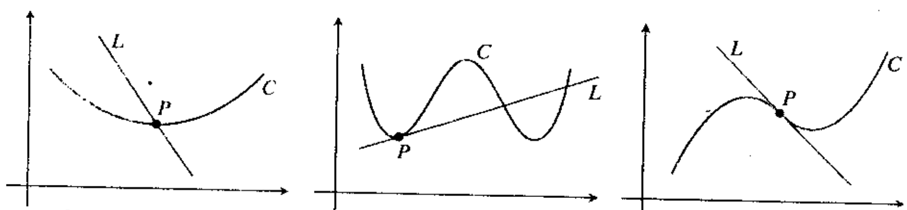
$\therefore f(x)$ discontinuous at $x = 3$

# Tangents and Derivatives

1. L passes through P perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.




$L$ is tangent to the circle at $P$ if it passes through $P$ perpendicular to radius $OP$.

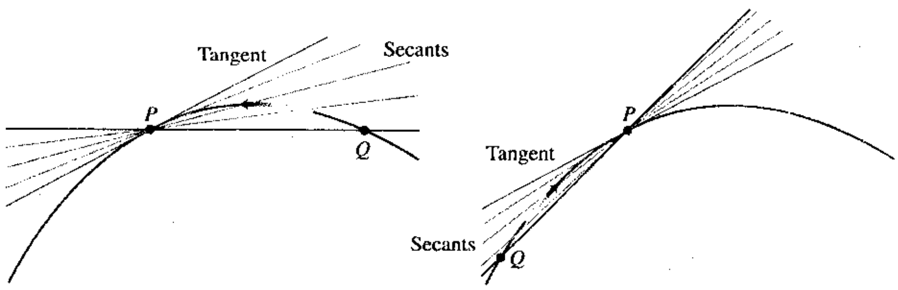



$L$ meets $C$ only at $P$ but is not tangent to $C$.

$L$ is tangent to $C$ at $P$ but meets $C$ at several points.

$L$ is tangent to $C$ at $P$ but lies on two sides of $C$, crossing $C$ at $P$.

To define tangency for general curves, we need a dynamic approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve.




1. We start with what we can calculate, namely the slope of the secant PQ.
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

Example : Find the slope of the parabola $y = x^2$ at the point $P(2,4)$. Write an equation for the tangent to the parabola at this point.

Solution :

Secant line $\Rightarrow$ $P(2,4)$ & $Q(2+h, (2+h)^2)$

$$\text{Secant slope } (PQ) = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} = \frac{h^2+4h+4-4}{h}$$

$$= \frac{h^2+4h}{h} = h+4$$

As $Q$ approaches $P$ along the curve, $h$ approaches zero

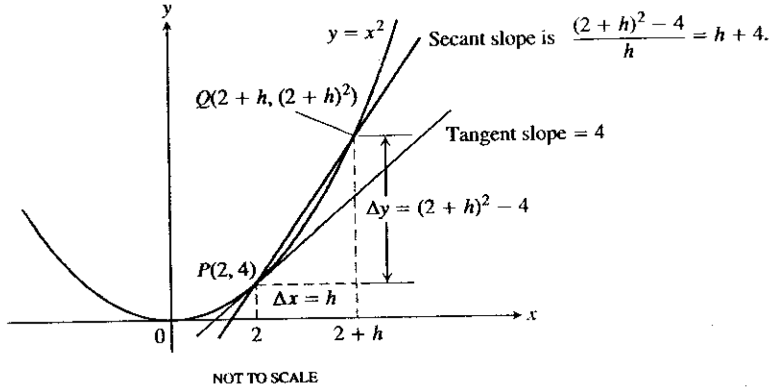
$$\lim_{h \to 0} (h+4) = 4$$

The parabola's slope at $P$ is 4.

The tangent to the parabola at $P$ is the line through $P$ with slope 4:

$$y = y_1 + m(x - x_1)$$

$$y = 4 + 4(x-2) \Rightarrow y = 4x - 4.$$



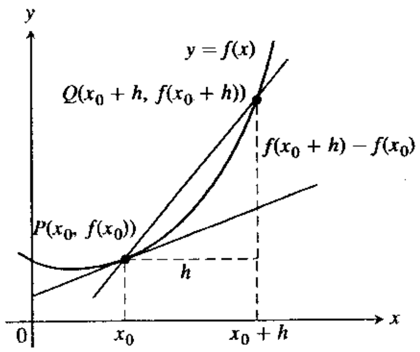
# Finding a Tangent to the Graph of a Function:

**DEFINITIONS** Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at $P$ is the line through $P$ with this slope.




Example: Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution:

let $f(x) = mx + b$

1. Find $f(x_0)$ & $f(x_0 + h)$

$$f(x_0) = mx_0 + b$$

$$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$$

2. Find the slope

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = m$$

3. Find the tangent line using the point-slope equation.

$$y = y_1 + m(x - x_1) \quad , \quad (x_0, mx_0 + b)$$

$$y = (mx_0 + b) + m(x - x_0)$$

$$y = mx_0 + b + mx - mx_0$$

$$y = mx + b$$

**Finding the Tangent to the Curve $y = f(x)$ at $(x_0, y_0)$**

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

Example ⓐ Find the slope of the curve $y = 1/x$ at $x = a \neq 0$

ⓑ Where does the slope equal $-1/4$ ?

ⓒ What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution :

a) $f(x) = 1/x$ , $(a, 1/a)$

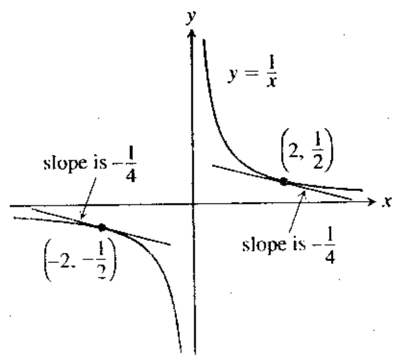
Slope $\Rightarrow \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \dfrac{\frac{1}{a+h} - \frac{1}{a}}{h}$

$= \lim_{h \to 0} \dfrac{1}{h} \dfrac{a - (a+h)}{a(a+h)} = \lim_{h \to 0} \dfrac{-h}{ha(a+h)}$

$= \lim_{h \to 0} \dfrac{-1}{a(a+h)} = -\dfrac{1}{a^2}$ , $a \neq 0$

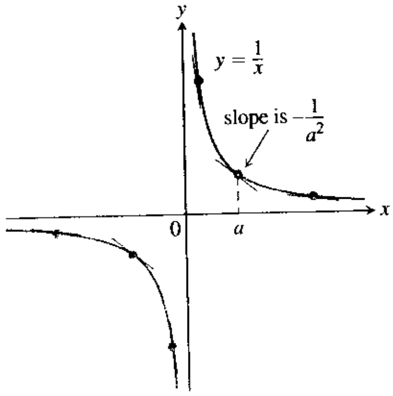
b) $-\dfrac{1}{a^2} = -\dfrac{1}{4} \Rightarrow a^2 = 4 \Rightarrow a = 2$ or $a = -2$

The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$



c) Notice that the slope $-1/a^2$ is always negative if $a \neq 0$.
As $a \to 0^+ \Rightarrow$ slope $\to -\infty \Rightarrow$ tangent increasingly steep.

As a moves away from the origin in either direction, the slope approaches $0^-$ and the tangent levels off to become horizontal.




## Differentiation :

The Derivative as a Function :

$$y = f(x) \quad , \quad x = x_0$$

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

**DEFINITION** Derivative Function
The **derivative** of the function $f(x)$ with respect to the variable $x$ is the function $f'$ whose value at $x$ is

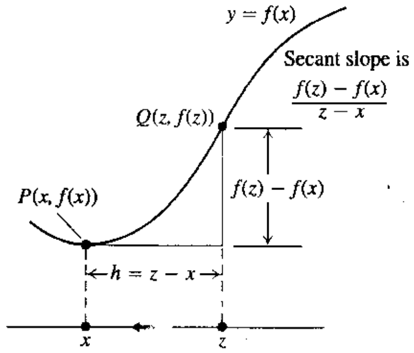
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

If we write $z = x + h \Rightarrow h = z - x$ & $h \to 0 \Rightarrow z \to x$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$




Example : Differentiate $f(x) = x/(x-1)$

Solution :

$$f(x) = \frac{x}{x-1} \quad , \quad f(x+h) = \frac{(x+h)}{(x+h)-1}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} = \lim_{h \to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}$$

Example : ⓐ Find the derivative of $y = \sqrt{x}$ for $x > 0$.
ⓑ Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution :

a) $$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} = \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}}$$

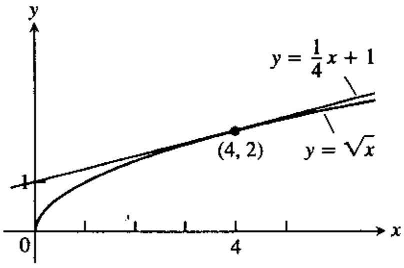
$$= \frac{1}{2\sqrt{x}} .$$

b) The slope of the curve at $x = 4$ is :

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The tangent is the line through the point $(4, 2)$ with slope $1/4$ :

$$y = 2 + \frac{1}{4}(x-4) \Rightarrow y = \frac{1}{4}x + 1$$




Notations

$y = f(x)$ $\qquad$ $x$ : independent variable.

$\qquad\qquad\qquad$ $y$ : dependent variable.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}$$

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

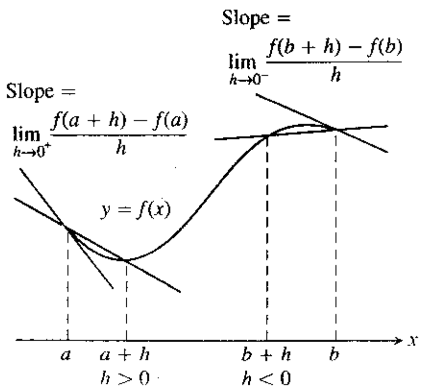
Differentiable on an Interval; One-Sided Derivatives:

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior $(a, b)$ and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{Left-hand derivative at } b$$

exist at the endpoints.




Example: Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ & $(0, \infty)$ but has no derivative at $x = 0$

Solution:

right $\Rightarrow \frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1$

left $\Rightarrow \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1$

There can be no derivative at the origin because the one-sided derivatives differ $(1 \neq -1)$

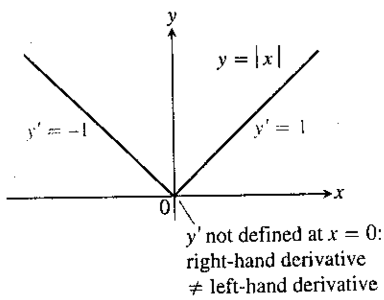
Right-hand derivative of $|x|$ at zero $= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h}$

$$= \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

Left-hand derivative of $|x|$ at zero $= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h}$

$$= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

* $|h| = h$ when $h > 0$.
* $|h| = -h$ when $h < 0$.




Note: A function is continuous at every point where it has a derivative. (Differentiable functions are continuous).

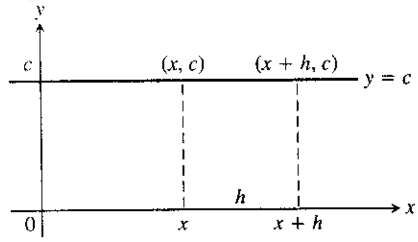
# Differentiation Rules :

**RULE 1 Derivative of a Constant Function**
If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example : $f(x) = 8 \Rightarrow \frac{df}{dx} = \frac{d}{dx}(8) = 0$

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \& \quad \frac{d}{dx}(\sqrt{3}) = 0$$




The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

**Proof of Rule 1** We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value $c$. At every value of $x$, we find that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0. \quad \blacksquare$$

**RULE 2 Power Rule for Positive Integers**
If $n$ is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Example :

| $f$ | $x$ | $x^2$ | $x^3$ | $x^4$ | ... |
|---|---|---|---|---|---|
| $f'$ | $1$ | $2x$ | $3x^2$ | $4x^3$ | ... |

**First Proof of Rule 2** The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative form for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} = \lim_{z\to x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z\to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \\ &= nx^{n-1} \end{aligned}$$

**Second Proof of Rule 2** If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Since $n$ is a positive integer, we can expand $(x+h)^n$ by the Binomial Theorem to get

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\
&= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\
&= nx^{n-1}
\end{aligned}$$

**RULE 3 Constant Multiple Rule**
If $u$ is a differentiable function of $x$, and $c$ is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if $n$ is a positive integer, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}$$

Example :

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}$$

**Proof of Rule 3**

$$\begin{aligned}
\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} && \text{Derivative definition with } f(x) = cu(x) \\
&= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} && \text{Limit property} \\
&= c\frac{du}{dx} && u \text{ is differentiable.}
\end{aligned}$$

**RULE 4 Derivative Sum Rule**
If $u$ and $v$ are differentiable functions of $x$, then their sum $u+v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example : $y = x^4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12$$

**Proof of Rule 4** We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x) + v(x)] = \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}$$

$$= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.$$

Example : $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

Example: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution :

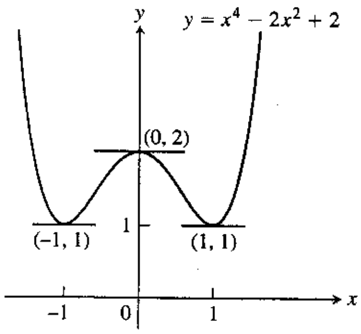
The horizontal tangents, if any, occur where the slope $dy/dx$ is zero.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$$

$$\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 4x = 0 \Rightarrow 4x(x^2 - 1) = 0$$

$$\therefore \quad x = 0, 1, -1$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1, \& -1$. The corresponding points on the curve are $(0,2), (1,1) \& (-1,1)$




Note : 1. The derivative of the sum of two functions is the sum of their derivatives.

2. The derivative of the product of two functions in not the product of their derivatives.

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$$

**RULE 5 Derivative Product Rule**
If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example: Find the derivative of $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$

Solution: $u = \frac{1}{x}$ , $v = x^2 + \frac{1}{x}$

$$\frac{d}{dx}\left[\frac{1}{x}\left(x^2 + \frac{1}{x}\right)\right] = \frac{1}{x}\left(2x - \frac{1}{x^2}\right) + \left(x^2 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3}$$

$$= 1 - \frac{2}{x^3}$$

**Proof of Rule 5**

$$\frac{d}{dx}(uv) = \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of $u$ and $v$, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}
\frac{d}{dx}(uv) &= \lim_{h \to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h \to 0}\left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right] \\
&= \lim_{h \to 0} u(x+h) \cdot \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}.
\end{aligned}$$

As $h$ approaches zero, $u(x+h)$ approaches $u(x)$ because $u$, being differentiable at $x$, is continuous at $x$. The two fractions approach the values of $dv/dx$ at $x$ and $du/dx$ at $x$. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ■

Example: Let $y = uv$ be the product of the functions $u$ and $v$. Find $y'(2)$ if
$u(2) = 3$, $u'(2) = -4$, $V(2) = 1$, & $V'(2) = 2$

Solution: $y' = (uv)' = uv' + vu'$

$$y'(2) = u(2)v'(2) + V(2) + u'(2)$$
$$= (3)(2) + (1)(-4) = 6 - 4 = 2$$

Example: Find the derivative of $y=(x^2+1)(x^3+3)$

Solution: a) $u=x^2+1$ & $v=x^3+3$

$$\frac{d}{dx}\left[(x^2+1)(x^3+3)\right]=(x^2+1)(3x^2)+(x^3+3)(2x)$$

$$=3x^4+3x^2+2x^4+6x$$
$$=5x^4+3x^2+6x.$$

b) $y=(x^2+1)(x^3+3)=x^5+x^3+3x^2+3$

$$\frac{dy}{dx}=5x^4+3x^2+6x.$$

Note: The derivative of the quotient of two functions is not the quotient of their derivatives.

**RULE 6 Derivative Quotient Rule**

If $u$ and $v$ are differentiable at $x$ and if $v(x)\neq 0$, then the quotient $u/v$ is differentiable at $x$, and

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

Example: Find the derivative of

$$y=\frac{t^2-1}{t^2+1}$$

Solution: $u=t^2-1$, $v=t^2+1$

$$\frac{dy}{dt}=\frac{(t^2+1)\cdot 2t-(t^2-1)\cdot 2t}{(t^2+1)^2}$$

$$=\frac{2t^3+2t-2t^3+2t}{(t^2+1)^2}$$

$$=\frac{4t}{(t^2+1)^2}$$

**Proof of Rule 6**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0}\frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of $u$ and $v$, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0}\frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0}\frac{v(x)\dfrac{u(x+h)-u(x)}{h} - u(x)\dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}.$$

Taking the limit in the numerator and denominator now gives the Quotient Rule. ∎

**RULE 7 Power Rule for Negative Integers**

If $n$ is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Examples :

1. $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\dfrac{1}{x^2}$

2. $\dfrac{d}{dx}\left(\dfrac{4}{x^3}\right) = 4\dfrac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\dfrac{12}{x^4}$

**Proof of Rule 7** The proof uses the Quotient Rule. If $n$ is a negative integer, then $n = -m$, where $m$ is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$, and

$$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right)$$

$$= \frac{x^m \cdot \frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(x^m)}{(x^m)^2} \qquad \text{Quotient Rule with } u = 1 \text{ and } v = x^m$$

$$= \frac{0 - mx^{m-1}}{x^{2m}} \qquad \text{Since } m > 0,\ \frac{d}{dx}(x^m) = mx^{m-1}$$

$$= -mx^{-m-1}$$

$$= nx^{n-1}. \qquad \text{Since } -m = n$$ ∎

Example : Find an equation for the tangent to the curve $y = x + \dfrac{2}{x}$ at the point $(1, 3)$.

Solution :

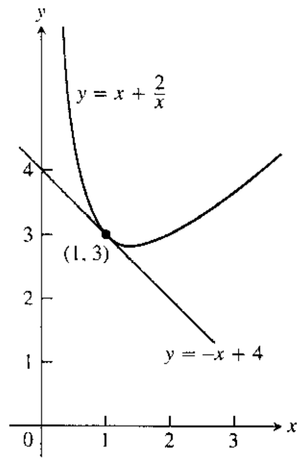
$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}$$

The slope of the curve at $x=1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1$$

The line through $(1,3)$ with slope $m=-1$ is

$$y = y_1 + m(x - x_1)$$
$$y = 3 + (-1)(x-1)$$
$$y = -x + 1 + 3$$
$$y = -x + 4$$




Example : $y = \dfrac{(x-1)(x^2-2x)}{x^4}$

Solution : $y = \dfrac{x^3 - 2x^2 - x^2 + 2x}{x^4} = \dfrac{x^3 - 3x^2 + 2x}{x^4}$

$$y = x^{-1} - 3x^{-2} + 2x^{-3}$$

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

## Second- and Higher-Order Derivatives :

If $y=f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If $f'$ is also differentiable, then we can differentiate $f'$ to get a new function of $x$ denoted by $f''$. So $f''=(f')'$. The function $f''$ is called the second derivative of $f$ because it is the derivative of the first derivative.

$$f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dy'}{dx}=y''=D^2(f)(x)=D_x^2 f(x).$$

The symbol $D^2$ means the operation of differentiation.

$$y=x^6 \Rightarrow y'=6x^5 \Rightarrow y''=30x^4.$$

If $y''$ is differentiable, its derivative, $y'''=\frac{dy''}{dx}=\frac{d^3y}{dx^3}$ is the third derivative of $y$ with respect to $x$.

$$y^{(n)}=\frac{d}{dx}y^{(n-1)}=\frac{d^ny}{dx^n}=D^ny.$$

the nth derivative of $y$ with respect to $x$ for any positive integer $n$.

Example : Find the first four derivatives of $y=x^3-3x^2+2$

Solution :
$$y=x^3-3x^2+2$$
$$y'=3x^2-6x$$
$$y''=6x-6$$
$$y'''=6$$
$$y^{(4)}=0$$

* How to read the symbols for derivative

$y' \rightarrow$ y prime.

$y'' \rightarrow$ y double prime.

$\frac{d^2y}{dx^2} \rightarrow$ d squared y dx squared.

$y''' \rightarrow$ y triple prime.

$y^{(n)} \rightarrow$ y super n.

# The Derivative as a Rate of Change :

## Instantaneous Rates of Change :

**DEFINITION** Instantaneous Rate of Change
The **instantaneous rate of change** of $f$ with respect to $x$ at $x_0$ is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Example: The area $A$ of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2$$

How fast does the area change with respect to the diameter when the diameter is 10m ?

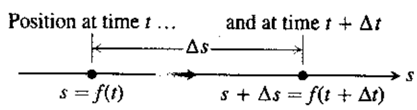
Solution : The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2} .$$

$$D = 10 \Rightarrow \frac{dA}{dD} = \frac{\pi D}{2} = \frac{\pi (10)}{2} = 5\pi \quad m^2/m .$$

## Motion Along a Line : Displacement, Velocity, Speed, Acceleration, and Jerk :

$$s = f(t)$$
$$\Delta s = f(t + \Delta t) - f(t)$$




**DEFINITION** Velocity
**Velocity (instantaneous velocity)** is the derivative of position with respect to time. If a body's position at time $t$ is $s = f(t)$, then the body's velocity at time $t$ is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

**DEFINITION** Speed
**Speed** is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

**DEFINITIONS** Acceleration, Jerk

**Acceleration** is the derivative of velocity with respect to time. If a body's position at time $t$ is $s = f(t)$, then the body's acceleration at time $t$ is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

**Jerk** is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of the earth all bodies fall with the same constant acceleration.

$$s = \frac{1}{2} g t^2$$

s : distance

g : the acceleration due to earth's gravity.

$g = 32 \ ft/sec^2$ (English units)

$g = 9.8 \ m/sec^2$ (Metric units)

The jerk of the constant acceleration of gravity is zero:

$$j = \frac{d}{dt}(g) = 0 .$$

Example : Figure shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

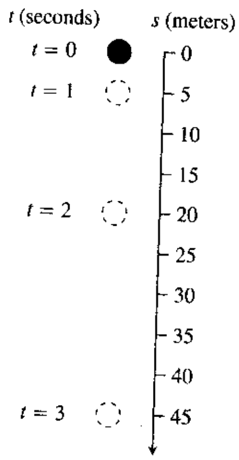
a) How many meters does the ball fall in the first 2 sec?

b) What is its velocity, speed, and acceleration then?

Solution :

a) $s = \frac{1}{2} g t^2$ ; $g = 9.8 \ m/sec^2$

$s = 4.9 t^2 \Rightarrow s(2) = 4.9(2)^2 = 19.6 \ m .$




b) $v(t) = s'(t) = \frac{d}{dt}(4.9 t^2) = 9.8 t$

At $t = 2 \Rightarrow v(2) = 9.8 \times 2 = 19.6 \ m/sec$

Speed $= |v(2)| = 19.6 \ m/sec$

$a(t) = v'(t) = s''(t) = 9.8 \ m/sec^2$

At $t = 2 \Rightarrow a(2) = 9.8 \ m/sec^2$

Example: A dynamic blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (Figure below). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

a) How high does the rock go?

b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

c) What is the acceleration of the rock at any time t during its flight (after the blast)?

d) When does the rock hit the ground again?

Solution:

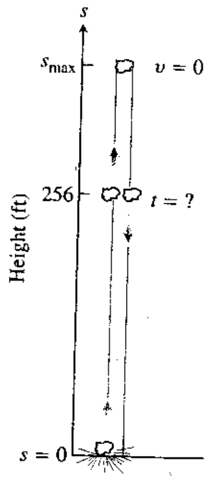
a) $v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2)$

$v = 160 - 32t$ ft/sec.

$160 - 32t = 0 \Rightarrow t = 5$ sec

∴ The rock's height at t=5 sec is

$S_{max} = S(5) = 160(5) - 16(5)^2 = 400$ ft.




b) $S(t) = 160t - 16t^2 = 256$

$16t^2 - 160t + 256 = 0$

$16(t^2 - 10t + 16) = 0$

$(t-2)(t-8) = 0$

$t = 2$ sec , $t = 8$ sec.

$V(2) = 160 - 32(2) = 96$ ft/sec

$V(8) = 160 - 32(8) = -96$ ft/sec

$V(2) > 0 \Rightarrow$ the rock is moving upward ( s is increasing ).

$V(8) < 0 \Rightarrow$ 〃 〃 〃 〃 downward ( s is decreasing).

c) $a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32$ ft/sec$^2$

d) The rock hits the ground at the positive time t for which $S = 0$.

$160t - 16t^2 = 0 \Rightarrow 16t(10 - t) = 0 \Rightarrow t = 0, t = 10$

the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later.

# Derivatives of Trigonometric Functions.

## Derivative of the Sine Function :

$\sin(x+h) = \sin x \cosh + \cos x \sin h$

If $f(x) = \sin x$, then

If $f(x) = \sin x$, then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left( \sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left( \cos x \cdot \frac{\sin h}{h} \right) \\
&= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}
$$

**The derivative of the sine function is the cosine function:**

$$\frac{d}{dx}(\sin x) = \cos x.$$

Example :

1. $y = x^2 - \sin x$

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) = 2x - \cos x$$

2. $y = x^2 \sin x$

$$
\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \cdot \sin x \\
&= x^2 \cos x + 2x \sin x
\end{aligned}
$$

3. $y = \dfrac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$$

$$\frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$$

# Derivative of the Cosine Function :

$\cos(x+h) = \cos x \cos h - \sin x \sin h$

If $f(x) = \cos x$

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

**The derivative of the cosine function is the negative of the sine function:**

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example :

1. $y = 5x + \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) = 5 - \sin x.$$

2. $y = \sin x \cos x$

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

3. $y = \dfrac{\cos x}{1 - \sin x}$

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{(1-\sin x)(-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{1-\sin x}{(1-\sin x)^2} = \frac{1}{1-\sin x} \quad ; \quad \sin^2 x + \cos^2 x = 1$$

# Derivatives of the Other Basic Trigonometric Functions :

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \text{ and } \quad \csc x = \frac{1}{\sin x}$$

**Derivatives of the Other Trigonometric Functions**

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example :

1. Find $d(\tan x)/dx$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x .$$

2. Find $y''$ if $y = \sec x$.

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x .$$

3. Find $\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)}$

$$= \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

Example: A body hanging from a spring is stretched 5 units beyond its rest position and released at time $t=0$ to bob up and down. Its position at any later time $t$ is

$$s = 5\cos t.$$

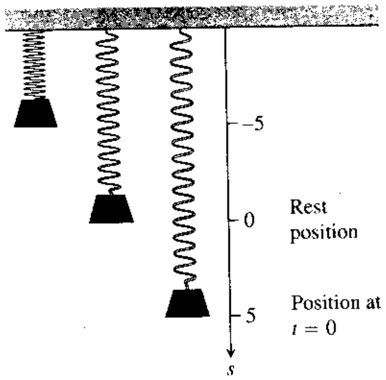
1. What are its velocity and acceleration at time $t$?
2. Find the jerk at time $t$?

Solution:

1. $s = 5\cos t$

$$v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = -5\sin t.$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\cos t.$$




2. $$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t) = 5\sin t.$$

## The Chain Rule and Parametric Equations:

### Derivative of a Composite Function:

Example: The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. How are the derivatives of these functions related?

Solution: $\frac{dy}{dx} = \frac{3}{2}$, $\frac{dy}{du} = \frac{1}{2}$, and $\frac{du}{dx} = 3$.

$$\because \frac{3}{2} = \frac{1}{2}\cdot 3 \Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Example:

The function $y = 9x^4 + 6x^2 + 1 = (3x^2+1)^2$ is the composite of $y = u^2$ and $u = 3x^2+1$. Calculating derivatives?:

$$\frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x$$
$$= 2(3x^2+1) \cdot 6x$$
$$= 36x^3 + 12x$$

$$\frac{dy}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1)$$
$$= 36x^3 + 12x$$

$$\therefore \quad \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

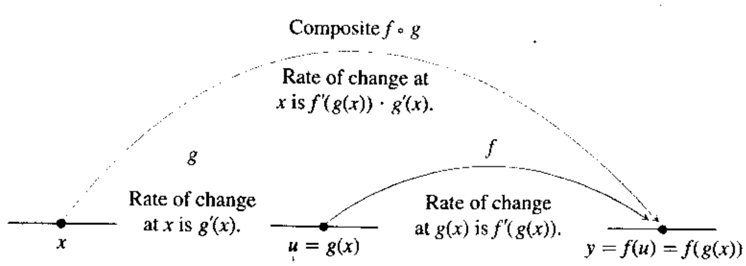



**FIGURE 3.27** Rates of change multiply: The derivative of $f \circ g$ at $x$ is the derivative of $f$ at $g(x)$ times the derivative of $g$ at $x$.

> **THEOREM 3 The Chain Rule**
> If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at $x$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at $x$, and
> $$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$
> In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then
> $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$
> where $dy/du$ is evaluated at $u = g(x)$.

Intuitive "Proof" of the Chain Rule:

Let $\Delta u$ be the change in $u$ corresponding to a change of $\Delta x$ in $x$, that is

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in $y$ is

$$\Delta y = f(u + \Delta u) - f(u)$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

take the limit as $\Delta x \to 0$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \qquad (\Delta u \to 0 \text{ as } \Delta x \to 0)$$

$$= \frac{dy}{du} \frac{du}{dx}.$$

Example: An object moves along the $x$-axis so that its position at any time $t \geqslant 0$ is given by $x(t) = \cos(t^2+1)$. Find the velocity of the object as a function of $t$.

Solution: We know that the velocity is $dx/dt$. In this instance, $x$ is a composite function:

$x = \cos(u)$ and $u = t^2+1$

$$\frac{dx}{du} = -\sin(u)$$

$$\frac{du}{dt} = 2t$$

By the Chain Rule,

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$

$$= -\sin(u) \cdot 2t$$

$$= -\sin(t^2+1) \cdot 2t$$

$$= -2t\sin(t^2+1)$$

"Outside - Inside" Rule :

If $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.

Example: Differentiate $\sin(x^2+x)$ with respect to $x$.

$$\frac{d}{dx} \sin(x^2+x) = \cos(x^2+x) \cdot (2x+1)$$

# Repeated Use of the Chain Rule:

Example: Find the derivative of $g(t) = \tan(5 - \sin 2t)$

Solution:

tangent is a function of $5 - \sin 2t$
sine is a function of $2t$

by the Chain Rule

$$g'(t) = \frac{d}{dt}(\tan(5 - \sin 2t))$$
$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t)$$
$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right)$$
$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$
$$= -2(\cos 2t)\sec^2(5 - \sin 2t)$$

## The Chain Rule with Powers of a Function:

If $f$ is a differentiable function of $u$ and if $u$ is a differentiable function of $x$, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{d}{dx} f(u) = f'(u) \frac{du}{dx}$$

If $n$ is a positive or negative integer and $f(u) = u^n \Rightarrow f'(n) = n u^{n-1}$. If $u$ is a differentiable function of $x$, then we can use the Chain Rule to extend this to the power Chain Rule:

$$\frac{d}{dx} u^n = n u^{n-1} \frac{du}{dx}$$

Example:

1. $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$
$$= 7(5x^3 - x^4)^6 \cdot (5 \cdot 3x^2 - 4x^3)$$
$$= 7(5x^3 - x^4)^6 (15x^2 - 4x^3)$$

2. $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2} \cdot \frac{d}{dx}(3x-2)$

$$= -1(3x-2)^{-2} \cdot (3) = -\frac{3}{(3x-2)^2}$$

Example:

a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

b) Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

Solution:

a) $\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x = 5\sin^4 x \cos x$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

b) $\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3} = -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x)$

$$= -3(1-2x)^{-4} \cdot (-2) = \frac{6}{(1-2x)^4}.$$

# Slopes of Parametrized Curves

A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at $t$ if $f$ and $g$ are differentiable at $t$.

Chain Rule : $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Parametric Formula for $dy/dx$

If all three derivatives exist and $dx/dt \neq 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example : If $x = 2t + 3$ and $y = t^2 - 1$ , find the value of $dy/dx$ at $t = 6$ .

Solution : $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t = \frac{x-3}{2} .$$

$$\left( x = 2t + 3 \Rightarrow 2t = x - 3 \Rightarrow t = \frac{x-3}{2} \right).$$

When $t = 6 \Rightarrow dy/dx = t \Rightarrow dy/dx = 6$ .

Example : Describe the motion of a particle whose position $P(x, y)$ at time $t$ is given by

$x = a\cos t$ , $y = b\sin t$ , $0 \leqslant t \leqslant 2\pi$ .

Find the line tangent to the curve at the point $(a/\sqrt{2}\, , b/\sqrt{2})$, where $t = \pi/4$ (the constants $a$ and $b$ are both positive).

Solution : We find a Cartesian equation for the particle's coordinates by eliminating $t$ between the equations

$$\cos t = \frac{x}{a} , \qquad \sin t = \frac{y}{b}$$

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

When $t = 0 \Rightarrow x = a\cos(0) = a$ , $y = b\sin(0) = 0$

so the motion starts at $(a, 0)$. As $t$ increases, the particle rises and moves toward the left, moving counterclockwise. It traverses the ellipse once, returning to its starting position $(a, 0)$ at $t = 2\pi$

The slope of the tangent line to the ellipse when $t = \pi/4$

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/4} = \left.\frac{b\cos t}{-a\sin t}\right|_{t=\pi/4} = \frac{b/\sqrt{2}}{-a/\sqrt{2}}$$

$$= -b/a.$$

The tangent line is

$$(y - y_0) = m(x - x_0)$$

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

$$y = \frac{b}{\sqrt{2}} - \frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

$$y = -\frac{b}{a}x + \sqrt{2}\, b$$

Parametric Formula for $d^2y/dx^2$

If the equations $x = f(t)$, $y = g(t)$ define $y$ as a twice-differentiable function of $x$, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Example: Find $d^2y/dx^2$ as a function of $t$ if $x = t - t^2$, $y = t - t^3$.

Solution:

1. Express $y' = dy/dx$ in terms of $t$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

2. Differentiate $y'$ with respect to $t$

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$$

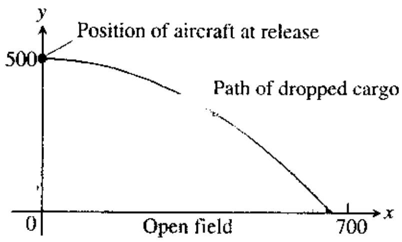
3. Divide $dy'/dt$ by $dx/dt$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2-6t+6t^2)/(1-2t)^2}{(1-2t)} = \frac{2-6t+6t^2}{(1-2t)^3}$$

Example: A Red Cross الصليب الاحمر aircraft is dropping emergency food and medical supplies into a disaster كارثة area. If the aircraft releases the supplies immediately above the edge of an open field 700 ft long and if the cargo حمولة moves along the path

$$x = 120t \quad \text{and} \quad y = -16t^2 + 500, \quad t \geq 0$$

does the cargo land in the field? The coordinates $x$ and $y$ are measured in feet, and the parameter $t$ (time since release) in seconds. Find a Cartesian equation for the path of the falling cargo and the cargo's rate of descent هبوط، نزول relative to its forward motion when it hits the ground.




Solution: The cargo hits the ground when $y = 0$

$$-16t^2 + 500 = 0 \Rightarrow t = \sqrt{\frac{500}{16}} = \frac{5\sqrt{5}}{2} \text{ sec.}$$

The $x$-coordinate at the time of the release is $x = 0$. At the time the cargo hits the ground

$$x = 120t = 120\left(\frac{5\sqrt{5}}{2}\right) = 300\sqrt{5} \text{ ft.}$$

$(300\sqrt{5} \approx 670.8 < 700)$, the cargo does land in the field.

$$y = -16t^2 + 500 \Rightarrow y = -16\left(\frac{x}{120}\right)^2 + 500$$

$$y = -\frac{1}{900}x^2 + 500$$

The rate of descent relative to its forward motion when the cargo hits the ground is

$$\left.\frac{dy}{dx}\right|_{t=5\sqrt{5}/2} = \left.\frac{dy/dt}{dx/dt}\right|_{t=5\sqrt{5}/2} = \left.\frac{-32t}{120}\right|_{t=5\sqrt{5}/2}$$

$$= -\frac{2\sqrt{5}}{3} = -1.49$$

Thus, it is falling about 1.5 feet for every foot of forward motion when it hits the ground.

**Standard Parametrizations and Derivative Rules**

CIRCLE $x^2 + y^2 = a^2$:

$x = a\cos t$
$y = a\sin t$
$0 \le t \le 2\pi$

ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

$x = a\cos t$
$y = b\sin t$
$0 \le t \le 2\pi$

FUNCTION $y = f(x)$:

$x = t$
$y = f(t)$

DERIVATIVES

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

## Implicit Differentiation:

Example: Find $dy/dx$ if $y^2 = x$.

Solution:

$$y^2 = x \Rightarrow y_1 = \sqrt{x} \text{ and } y_2 = -\sqrt{x}$$

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \text{ and } \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}$$

To find $dy/dx$, we simply differentiate both sides of the equation $y^2 = x$ with respect to $x$

$$y^2 = x \Rightarrow 2y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$y_1 = \sqrt{x} \quad \& \quad y_2 = -\sqrt{x}$$

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \text{ and } \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}$$

Example: Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution: $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$

The point $(3, -4)$ lies on the graph of $y_2$

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25-x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25-9}} = \frac{3}{4}.$$

But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to $x$:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope at $(3,-4)$ is $\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Example: Find $dy/dx$ if $y^2 = x^2 + \sin xy$

Solution: $y^2 = x^2 + \sin xy$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$

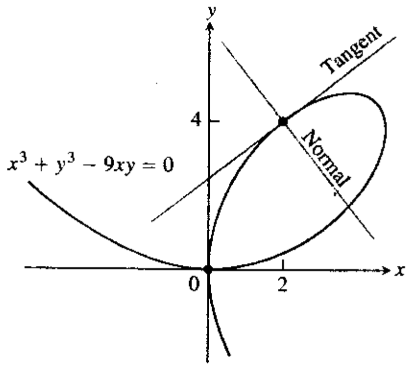
$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

**Implicit Differentiation**

1. Differentiate both sides of the equation with respect to $x$, treating $y$ as a differentiable function of $x$.
2. Collect the terms with $dy/dx$ on one side of the equation.
3. Solve for $dy/dx$.

Example: Show that the point $(2,4)$ lies on the curve $x^3+y^3-9xy=0$. Then find the tangent and normal to the curve




Solution: $x^3+y^3-9xy=0$ & $(2,4)$

$$(2)^3+(4)^3-9(2)(4) = 8+64-72=0$$

$$x^3+y^3-9xy=0$$

$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)-\frac{d}{dx}(9xy)=\frac{d}{dx}(0)$$

$$3x^2+3y^2\frac{dy}{dx}-9\left(x\frac{dy}{dx}+y\frac{dx}{dx}\right)=0$$

$$(3y^2-9x)\frac{dy}{dx}+3x^2-9y=0$$

$$3(y^2-3x)\frac{dy}{dx}=9y-3x^2$$

$$\frac{dy}{dx}=\frac{3y-x^2}{y^2-3x}$$

$$\left.\frac{dy}{dx}\right|_{(2,4)}=\left.\frac{3y-x^2}{y^2-3x}\right|_{(2,4)}=\frac{3(4)-2^2}{4^2-3(2)}=\frac{4}{5}$$

The tangent at $(2,4)$ is the line through $(2,4)$ with slope $4/5$:

$$y-y_0=m(x-x_0)$$

$$y=4+\frac{4}{5}(x-2)$$

$$y=\frac{4}{5}x+\frac{12}{5}$$

The normal to the curve at $(2,4)$ is the line perpendicular to the tangent there, the line through $(2,4)$ with slope $-5/4$

$$(y-y_0) = m(x-x_0)$$

$$y = 4 - \frac{5}{4}(x-2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}$$

## Derivatives of Higher Order:

*Example:* Find $d^2y/dx^2$ if $2x^3 - 3y^2 = 8$

Solution: $\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$

$$6x^2 - 6yy' = 0 \qquad (y' = dy/dx)$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y} \qquad \text{when } y \neq 0$$

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

substitute $y' = x^2/y$

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3} \qquad (y \neq 0)$$

## Rational Powers of Differentiable Functions:

$$\frac{d}{dx}x^n = nx^{n-1}$$

If $p/q$ is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}$$

*Example:*

1. $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ for $x > 0$

2. $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{-1/3}$ for $x \neq 0$

3. $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-7/3}$ for $x \neq 0$

Example:

1. $$\frac{d}{dx}(1-x^2)^{1/4} = \frac{1}{4}(1-x^2)^{-3/4}(-2x)$$
$$= \frac{-x}{2(1-x^2)^{3/4}}$$

2. $$\frac{d}{dx}(\cos x)^{-1/5} = -\frac{1}{5}(\cos x)^{-6/5}\frac{d}{dx}(\cos x)$$
$$= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$
$$= \frac{1}{5}(\sin x)(\cos x)^{-6/5}$$

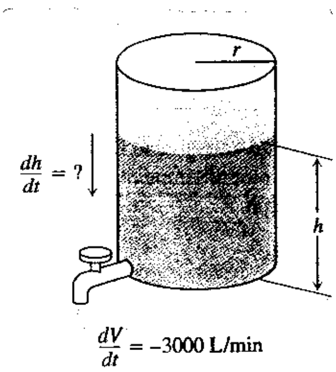
## Related Rates:

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$

Example: How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Solution: As time passes, the radius remains constant, but $V$ and $h$ change.
where
$V$ : volume of the fluid
$h$ : height of the fluid
$r$ : radius.




$$\frac{dV}{dt} = -3000$$ (The rate is negative because the volume is decreasing)

$V = 1000\pi r^2 h$ $\qquad (1\ m^3 = 1000\ L)$

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = 1000\pi r^2 \frac{dh}{dt} \qquad (r \text{ is a constant})$$

$$\therefore \quad \frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt} = -3000$$

$$\frac{dh}{dt} = -\frac{3000}{1000\pi r^2} = -\frac{3}{\pi r^2} \quad m/min.$$

The equation $dh/dt = -3/\pi r^2$ shows how the rate at which the fluid level drops depends on the tank's radius. If $r$ is small, $dh/dt$ will be large; if $r$ is large, $dh/dt$ will be small.

If $r = 1\ m \Rightarrow \frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95\ m/min = -95\ cm/min.$

If $r = 10\ m \Rightarrow \frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095\ m/min = -0.95\ cm/min.$

**Related Rates Problem Strategy**

1. *Draw a picture and name the variables and constants.* Use $t$ for time. Assume that all variables are differentiable functions of $t$.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

منطاد الهواء الساخن　　　مستوى ميداني

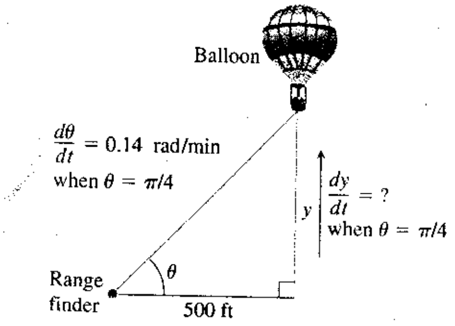
Example : A hot air balloon rising straight up from a level field is

جهاز تحديد المدى　　　انطلاق

tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution :

1. Draw a picture and name the variables and constants.




$\theta$ = the angle in radians the range finder makes with the ground.

$y$ = the height in feet of the balloon.

We let $t$ represent time in minutes and assume that $\theta$ and $y$ are differentiable functions of $t$.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. Write down the additional numerical information.

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. Write down what we are to find. We want $dy/dt$ when $\theta = \pi/4$.

4. Write an equation that relates the variables $y$ and $\theta$

$$\frac{y}{500} = \tan\theta \qquad \text{or} \qquad y = 500\tan\theta$$

5. Differentiate with respect to $t$ using the Chain Rule. The result tells how $dy/dt$ (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500(\sec^2\theta)\frac{d\theta}{dt}$$

6. Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find $dy/dt$.

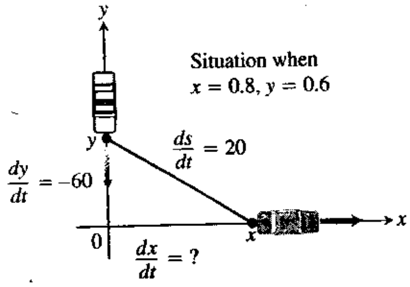
$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = 140 \qquad , \quad \sec\frac{\pi}{4} = \sqrt{2}$$

The balloon is rising at the rate of 140 ft/min.

تقاطع قائم الزاوية

**Example:** A police cruiser, approaching a right-angled intersection from the north, is chasing (مطاردة) a speeding car that has turned the corner and is now moving straight east (شرقاً). When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

**Solution:** We picture the car and cruiser in the coordinate plane, using the positive $x$-axis as the eastbound highway and the positive $y$-axis as the southbound highway. We let $t$ represent time and set




$x$ = position of car at time $t$.

$y$ = position of cruiser at time $t$.

$s$ = distance between car and cruiser at time $t$.

We assume that $x$, $y$, and $s$ are differentiable function of $t$

We want to find $dx/dt$ when

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \text{and}$$

$$\frac{ds}{dt} = 20 \text{ mph}.$$

Note that $dy/dt$ is negative because $y$ is decreasing.

$$s^2 = x^2 + y^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

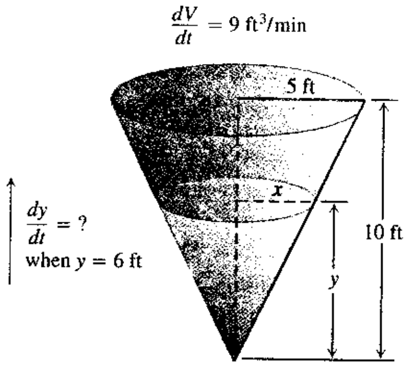
$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\frac{ds}{dt} = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$20 = \frac{1}{\sqrt{(0.8)^2+(0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\therefore \frac{dx}{dt} = 70 \text{ mph} \quad \text{The car's speed.}$$

Example : Water runs into a conical tank at the rate of $9\ ft^3/min$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?




Solution : The variables are

$V$ = volume ($ft^3$) of the water in the tank at time $t$ (min).

$x$ = radius (ft) of the surface of the water at time $t$.

$y$ = depth (ft) of water in tank at time $t$.

We assume that $V$, $x$, and $y$ are differentiable functions of $t$. The constants are the dimensions of the tank.

$$y = 6\ ft \quad \text{and} \quad \frac{dV}{dt} = 9\ ft^3/min$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y$$

This equation involves $x$ as well as $V$ and $y$. Because no information is given about $x$ and $dx/dt$ at the time in question, we need to eliminate $x$. The similar triangles give us a way to express $x$ in term of $y$:

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}$$

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

use $y = 6$ and $dV/dt = 9$

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{\pi} \approx 0.32\ ft/min.$$

The water level is rising at about 0.32 ft/min.

# Indeterminate Forms and L'Hôpital's Rule :

**L'Hôpital's Rule (First Form)**
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example :

1. $$\lim_{x \to 0} \frac{3x - \sin x}{x} = \left. \frac{3 - \cos x}{1} \right|_{x=0} = 2$$

2. $$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \left. \frac{\frac{1}{2\sqrt{1+x}}}{1} \right|_{x=0} = \frac{1}{2}$$

**THEOREM** **L'Hôpital's Rule (Stronger Form)**
Suppose that $f(a) = g(a) = 0$, that $f$ and $g$ are differentiable on an open interval $I$ containing $a$, and that $g'(x) \neq 0$ on $I$ if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Sometimes after differentiation, the new numerator and denominator both equal zero at $x = a$ .

Example :

1. $$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \qquad \text{still } \frac{0}{0} \text{ ; differentiate again.}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} \qquad \text{Not } \frac{0}{0} \text{ ; limit is found.}$$

$$= -\frac{1}{8}$$

2. $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$ $\quad \frac{0}{0}$

$= \lim_{x\to 0} \dfrac{1-\cos x}{3x^2}$ $\quad$ still $\frac{0}{0}$

$= \lim_{x\to 0} \dfrac{\sin x}{6x}$ $\quad$ still $\frac{0}{0}$

$= \lim_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}$ $\quad$ Not $\frac{0}{0}$; limit is found.

3. $\lim_{x\to 0} \dfrac{1-\cos x}{x+x^2}$ $\quad \frac{0}{0}$

$= \lim_{x\to 0} \dfrac{\sin x}{1+2x} = \dfrac{0}{1} = 0$ $\quad$ Not $\frac{0}{0}$; limit is found.

## Using L'Hôpital's Rule with One-Sided Limits:

Example:

1. $\lim_{x\to 0^+} \dfrac{\sin x}{x^2}$ $\quad \frac{0}{0}$

$= \lim_{x\to 0^+} \dfrac{\cos x}{2x} = \infty$ $\quad$ Positive for $x>0$

2. $\lim_{x\to 0^-} \dfrac{\sin x}{x^2}$ $\quad \frac{0}{0}$

$= \lim_{x\to 0^-} \dfrac{\cos x}{2x} = -\infty$ $\quad$ Negative for $x<0$

## Indeterminate Forms $\infty/\infty$, $\infty\cdot 0$, $\infty-\infty$

Example:

1. $\lim_{x\to \pi/2} \dfrac{\sec x}{1+\tan x}$

The numerator and denominator are discontinuous at $x=\pi/2$, so we investigate the one sided limits there.

$\lim_{x\to (\pi/2)^-} \dfrac{\sec x}{1+\tan x}$ $\quad \frac{\infty}{\infty}$ from the left

$$\lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

2. $\displaystyle \lim_{x\to\infty} \frac{x-2x^2}{3x^2+5x}$

$$= \lim_{x\to\infty} \frac{1-4x}{6x+5} = \lim_{x\to\infty} \frac{-4}{6} = -\frac{2}{3}.$$

3. $\displaystyle \lim_{x\to\infty} \left(x \sin\frac{1}{x}\right)$ $\qquad \infty \cdot 0$

$$= \lim_{h\to 0^+} \left(\frac{1}{h}\sin h\right), \quad \text{Let } h = \frac{1}{x}$$

$$= 1$$

4. $\displaystyle \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \longrightarrow \infty - \infty$$

If $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \longrightarrow -\infty - (-\infty) = -\infty + \infty$$

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \qquad \text{still } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

# Derivatives and Integrals :

| Derivatives of hyperbolic functions |
|---|
| $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$ |
| $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$ |
| $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$ |
| $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$ |
| $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$ |
| $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$ |

| Integral formulas for hyperbolic functions |
|---|
| $\int \sinh u\, du = \cosh u + C$ |
| $\int \cosh u\, du = \sinh u + C$ |
| $\int \operatorname{sech}^2 u\, du = \tanh u + C$ |
| $\int \operatorname{csch}^2 u\, du = -\coth u + C$ |
| $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$ |
| $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$ |

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right)$$
$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2}$$
$$= \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right)$$
$$= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx}$$
$$= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx}$$
$$= -\operatorname{csch} u \coth u \frac{du}{dx}.$$

Examples :

1. $$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$
$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

2. $$\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x}\, dx = \frac{1}{5}\int \frac{du}{u}$$

$u = \sinh 5x$
$du = 5\cosh 5x\, dx$

$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

3. $$\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx = \frac{1}{2}\int_0^1 (\cosh 2x - 1)\,dx$$

$$= \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1 = \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.407$$

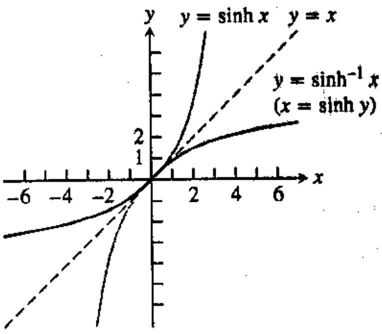
4. $$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2} (2e^{2x} - 2)\,dx$$

$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$

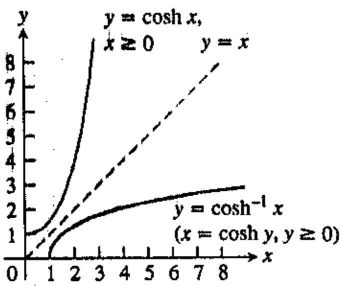
$$= 4 - 2\ln 2 - 1 \approx 1.614$$

## Inverse Hyperbolic Functions:

$y = \sinh^{-1} x$, $-\infty < x < \infty$ , $y = \cosh^{-1} x$, $x \geq 1$ , $y = \operatorname{sech}^{-1} x$, $x : (0, 1]$



(a)



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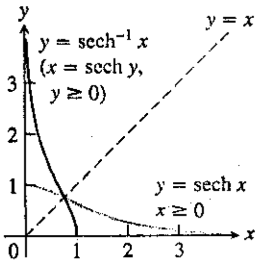
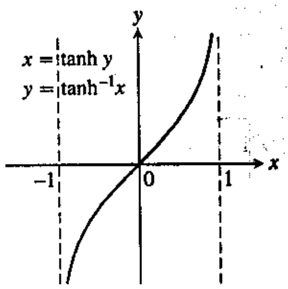
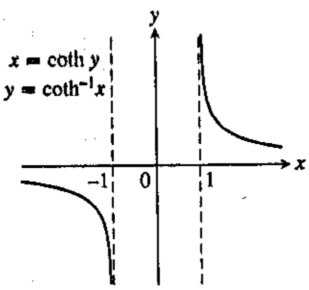


(c)

$y = \tanh^{-1} x$ , $y = \coth^{-1} x$ , $y = \operatorname{csch}^{-1} x$



(a)



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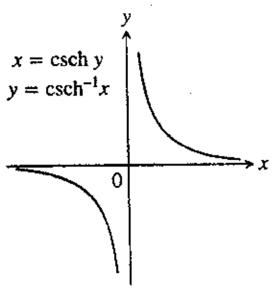


(c)

## Useful Identities:

$\operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$ , $\operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$ , $\coth^{-1} x = \tanh^{-1}\frac{1}{x}$

# Derivatives and Integrals :

**Derivatives of inverse hyperbolic functions**

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \qquad u \neq 0$$

**Integrals leading to inverse hyperbolic functions**

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$

3. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$

Example : Show that if $u$ is a differentiable function of $x$ whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}.$$

Solution.

$$y = \cosh^{-1} x, \quad f(x) = \cosh x, \quad f^{-1}(x) = \cosh^{-1} x.$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sinh(\cosh^{-1} x)} = \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}}$$

$$\left(\cosh^2 u - \sinh^2 u = 1, \quad \sinh u = \sqrt{\cosh^2 u - 1}\right)$$

$$(f^{-1})'(x) = \frac{1}{\sqrt{x^2-1}} \qquad (\cosh(\cosh^{-1}x) = x).$$

$$\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cosh^{-1}u) = \frac{1}{\sqrt{u^2-1}}\,\frac{du}{dx}.$$

Alternate Derivation:

$$y = \cosh^{-1}x$$

$$x = \cosh y \Rightarrow 1 = \sinh y\,\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y-1}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}} \qquad (\cosh y = x).$$

Example: Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$$

Solution:

$$u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

$$\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

$$\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.987.$$

# Techniques of Integration:

$$\int f(x)\,dx = F(x) + C$$

## Basic Integration Formulas:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

**Basic integration formulas**

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number $k$)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C$ $(a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C$ $(u > a > 0)$

## Examples:

1. $$\int \frac{2x-9}{\sqrt{x^2-9x+1}}\,dx$$

$$\boxed{\begin{aligned} u &= x^2 - 9x + 1 \\ du &= (2x-9)\,dx \end{aligned}}$$

$$\int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2u^{1/2} + C$$

$$2\sqrt{x^2-9x+1} + C$$

2. $\displaystyle\int \frac{dx}{\sqrt{8x-x^2}}$

$8x-x^2 = -(x^2-8x) = -(x^2-8x+16-16) = -(x^2-8x+16)+16$
$16-(x-4)^2.$

$$\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}}$$

$$\boxed{\begin{array}{ll} u = x-4 & , a=4 \\ du = dx & \end{array}}$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+C = \sin^{-1}\left(\frac{x-4}{4}\right)+C$$

3. $\displaystyle\int (\sec x + \tan x)^2\, dx$

$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$
$\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$$\int (\sec x + \tan x)^2\, dx = \int (\sec^2 x + 2\sec x \tan x + \sec^2 x - 1)\, dx$$

$$2\int \sec^2 x\, dx + 2\int \sec x \tan x\, dx - \int 1\, dx$$

$$2\tan x + 2\sec x - x + C$$

4. $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow 1+\cos 2\theta = 2\cos^2\theta$$

$$\theta = 2x \Rightarrow 1+\cos 4x = 2\cos^2 2x$$

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx$$

$$\sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$$\boxed{\begin{array}{l} \text{On } [0, \pi/4],\ \cos 2x \geq 0 \\ \text{So } |\cos 2x| = \cos 2x \end{array}}$$

$$\sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \sqrt{2}\left[\frac{1}{2} - 0\right] = \frac{\sqrt{2}}{2}$$

$$\boxed{\begin{array}{l} u = 2x \\ du = 2dx \end{array}}$$

5. $\displaystyle\int \frac{3x^2-7x}{3x+2}\,dx$

* ( Degree of numerator greater than or equal to degree of denominator ).

$$\begin{array}{r|l} & x-3 \\ \hline 3x+2 & 3x^2-7x \\ & \mp 3x^2 \mp 2x \\ \hline & \quad -9x \\ & \quad \pm 9x \pm 6 \\ \hline & \qquad +6 \end{array}$$

$$\frac{3x^2-7x}{3x+2} = x-3+\frac{6}{3x+2}$$

$$\int \frac{3x^2-7x}{3x+2}\,dx = \int \left(x-3+\frac{6}{3x+2}\right)dx$$

$$\frac{x^2}{2} - 3x + 2\ln|3x+2| + C.$$

6. $\displaystyle\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}$$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \\ x\,dx &= -\tfrac{1}{2}\,du \end{aligned}}$$

$$3\int \frac{x\,dx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)\,du}{\sqrt{u}}$$

$$-\frac{3}{2}\int u^{-1/2}\,du = -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1$$

$$-3\sqrt{1-x^2} + C_1$$

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1}x + C_2$$

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + C.$$

7. $\int \sec x \, dx = \int (\sec x)(1)\, dx = \int \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}\, dx$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$u = \tan x + \sec x$

$du = (\sec^2 x + \sec x \tan x)dx$

$$\int \frac{du}{u} = \ln|u| + c = \ln|\sec x + \tan x| + C$$

$$\boxed{\begin{aligned} \int \sec u \, du &= \ln|\sec u + \tan u| + c \\ \int \csc u \, du &= -\ln|\csc u + \cot u| + c \end{aligned}}$$

<u>Integration by Parts</u>:

$$\int x\,dx = \frac{1}{2}x^2 + c$$

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx$$

$$\int f(x)\,g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$$

$$\int f(x)\,g(x)\,dx$$

$$\int x e^x\,dx \quad , \quad \int e^x \sin x\,dx$$

$$\boxed{\begin{gathered} \int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx \\ \int u\,dv = uv - \int v\,du \\ \int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big]_a^b - \int_a^b f'(x)\,g(x)\,dx \end{gathered}}$$

Examples :

1. $\int x\cos x\,dx$.

$$\boxed{\begin{array}{ll} u=x\,, & dv=\cos x\,dx \\ du=dx\,, & v=\sin x \end{array}}$$

$$\int u\,dv = uv-\int v\,du$$

$$\int x\cos x\,dx = x\sin x-\int \sin x\,dx = x\sin x+\cos x+C.$$

2. $\int \ln x\,dx = \int \ln x\cdot 1\,dx$

$$\boxed{\begin{array}{ll} u=\ln x\,, & dv=dx \\ du=\frac{1}{x}\,dx\,, & v=x \end{array}}$$

$$\int \ln x\,dx = x\ln x-\int x\cdot\frac{1}{x}\,dx = x\ln x-\int dx$$

$$x\ln x-x+C.$$

3. $\int x^2e^x\,dx$

$$\boxed{\begin{array}{ll} u=x^2\,, & dv=e^x\,dx \\ du=2x\,dx\,, & v=e^x \end{array}}$$

$$\int x^2e^x\,dx = x^2e^x-2\int xe^x\,dx$$

$$\boxed{\begin{array}{ll} u=x\,, & dv=e^x\,dx \\ du=dx\,, & v=e^x \end{array}}$$

$$\int xe^x\,dx = xe^x-\int e^x\,dx = xe^x-e^x+C$$

$$\therefore \int x^2e^x\,dx = x^2e^x-2\int xe^x\,dx$$

$$= x^2e^x-2(xe^x-e^x)+C$$

$$= x^2e^x-2xe^x+2e^x+C.$$

4. $\int e^{x} \cos x \, dx$

$$\boxed{\begin{array}{ll} u = e^{x} & , \quad dv = \cos x \, dx \\ du = e^{x} dx & , \quad v = \sin x \end{array}}$$

$$\int e^{x} \cos x \, dx = e^{x} \sin x - \int e^{x} \sin x \, dx$$

$$\boxed{\begin{array}{ll} u = e^{x} & , \quad dv = \sin x \, dx \\ du = e^{x} dx & , \quad v = -\cos x \end{array}}$$

$$\int e^{x} \cos x \, dx = e^{x} \sin x - \left(-e^{x} \cos x - \int (-\cos x)(e^{x} dx)\right)$$

$$\int e^{x} \cos x \, dx = e^{x} \sin x + e^{x} \cos x - \int e^{x} \cos x \, dx$$

$$2 \int e^{x} \cos x \, dx = e^{x} \sin x + e^{x} \cos x$$

$$\int e^{x} \cos x \, dx = \frac{e^{x} \sin x + e^{x} \cos x}{2} + C$$

no .....

5. $\int x^2 \sin x \, dx$

$$\boxed{\begin{array}{ll} u = x^2 & , \quad dv = \sin x \, dx \\ du = 2x \, dx & , \quad v = -\cos x \end{array}}$$

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int (-\cos x)(2x \, dx)$$

$$= -x^2 \cos x + 2 \int x \cos x \, dx$$

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

$$\therefore \int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

$$= (-x^2 + 2) \cos x + 2x \sin x + C$$

6. $\int \sin^{-1}x \, dx$

$$\boxed{\begin{array}{ll} u = \sin^{-1}x \quad , & dv = dx \\ du = \dfrac{dx}{\sqrt{1-x^2}} \quad , & v = x \end{array}}$$

$$\int \sin^{-1}x \, dx = x \sin^{-1}x - \int x \cdot \frac{dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1}x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} \, dx$$

$$= x \sin^{-1}x + \frac{1}{2} \; \frac{(1-x^2)^{1/2}}{1/2} + C$$

$$= x \sin^{-1}x + \sqrt{1-x^2} + C$$

Example :

Find the area of the region bounded by the curve $y = x e^{-x}$ and the $x$-axis from $x = 0$ to $x = 4$.

Solution :

$$\int_0^4 x e^{-x} \, dx$$

$$\boxed{\begin{array}{ll} u = x \quad , & dv = e^{-x} dx \\ du = dx \quad , & v = -e^{-x} \end{array}}$$

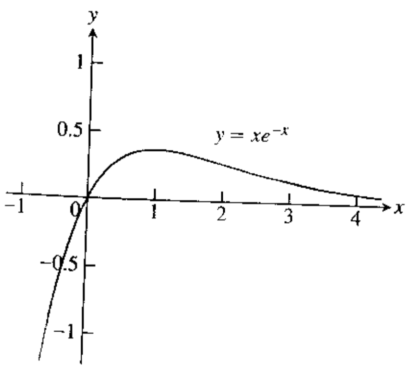
$$\int_0^4 x e^{-x} dx = -x e^{-x} \Big]_0^4 - \int_0^4 (-e^{-x}) \, dx$$

$$= \left[-4e^{-4} - (0)\right] + \int_0^4 e^{-x} \, dx$$

$$= -4e^{-4} - e^{-x} \Big]_0^4$$

$$= -4e^{-4} - e^{-4} - (-e^{0})$$

$$= 1 - 5e^{-4} \approx 0.91$$



## Tabular Integration:

Examples:

1. $\int x^2 e^x \, dx$

$f(x) = x^2 \quad , \quad g(x) = e^x$

| $f(x)$ and its derivatives | | $g(x)$ and its integrals |
|---|---|---|
| $x^2$ | $\xrightarrow{(+)}$ | $e^x$ |
| $2x$ | $\xrightarrow{(-)}$ | $e^x$ |
| $2$ | $\xrightarrow{(+)}$ | $e^x$ |
| $0$ | | $e^x$ |

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2 e^x + C .$$

2. $\int x^3 \sin x \, dx$

$f(x) = x^3 \quad , \quad g(x) = \sin x$

| $f(x)$ and its derivatives | | $g(x)$ and its integrals |
|---|---|---|
| $x^3$ | $\xrightarrow{(+)}$ | $\sin x$ |
| $3x^2$ | $\xrightarrow{(-)}$ | $-\cos x$ |
| $6x$ | $\xrightarrow{(+)}$ | $-\sin x$ |
| $6$ | $\xrightarrow{(-)}$ | $\cos x$ |
| $0$ | | $\sin x$ |

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C .$$

# Integration of Rational Functions by Partial Fractions :

Examples:

1. $\displaystyle\int \frac{5x-3}{(x+1)(x-3)}\,dx$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

$$5x-3 = A(x-3)+B(x+1)$$

$$5x-3 = (A+B)x + (B-3A)$$

$$A+B = 5 \quad \text{———} \quad ①$$

$$-3A+B = -3 \quad \text{———} \quad ②$$

$$\therefore B = 3A-3 \Rightarrow A+3A-3 = 5 \Rightarrow A = 2 \quad \& \quad B = 3$$

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \left( \frac{2}{x+1} + \frac{3}{x-3} \right) dx$$

$$2\int \frac{dx}{x+1} + 3\int \frac{dx}{x-3} = 2\ln|x+1| + 3\ln|x-3| + C.$$

2. $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A(x+1)(x+3)+B(x-1)(x+3)+C((x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$x^2+4x+1 = A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1)$$

$$x^2+4x+1 = (A+B+C)x^2+(4A+2B)x+(3A-3B-C)$$

$$A+B+C = 1$$

$$4A+2B = 4$$

$$3A-3B-C = 1$$

$$A = 3/4, \quad B = 1/2, \quad C = -1/4$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$

$$\frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K \,.$$

3. $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\frac{6x+7}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

$$6x+7 = A(x+2)+B$$
$$6x+7 = Ax + (2A+B)$$
$$A = 6 \quad \& \quad 2A+B = 7 \Rightarrow 2(6)+B = 7 \Rightarrow B = -5$$

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int\left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right)dx$$

$$6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx = 6\ln|x+2| + 5(x+2)^{-1} + C$$

4. $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

$$\begin{array}{r|l} & \quad 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & \mp 2x^3 \pm 4x^2 \pm 6x \\ \hline & \qquad\qquad 5x-3 \end{array}$$

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{5x-3}{(x+1)(x-3)}\,dx$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

$$5x-3 = A(x-3)+B(x+1) \Rightarrow 5x-3 = (A+B)x + (B-3A)$$

$$\left.\begin{array}{l} A+B = 5 \\ -3A+B = -3 \end{array}\right\} \quad \begin{array}{l} A = 2 \\ B = 3 \end{array}$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = x^2 + 2\ln|x+1| + 3\ln|x-3| + C .$$

5. $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{(Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$

$$-2x+4 = (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D) .$$

$$0 = A+C$$
$$0 = -2A+B-C+D$$
$$-2 = A-2B+C$$
$$4 = B-C+D$$

$\therefore\ A=2\ ,\quad C=-2\ ,\quad B=1\ ,\quad D=1$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2} .$$

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int\left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \int\left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)dx$$

$$= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C .$$

6. $\displaystyle\int \frac{dx}{x(x^2+1)^2}$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$1 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$A+B=0$ , $C=0$ , $2A+B+D=0$ , $C+E=0$ , $A=1$

$A=1$ , $B=-1$ , $C=0$ , $D=-1$ , & $E=0$

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[ \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right] dx$$

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

$u = x^2+1$

$du = 2x\,dx$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

$$= \ln \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K .$$

## Trigonometric Integrals :

### Products of Powers of Sines and Cosines :

$$\int \sin^m x \cos^n x \, dx$$

where m and n are nonnegative integers ( positive or zero ).

case 1 If m is odd, we write m as $2K+1$ and use the identity $\sin^2 x = 1-\cos^2 x$ to obtain

$$\sin^m x = \sin^{2K+1} x = (\sin^2 x)^K \sin x = (1-\cos^2 x)^K \sin x .$$

case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2K+1$ and use the identity $\cos^2 x = 1-\sin^2 x$ to obtain

$$\cos^n x = \cos^{2K+1} x = (\cos^2 x)^K \cos x = (1-\sin^2 x)^K \cos x .$$

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = (1-\cos 2x)/2 \quad , \quad \cos^2 x = (1+\cos 2x)/2$$

to reduce the integrand to one in lower powers of $\cos 2x$ .

Examples :

1. $\int \sin^3 x \cos^2 x \, dx$

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx = \int (1-\cos^2 x)\cos^2 x \, (-d(\cos x))$$

$u = \cos x$

$$\int (1-u^2)(u^2)(-du) = \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C \; .$$

2. $\int \cos^5 x \, dx$ $\qquad m = 0$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1-\sin^2 x)^2 \, d(\sin x)$$

$u = \sin x$

$$\int (1-u^2)^2 du = \int (1 - 2u^2 + u^4)\,du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \; .$$

3. $\int \sin^2 x \cos^4 x \, dx$

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\frac{1}{8}\int (1-\cos 2x)(1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$\frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\,dx$$

$$\frac{1}{8}\left[ x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\,dx \right]$$

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1+\cos 4x)\,dx = \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)$$

$$\int \cos^3 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x \quad , \quad du = 2\cos 2x \, dx$

$$\frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)$$

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C \; .$$

# Integrals of Powers of $\tan x$ and $\sec x$:

$$\tan^2 x = \sec^2 x - 1 \quad , \quad \sec^2 x = \tan^2 x + 1$$

Examples:

1. $\int \tan^4 x \, dx$

$$\int \tan^4 x \, dx = \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \, (\sec^2 x - 1) \, dx$$

$$\int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$\int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$\int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx$$

$$u = \tan x \quad , \quad du = \sec^2 x \, dx$$

$$\int u^2 du = \frac{1}{3} u^3 + C_1 = \frac{1}{3} \tan^3 x + C_1$$

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C .$$

2. $\int \sec^3 x \, dx$

$$\boxed{\begin{array}{ll} u = \sec x & , \; dv = \sec^2 x \, dx \\ du = \sec x \tan x \, dx \; , & v = \tan x \end{array}}$$

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx = \sec x \tan x - \int (\tan x)(\sec x \tan x \, dx)$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln | \sec x + \tan x | + C .$$

$$* \quad \int \sec u \, du = \ln | \sec u + \tan u | + C$$

## Products of Sines and Cosines:

$\int \sin mx \sin nx \, dx$ , $\int \sin mx \cos nx \, dx$ , $\int \cos mx \cos nx \, dx$

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

Example:

$$\int \sin 3x \cos 5x \, dx$$

Solution:

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$

$$m = 3 \quad \& \quad n = 5$$

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx$$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx = -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

## Trigonometric Substitutions:

### Three Basic Substitutions:

$x = a\tan\theta$

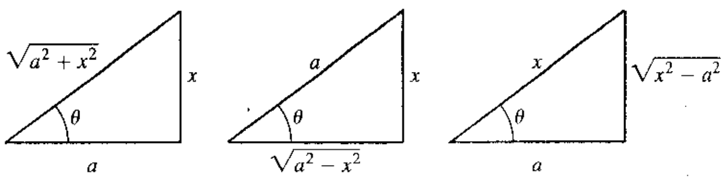
$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$x = a\sin\theta$

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

$x = a\sec\theta$

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$



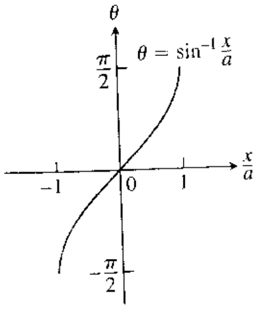

$x = a\tan\theta$ $\quad$ $x = a\sin\theta$ $\quad$ $x = a\sec\theta$

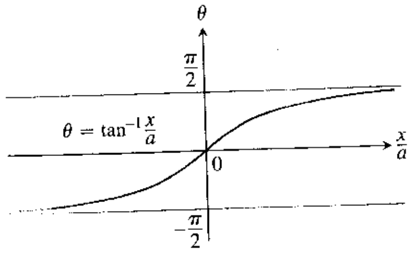
$\sqrt{a^2+x^2} = a|\sec\theta|$ $\quad$ $\sqrt{a^2-x^2} = a|\cos\theta|$ $\quad$ $\sqrt{x^2-a^2} = a|\tan\theta|$

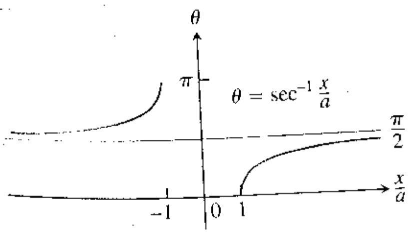
$x = a\tan\theta \quad , \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$x = a\sin\theta \quad , \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

$$x = a\sec\theta \quad , \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \begin{cases} 0 \leqslant \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geqslant 1 \\ \frac{\pi}{2} < \theta \leqslant \pi & \text{if } \frac{x}{a} \leqslant -1 \end{cases}$$




Examples:

1. $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

$$\boxed{\begin{aligned} &x = 2\tan\theta \ , \quad dx = 2\sec^2\theta\, d\theta \ , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta \end{aligned}}$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} = \int \sec\theta\, d\theta$$

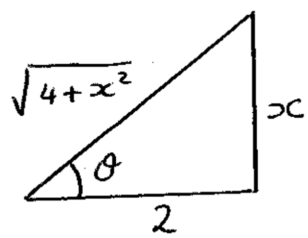
$$\boxed{\sqrt{\sec^2\theta} = |\sec\theta| \quad , \quad \sec\theta > 0 \ \text{ for } \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}}.$$

$\ln|\sec\theta + \tan\theta| + C$

$\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

$\ln\left|\sqrt{4+x^2} + x\right| + C'$

$C' = C - \ln 2$




$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$

$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

2. $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

$$\boxed{\begin{array}{l} x = 3\sin\theta \ , \quad dx = 3\cos\theta\, d\theta \ , \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta \end{array}}$$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} = 9\int \sin^2\theta\, d\theta$$

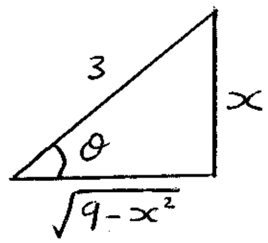
$$\boxed{\sqrt{\sec^2\theta} = |\sec\theta| \ , \quad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad , \quad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C$$




$x = 3\sin\theta$

$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

3. $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}} \quad , \quad x > \frac{2}{5}$

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)} = 2\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

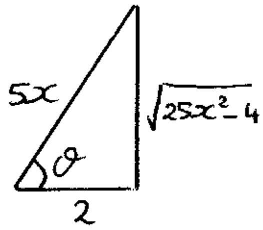
$$\boxed{\begin{array}{l} x = \frac{2}{5}\sec\theta \ , \quad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta \ , \quad 0 < \theta < \frac{\pi}{2} \\ x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta \end{array}}$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta \qquad \tan\theta > 0 \text{ for } 0 < \theta < \frac{\pi}{2}$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2 - \left(\frac{4}{25}\right)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta \, d\theta = \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$$




$x = \frac{2}{5}\sec\theta$

$\theta = \sec^{-1}\left(\frac{5x}{2}\right)$

# Integration of Rational Function of $\sin x$ and $\cos x$:

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$\boxed{Z = \tan\frac{x}{2}}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2} - \left(1 - \cos^2\frac{x}{2}\right) = 2\cos^2\frac{x}{2} - 1$$

$$\frac{2}{\sec^2\frac{x}{2}} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+Z^2} - 1$$

$$\frac{2-1-Z^2}{1+Z^2} = \frac{1-Z^2}{1+Z^2}$$

$$\boxed{\cos x = \frac{1-Z^2}{1+Z^2}}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\,\frac{1}{\sec^2\frac{x}{2}}$$

$$\frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2Z}{1+Z^2}$$

$$\boxed{\sin x = \frac{2Z}{1+Z^2}}$$

$$Z = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \tan^{-1} Z \Rightarrow x = 2\tan^{-1} Z$$

$$\boxed{dx = \frac{2\,dZ}{1+Z^2}}$$

Examples :

1. $\int \frac{1}{1+\cos x}\,dx$

$$\cos x = \frac{1-z^2}{1+z^2} \quad , \quad dx = \frac{2dz}{1+z^2}$$

$$\int \frac{1}{1+\left(\frac{1-z^2}{1+z^2}\right)} \cdot \frac{2dz}{1+z^2} = \int \frac{2dz}{1+z^2+1-z^2} = \int \frac{2}{2}\,dz$$

$$\int dz = z + C = \tan\frac{x}{2} + C \,.$$

2. $\int \frac{1}{2+\sin x}\,dx$

$$\sin x = \frac{2z}{1+z^2} \quad , \quad dx = \frac{2dz}{1+z^2}$$

$$\int \frac{1}{2+\left(\frac{2z}{1+z^2}\right)} \left(\frac{2dz}{1+z^2}\right) = \int \frac{2dz}{2+2z^2+2z} = \frac{2}{2}\int \frac{dz}{z^2+z+1}$$

$$\int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+\frac{1}{2})^2+\frac{3}{4}}$$

$$\boxed{u = z+\tfrac{1}{2} \;,\; du = dz \qquad a = \tfrac{\sqrt{3}}{2}}$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z+\frac{1}{2}}{\sqrt{3}/2}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right) + C$$

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(\frac{x}{2})+1}{\sqrt{3}}\right) + C \,.$$

3. $\int \frac{dx}{\sin x - \cos x}$

$$\sin x = \frac{2z}{1+z^2} \quad , \quad \cos x = \frac{1-z^2}{1+z^2} \quad , \quad dx = \frac{2dz}{1+z^2}$$

$$\int \frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}} = \int \frac{\frac{2}{1+z^2}}{\frac{2z-1+z^2}{1+z^2}}\,dz = \int \frac{2}{z^2+2z-1}\,dz$$

$$2\int \frac{dz}{(z+1)^2-2} = 2\int \frac{dz}{(z+1)^2-(\sqrt{2})^2}$$

$$\boxed{u = z+1 \ , \quad du = dz \quad a=\sqrt{2}}$$

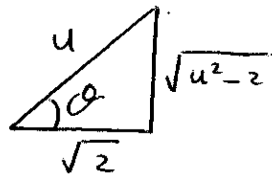
$$2\int \frac{du}{u^2-a^2} = 2\int \frac{\sqrt{2}\sec\theta\tan\theta\, d\theta}{2\sec^2\theta - 2}$$

$$\boxed{u = a\sec\theta = \sqrt{2}\sec\theta \quad du = \sqrt{2}\sec\theta\tan\theta\, d\theta}$$

$$2\sqrt{2}\int \frac{\sec\theta\tan\theta\, d\theta}{2(\sec^2\theta - 1)} = \sqrt{2}\int \frac{\sec\theta\tan\theta}{\tan^2\theta}\, d\theta$$

$$\sqrt{2}\int \frac{\sec\theta}{\tan\theta}\, d\theta = \sqrt{2}\int \frac{1/\cos\theta}{\sin\theta/\cos\theta}\, d\theta = \sqrt{2}\int \csc\theta\, d\theta$$

$$\sqrt{2}\ \ln\left|\csc\theta - \cot\theta\right| + c$$




$$\sqrt{2}\ \ln\left|\frac{u}{\sqrt{u^2-2}} - \frac{\sqrt{2}}{\sqrt{u^2-2}}\right| + c$$

$$\sqrt{2}\ \ln\left|\frac{z+1}{\sqrt{(z+1)^2-2}} - \frac{\sqrt{2}}{\sqrt{(z+1)^2-2}}\right| + c$$

$$\sqrt{2}\ \ln\left|\frac{\tan\frac{x}{2}+1}{\sqrt{(\tan\frac{x}{2}+1)^2-2}} - \frac{\sqrt{2}}{\sqrt{(\tan\frac{x}{2}+1)^2-2}}\right| + c$$

## Applications of Definite Integrals :

### Area between Curves :

If functions $f_1$ and $f_2$ are continuous and if $f_1(x) \geqslant f_2(x)$ throughout the interval $a \leqslant x \leqslant b$, then the area of the region between the curves $y=f_1(x)$ and $y=f_2(x)$ from a to b is the integral of $(f_1 - f_2)$ from a to b :

$$\boxed{\text{Area} = \int_a^b \left(f_1(x) - f_2(x)\right) dx}$$

Examples :

1. Find the area between the curves $y=\cos x$ and $y=-\sin x$ from $0$ to $\pi/2$ .

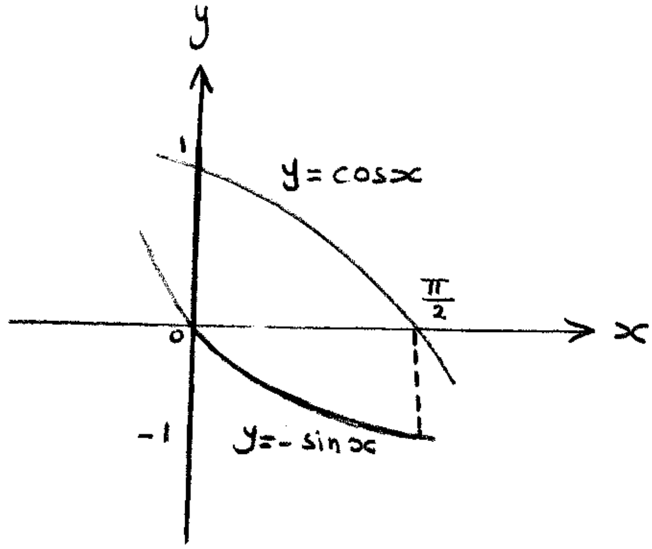
Solution:

$f_1(x) = \cos x$

$f_2(x) = -\sin x$

$f_1(x) - f_2(x) = \cos x - (-\sin x)$

$= \cos x + \sin x$



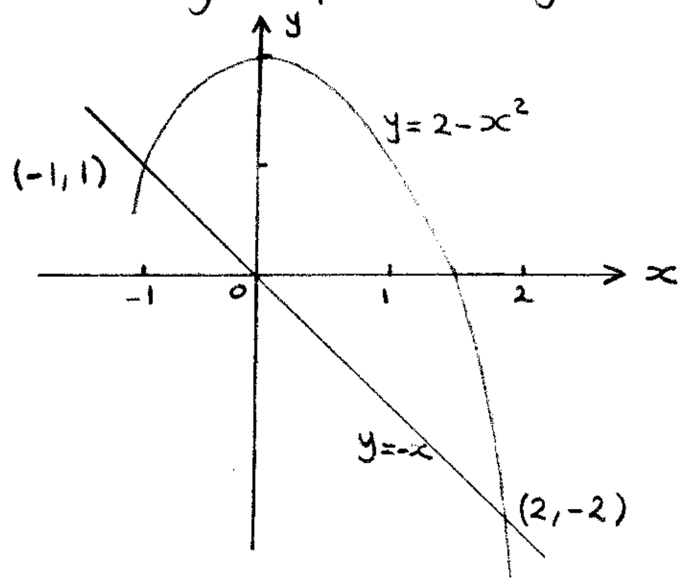

$$A = \int_0^{\pi/2} (\cos x + \sin x)\,dx$$

$$A = \left[\sin x - \cos x\right]_0^{\pi/2}$$

$$A = (1 - 0) - (0 - 1)$$

$$A = 2$$

2. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.




Solution:

$f_1(x) = f_2(x)$

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = -1, \; x = 2$

$f_1(x) - f_2(x) = 2 - x^2 - (-x)$

$= 2 - x^2 + x$

$= 2 + x - x^2$

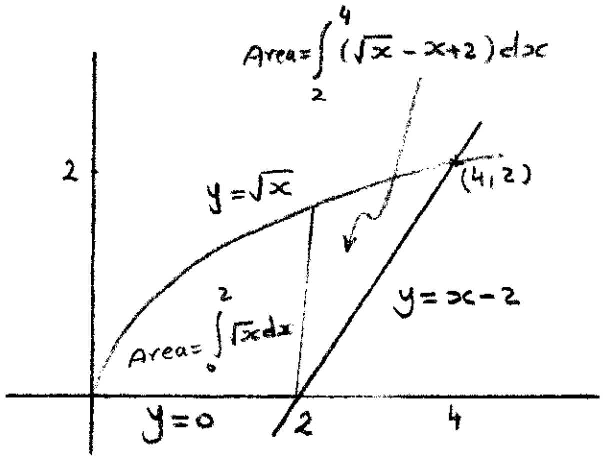
$$A = \int [f_1(x) - f_2(x)]\,dx$$

$$A = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$A = \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

3. Find the area of the region in the first quadrant bounded above by the curve $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$.

Solution :




$f_1(x) = \sqrt{x}$

$f_2(x) = 0 \quad \text{for} \quad 0 \leqslant x \leqslant 2$

$f_2(x) = x - 2 \quad \text{for} \quad 2 \leqslant x \leqslant 4$

$\sqrt{x} = x - 2 \Rightarrow x = (x-2)^2$

$x = x^2 - 4x + 4 \Rightarrow x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0 \Rightarrow x = 1 \,,\, x = 4$

$x = 1$ does not satisfy the equation $\sqrt{x} = x - 2$

for $0 \leqslant x \leqslant 2$ : $\quad f_1(x) - f_2(x) = \sqrt{x} - 0 = \sqrt{x}$

for $2 \leqslant x \leqslant 4$ : $\quad f_1(x) - f_2(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

$$\text{Area} = \int_0^4 [f_1(x) - f_2(x)]\,dx = \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} + \left(\frac{2}{3}(4)^{3/2} - \frac{16}{2} + 8\right) - \left(\frac{2}{3}(2)^{3/2} - \frac{4}{2} + 4\right) = \frac{10}{3} .$$

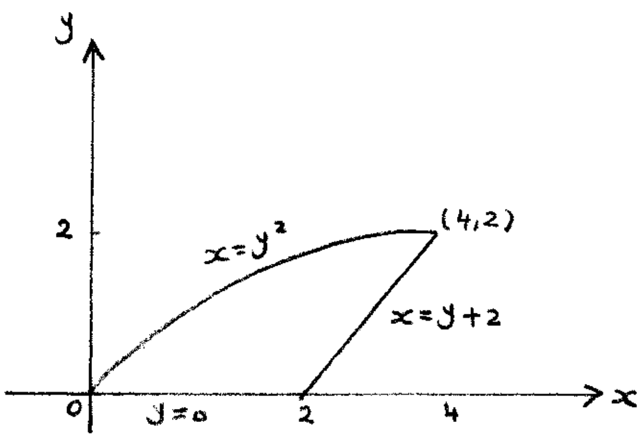
<u>Integration with respect to $y$</u> :

$$\boxed{\text{Area} = \int_c^d [f_1(y) - f_2(y)]\,dy}$$

Examples :

1. Find the area of the region between the curves $x = y^2$ and $x = y + 2$ in the first quadrant.

Solution :




$f_1(y) = y + 2$

$f_2(y) = y^2$

$y + 2 = y^2 \Rightarrow y^2 - y - 2 = 0$

$(y+1)(y-2) = 0$

$y = -1 \quad , \quad y = 2$

The value $y = -1$ gives the point of intersection below the $x$-axis.

$f_1(y) - f_2(y) = y + 2 - y^2 = 2 + y - y^2$

$$Area = \int_0^2 (2 + y - y^2)\, dy = \left[ 2y + \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2$$

$$Area = \frac{4}{1} + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}$$

2. Find the area between $y^2 = 4x$ and $4x - 3y = 4$

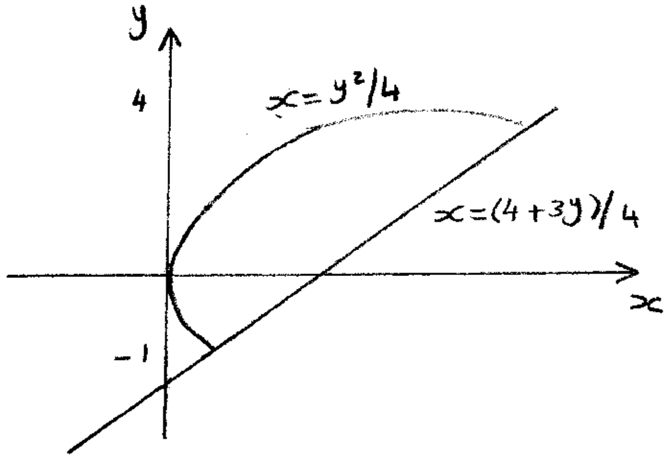
solution:

$$4x - 3y = 4 \Rightarrow 4x = 4 + 3y \Rightarrow x = (4+3y)/4$$

$$y^2 = 4x \Rightarrow x = y^2/4$$

$$f_1(y) = x = \frac{4+3y}{4}$$

$$f_2(y) = x = \frac{y^2}{4}$$




$$\frac{4+3y}{4} = \frac{y^2}{4} \Rightarrow y^2 - 3y - 4 = 0$$

$$(y-4)(y+1) = 0$$

$$y = 4 \quad , \quad y = -1$$

$$Area = \int_{-1}^{4} (f_1(y) - f_2(y))\, dy = \int_{-1}^{4} \left( \left(\frac{4+3y}{4}\right) - \left(\frac{y^2}{4}\right) \right) dy$$

$$Area = \left[ y + \frac{3y^2}{8} - \frac{y^3}{12} \right]_{-1}^{4} = \frac{125}{24}$$

## Volumes by Slicing and Rotation About an Axis:

Volume = Area × Height , $V = A \cdot h$

The volume of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of $A$ from $a$ to $b$,

$$\boxed{V = \int_a^b A(x)\, dx}$$

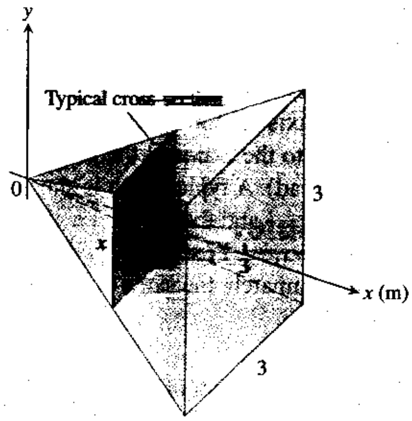
Calculating the volume of a solid:

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ using the Fundamental Theorem.

هرم

**Example:** A pyramid 3m high has a square base that is 3m on a side.
ارتفاع
The cross-section of the pyramid perpendicular to the altitude
قمة
$x$ m down from the vertex is a square $x$ m on a side.
Find the volume of the pyramid ?

solution:




1. A sketch.

2. A formula for $A(x)$.

$$A(x) = x^2$$

3. The limits of integration.

The squares lie on the planes from $x=0$ to $x=3$

4. Integrate to find the volume.

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right]_0^3 = 9\ m^3.$$

## Solids of Revolution: The Disk Method :

$$A(x) = \pi\,(\text{radius})^2 = \pi\,[R(x)]^2.$$

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.}$$ revolution about $x$-axis.

$$A(y) = \pi\,(\text{radius})^2 = \pi\,[R(y)]^2.$$

$$\boxed{V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.}$$ revolution about $y$-axis.

Examples :

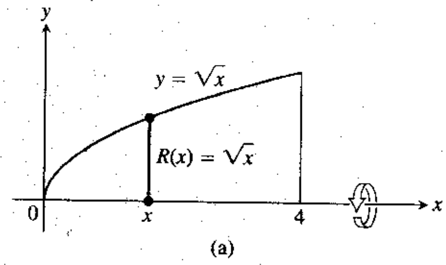
1. The region between the curve $y=\sqrt{x}$, $0 \leqslant x \leqslant 4$, and the $x$-axis is revolved about the $x$-axis to generate a solid. Find its volume ?

Solution:

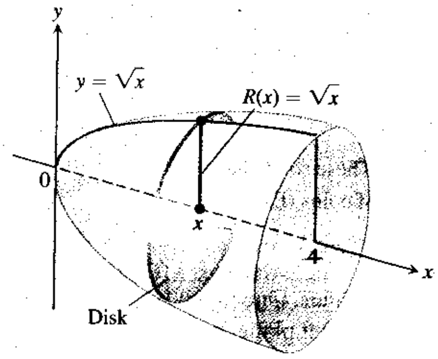
$$V = \int_a^b \pi[R(x)]^2\,dx \qquad , \qquad R(x) = \sqrt{x}.$$

$$V = \int_0^4 \pi [\sqrt{x}]^2 \, dx$$

$$V = \pi \int_0^4 x \, dx = \pi \left. \frac{x^2}{2} \right]_0^4 = 8\pi .$$



(a)



(b)

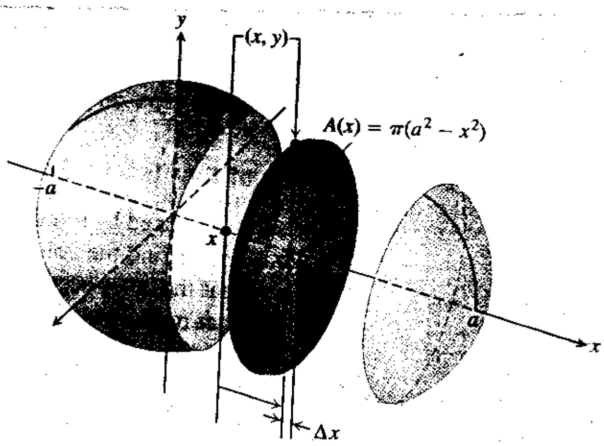
2. The circle $x^2 + y^2 = a^2$ is rotated about the $x$-axis to generate a sphere. Find its volume?

Solution:

We imagine the sphere cut into thin slices by planes perpendicular to the $x$-axis. The cross-sectional area at a typical point $x$ between $-a$ and $a$ is

$$A(x) = \pi y^2 = \pi (a^2 - x^2)$$

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi (a^2 - x^2)\,dx = \pi \left[ a^2 x - \frac{x^3}{3} \right]_{-a}^{a} = \frac{4}{3} \pi a^3 .$$



3. Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$?

solution:

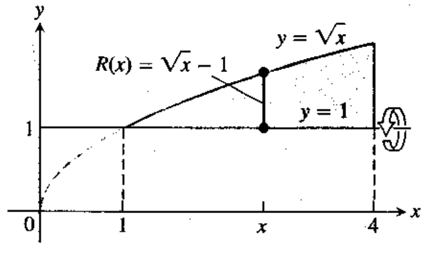
$$V=\int_1^4 \pi[R(x)]^2\,dx$$

$$=\int_1^4 \pi[\sqrt{x}-1]^2\,dx$$

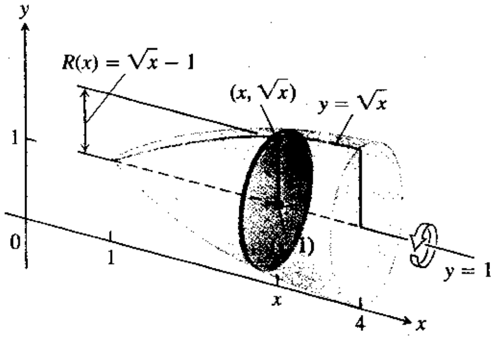
$$=\pi\int_1^4 [x-2\sqrt{x}+1]\,dx$$

$$=\pi\left[\frac{x^2}{2}-2\cdot\frac{2}{3}x^{3/2}+x\right]_1^4$$

$$=\frac{7\pi}{6}.$$



(a)



(b)

4. Find the volume of the solid generated by revolving the region between the y-axis and the curve $x=2/y$, $1\leqslant y\leqslant 4$, about the y-axis?

Solution:

$$V=\int_1^4 \pi[R(y)]^2\,dy$$

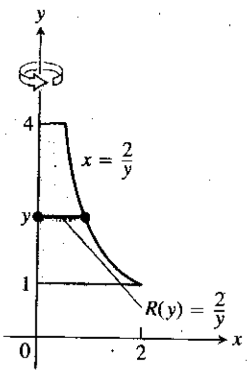
$$=\int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$=\pi\int_1^4 \frac{4}{y^2}\,dy$$

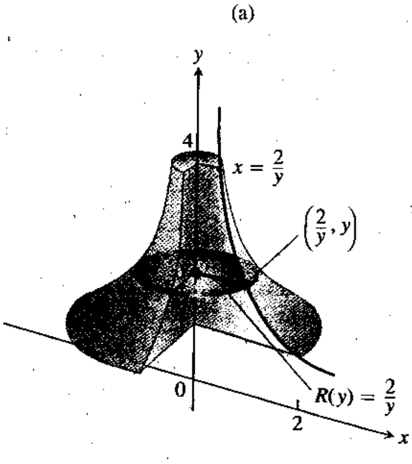
$$=4\pi\left[-\frac{1}{y}\right]_1^4$$

$$=4\pi\left[\frac{3}{4}\right]$$

$$=3\pi.$$



(a)



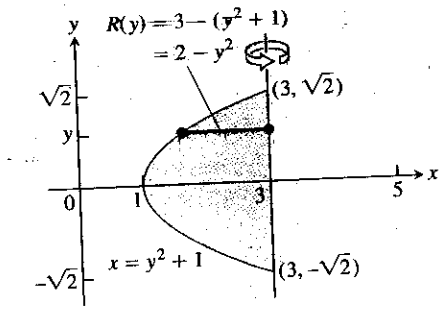
5. Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$ ?

solution :

$$R(y) = 3 - (y^2 + 1)$$

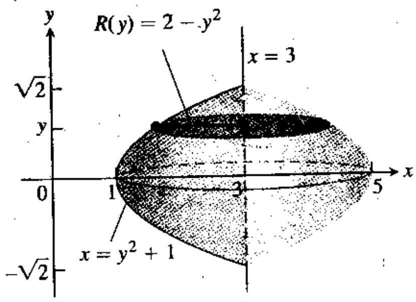
$$R(y) = 3 - y^2 - 1$$

$$R(y) = 2 - y^2$$




$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 dy$$




$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy$$

$$= \pi \left[ 4y - \frac{4}{3} y^3 + \frac{1}{5} y^5 \right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \frac{64 \pi \sqrt{2}}{15}$$

## Solids of Revolution : The Washer Method :

$R(x)$ : Outer radius.

$r(x)$ : Inner radius.

$$A(x) = \pi [R(x)]^2 - \pi [r(x)]^2 = \pi ([R(x)]^2 - [r(x)]^2).$$

$$\boxed{V = \int_a^b A(x) dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx}$$

Examples :

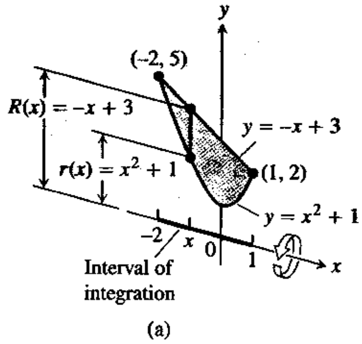
1. The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid ?

solution ;

Outer radius : $R(x) = -x + 3$

Inner radius : $r(x) = x^2 + 1$

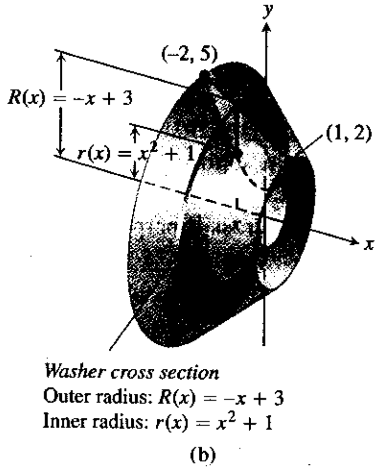
$$x^2+1=-x+3$$
$$x^2+x-2=0$$
$$(x+2)(x-1)=0$$
$$x=-2\ ,\ x=1$$



(a)

$$V=\int_a^b \pi([R(x)]^2-[r(x)]^2)\,dx$$

$$=\int_{-2}^{1}\pi\left((-x+3)^2-(x^2+1)^2\right)dx$$



*Washer cross section*
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$
(b)

$$=\int_{-2}^{1}\pi(8-6x-x^2-x^4)\,dx$$

$$=\pi\left[8x-3x^2-\frac{x^3}{3}-\frac{x^5}{5}\right]_{-2}^{1}$$

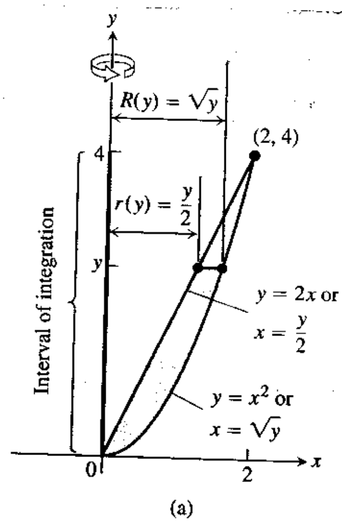
$$=\frac{117\pi}{5}$$

2. The region bounded by the parabola $y=x^2$ and the line $y=2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid?

solution:

$R(y)=\sqrt{y}\ ,\ y=x^2 \Rightarrow x=\sqrt{y}$

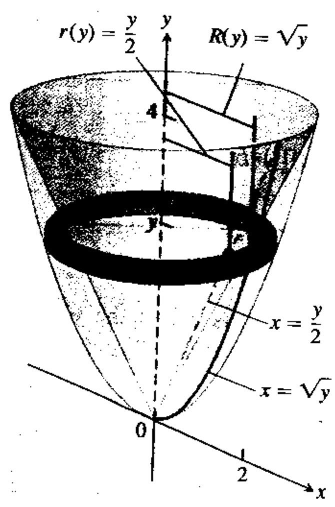
$r(y)=y/2\ ,\ y=2x \Rightarrow x=y/2$



(a)

$$x^2=2x \Rightarrow x^2-2x=0$$
$$x(x-2)=0\ ,\ x=0\ \&\ x=2$$
$$x=0 \Rightarrow y=2x \Rightarrow y=0$$
$$x=2 \Rightarrow y=2x \Rightarrow y=4$$

$$V=\int_c^d \pi([R(y)]^2-[r(y)]^2)\,dy$$




$$=\int_0^4 \pi\left([\sqrt{y}]^2-[y/2]^2\right)dy$$

$$=\pi\int_0^4\left(y-\frac{y^2}{4}\right)dy=\pi\left[\frac{y^2}{2}-\frac{y^3}{12}\right]_0^4$$

$$=\frac{8}{3}\pi.$$

# Volumes by Cylindrical Shells :

## The Shell Method :

The volume of the solid generated by revolving the region between the $x$-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$\boxed{V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx}$$

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region.

Examples :

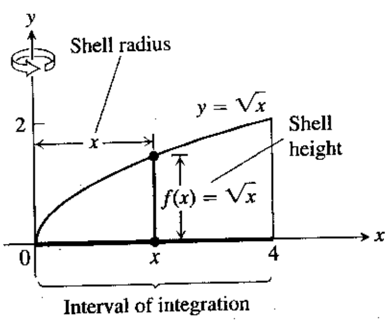
1. The region bounded by the curve $y = \sqrt{x}$, the $x$-axis, and the line $x = 4$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid ?

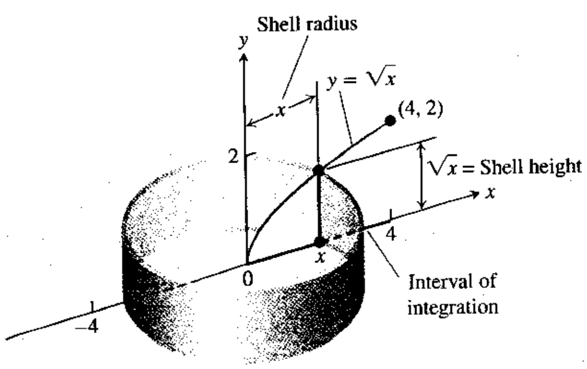
Solution :

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

$$V = \int_0^4 2\pi (x)(\sqrt{x})\, dx = 2\pi \int_0^4 x^{3/2}\, dx$$

$$V = 2\pi \left[ \frac{2}{5} x^{5/2} \right]_0^4 = \frac{128}{5}\pi .$$



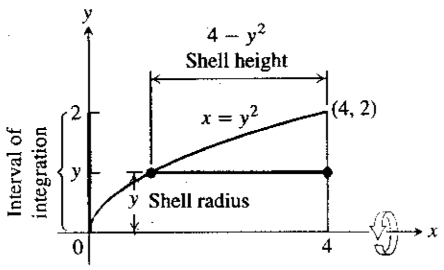
2. The region bounded by the curve $y = \sqrt{x}$, the $x$-axis, and the line $x = 4$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid?

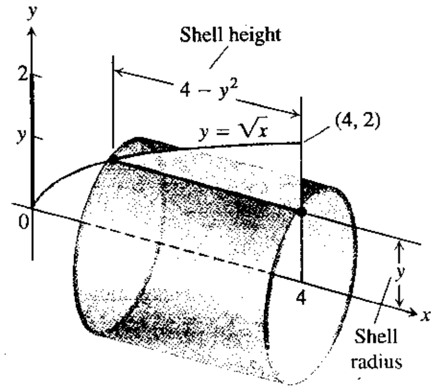
Solution:

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dy$$

$$V = \int_0^2 2\pi (y)(4 - y^2)\, dy = \int_0^2 2\pi (4y - y^3)\, dy$$

$$V = 2\pi \left[ 2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi$$




## Lengths of Plane Curves:

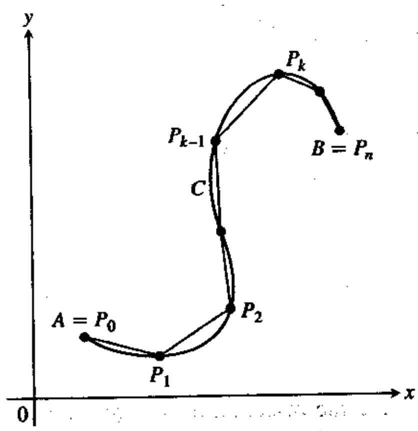
### Length of a Parametric Curve:

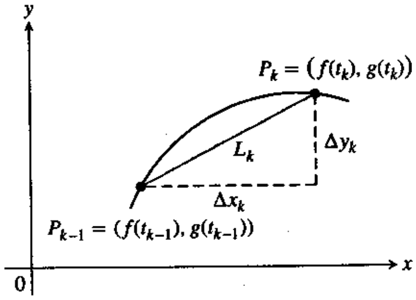
If a curve $C$ is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leqslant t \leqslant b$, where $f'$ and $g'$ are continuous and not simultaneously zero on $[a, b]$, and $C$ is traversed exactly once as $t$ increases from $t = a$ to $t = b$, then the length of $C$ is the definite integral:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

if $x = f(t)$ & $y = g(t)$

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$




Examples :

1. Find the length of the circle of radius $r$ defined parametrically by

$$x = r\cos t \quad \text{and} \quad y = r\sin t \quad , \quad 0 \leqslant t \leqslant 2\pi .$$

solution :

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$x = r\cos t \Rightarrow \frac{dx}{dt} = -r\sin t .$$

$$y = r\sin t \Rightarrow \frac{dy}{dt} = r\cos t .$$

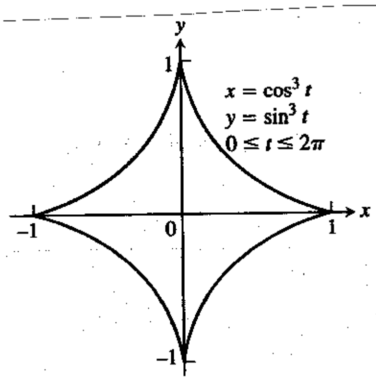
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2$$

$$\sin^2 t + \cos^2 t = 1$$

$$L = \int_0^{2\pi} \sqrt{r^2}\ dt = r\left[t\right]_0^{2\pi} = 2\pi r .$$

2. Find the length of the astroid ?

$$x = \cos^3 t \quad , \quad y = \sin^3 t \quad , \quad 0 \leqslant t \leqslant 2\pi$$



Solution: Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion.

$$x = \cos^3 t \;, \quad \left(\frac{dx}{dt}\right)^2 = \left[3\cos^2 t\,(-\sin t)\right]^2 = 9\cos^4 t \sin^2 t$$

$$y = \sin^3 t \;, \quad \left(\frac{dy}{dt}\right)^2 = \left[3\sin^2 t\,(\cos t)\right]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= \sqrt{9\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)} \;, \quad \cos^2 t + \sin^2 t = 1$$

$$= \sqrt{9\cos^2 t \sin^2 t} = 3\left|\cos t \sin t\right| \;, \quad \cos t \sin t \geqslant 0, \; 0 \leqslant t \leqslant \pi/2$$

$$= 3\cos t \sin t$$

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t \, dt$$

$$\sin 2t = 2\cos t \sin t \Rightarrow \cos t \sin t = \sin 2t \,/\, 2$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t \, dt$$

$$= -\frac{3}{2}\,\frac{\cos 2t}{2}\Bigg]_0^{\pi/2} = -\frac{3}{4}\cos 2t\Bigg]_0^{\pi/2} = \frac{3}{2}$$

The length of the astroid is four times this: $4\left(\frac{3}{2}\right) = 6$.

## Length of Curve $y = f(x)$:

If $f$ is continuously differentiable on the closed interval $[a,b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$\boxed{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Example:

Find the length of the curve ?

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \quad , \quad 0 \leqslant x \leqslant 1$$

Solution:

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \Rightarrow \frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2}\, x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}\, x^{1/2}\right)^2 = 8x$$

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$L = \int_0^1 (1+8x)^{1/2}\, dx = \frac{1}{8} \int_0^1 (1+8x)^{1/2} \cdot 8\, dx$$

$$L = \frac{1}{8} \frac{(1+8x)^{3/2}}{\frac{3}{2}} \Bigg]_0^1 = \frac{2}{3} \cdot \frac{1}{8} (1+8x)^{3/2} \Bigg]_0^1 = \frac{13}{6}$$

## Length of a Curve $x = g(y)$ : Dealing with Discontinuities in $dy/dx$

If $g$ is continuously differentiable on $[c,d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

$$\boxed{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

Example:

Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ ?

Solution:

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with $y = f(x)$. ( $\frac{dy}{dx}$ fails to exist )

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}} \Rightarrow y^{\frac{3}{2}} = \frac{x}{2} \Rightarrow x = 2y^{\frac{3}{2}}$$

$$\frac{dx}{dy} = 2 \cdot \frac{3}{2} y^{\frac{1}{2}} = 3y^{\frac{1}{2}}$$

$$x = 0 \Rightarrow y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y = \left(\frac{0}{2}\right)^{2/3} \Rightarrow y = 0$$

$$x = 2 \Rightarrow y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} \Rightarrow y = 1$$

$$0 \leqslant y \leqslant 1$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + (3y^{1/2})^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy$$

$$L = \frac{1}{9} \int_0^1 (1 + 9y)^{1/2} \cdot 9\, dy = \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big]_0^1$$

$$L = \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27$$

## Areas of Surfaces of Revolution :

### Surface Area for Revolution About the x-Axis :

If the function $f(x) \geqslant 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the curve $y = f(x)$ about the $x$-axis is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$$

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

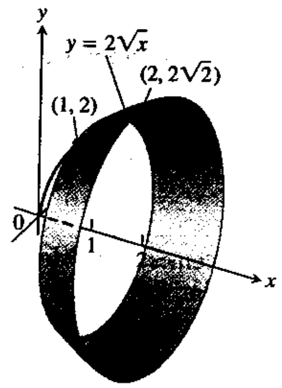
Example :

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$ , $1 \leqslant x \leqslant 2$ about the $x$-axis?

Solution :

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$a = 1 \quad , \quad b = 2 \quad , \quad y = 2\sqrt{x} \quad , \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$




$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1+\frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx$$

$$S = 4\pi\int_1^2 \sqrt{x+1}\,dx = 4\pi\int_1^2 (x+1)^{1/2}\,dx$$

$$S = 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}(3\sqrt{3}-2\sqrt{2}).$$

## Surface Area for Revolution About the y-Axis :

If $x = g(y) \geqslant 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

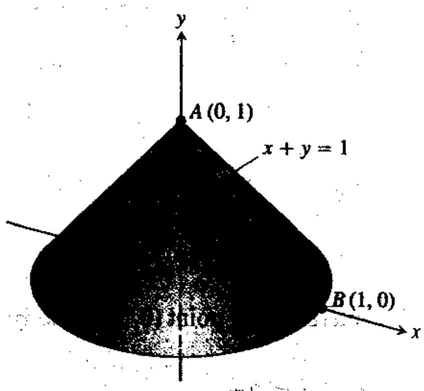
$$S = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$

$$\boxed{S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy}$$

Example :

The line segment $x = 1-y$ , $0 \leqslant y \leqslant 1$ , is revolved about the y-axis to generate the cone. Find its surface area ?

Solution :




$c = 0 \quad , \quad d = 1 \quad , \quad x = 1-y \quad , \quad \frac{dx}{dy} = -1$

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+(-1)^2} = \sqrt{2}$$

$$S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1-y)\sqrt{2}\, dy$$

$$S = 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1-\frac{1}{2}\right) = \pi\sqrt{2}.$$

## Surface Area of Revolution for Parametrized Curves:

If a smooth curve $x=f(t)$, $y=g(t)$, $a \leq t \leq b$, is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the $x$-axis ($y \geq 0$):

$$\boxed{S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

2. Revolution about the $y$-axis ($x \geq 0$):

$$\boxed{S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Example:

The standard parametrization of the circle of radius 1 centered at the point $(0,1)$ in the $xy$-plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi$$

Use this parametrization to find the area of the surface swept out by revolving the circle?

Solution:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

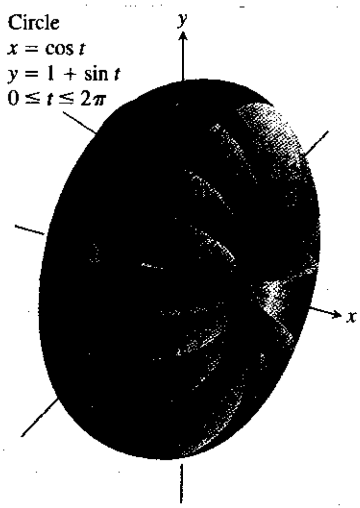
$a = 0$, $b = 2\pi$

$y = 1 + \sin t \Rightarrow \frac{dy}{dt} = \cos t$

$x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2+(\cos t)^2} = \sqrt{1} = 1$$

$$\sin^2 t + \cos^2 t = 1$$




$$S = \int_0^{2\pi} 2\pi(1+\sin t).1.dt$$

$$= 2\pi \int_0^{2\pi} (1+\sin t)dt$$

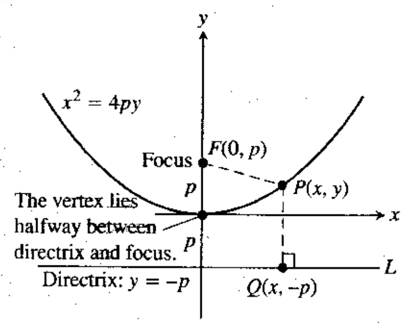
$$= 2\pi \Big[t - \cos t\Big]_0^{2\pi}$$

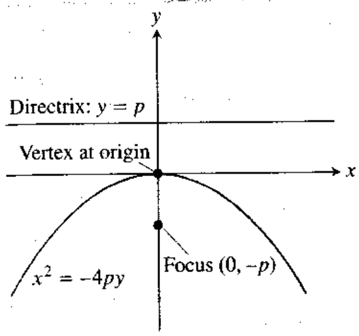
$$= 4\pi^2.$$

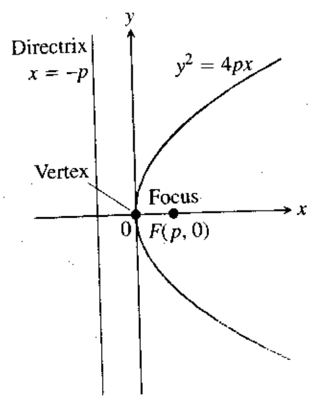
# Conic Sections and Quadratic Equations:

## Parabolas:

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a parabola. The fixed point is the focus of the parabola. The fixed line is the directrix.



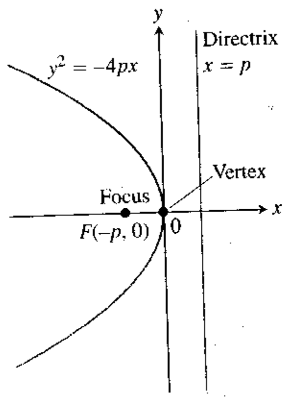




**TABLE** Standard-form equations for parabolas with vertices at the origin ($p > 0$)

| Equation | Focus | Directrix | Axis | Opens |
|---|---|---|---|---|
| $x^2 = 4py$ | $(0, p)$ | $y = -p$ | $y$-axis | Up |
| $x^2 = -4py$ | $(0, -p)$ | $y = p$ | $y$-axis | Down |
| $y^2 = 4px$ | $(p, 0)$ | $x = -p$ | $x$-axis | To the right |
| $y^2 = -4px$ | $(-p, 0)$ | $x = p$ | $x$-axis | To the left |

Example: Find the focus and directrix of the parabola $y^2 = 10x$?

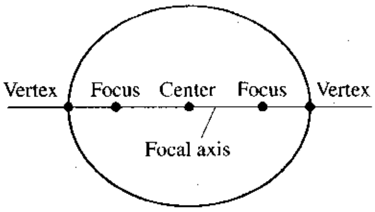
Solution: standard equation $y^2 = 4Px$ , $y^2 = 10x$

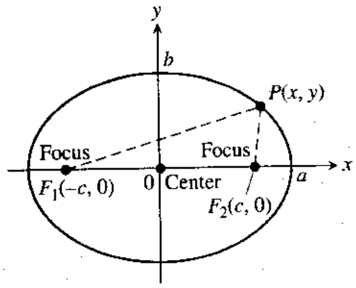
$4P = 10 \Rightarrow p = 5/2$

Focus: $(P, 0) = (5/2, 0)$, Directrix: $x = -p$ or $x = -5/2$.

# Ellipses :

An ellipse is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the foci of the ellipse. The line through the foci of an ellipse's focal axis. The point on the axis halfway between the foci is the center. The points where the focal axis and ellipse cross are the ellipse's vertices.




**Standard-Form Equations for Ellipses Centered at the Origin**

*Foci on the x-axis*: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

*Foci on the y-axis*: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

In each case, $a$ is the semimajor axis and $b$ is the semiminor axis.

Examples :

1. Major Axis Horizontal : The ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Semimajor axis : $a = \sqrt{16} = 4$

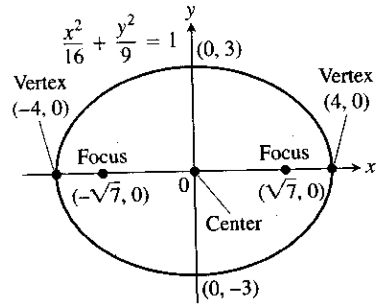
Seminimor axis : $b = \sqrt{9} = 3$

Center-to-focus distance : $c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$

Foci : $(\pm c, 0) = (\pm\sqrt{7}, 0)$

Vertices : $(\pm a, 0) = (\pm 4, 0)$

Center : $(0, 0)$




2. Major Axis Vertical : The ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Semimajor axis : $a = \sqrt{16} = 4$

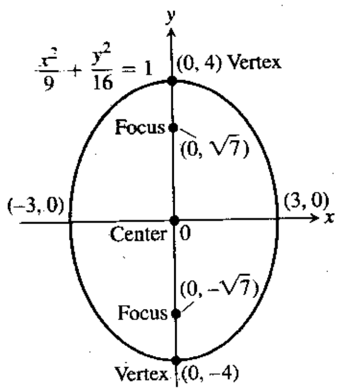
Semiminor axis : $b = \sqrt{9} = 3$

Center-to-focus distance : $c = \sqrt{16-9} = \sqrt{7}$

Foci : $(0, \pm c) = (0, \pm\sqrt{7})$

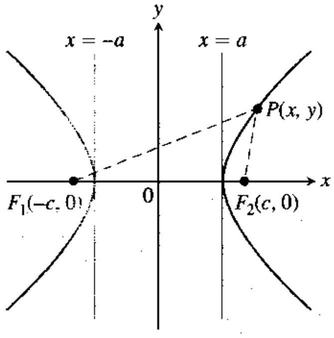
Vertices : $(0, \pm a) = (0, \pm 4)$

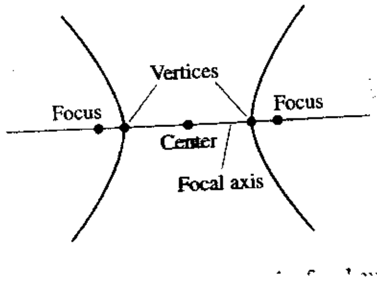
Center : $(0, 0)$




## Hyperbolas :

A hyperbola is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the foci of the hyperbola. The line through the foci of a hyperbola is the focal axis. The point on the axis halfway between the foci is the hyperbola's center. The points where the focal axis and hyperbola cross are the vertices.




**Standard-Form Equations for Hyperbolas Centered at the Origin**

*Foci on the x-axis:* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $y = \pm\frac{b}{a}x$

*Foci on the y-axis:* $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

Asymptotes: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0$ or $y = \pm\frac{a}{b}x$

Notice the difference in the asymptote equations ($b/a$ in the first, $a/b$ in the second).

Examples :

1. Foci on the $x$-axis : The equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

$a^2 = 4$ & $b^2 = 5$ ( $a = 2$ & $b = \sqrt{5}$ )

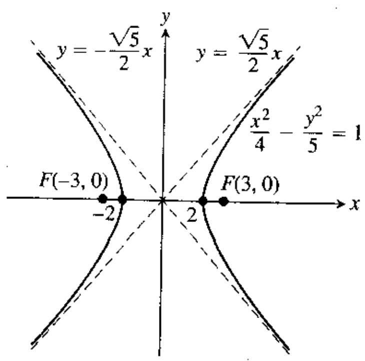
Center-to-focus distance : $c = \sqrt{a^2+b^2} = \sqrt{4+5} = 3$

Foci : $(\pm c, 0) = (\pm 3, 0)$

Vertices : $(\pm a, 0) = (\pm 2, 0)$

Center : $(0, 0)$

Asymptotes : $\frac{x^2}{4} - \frac{y^2}{5} = 0$ or $y = \pm\frac{\sqrt{5}}{2}x$



2. Foci on the y-axis: The equation

$$\frac{y^2}{4} - \frac{x^2}{5} = 1$$

$a^2 = 4$ & $b^2 = 5$ ( $a = 2$ & $b = \sqrt{5}$ )

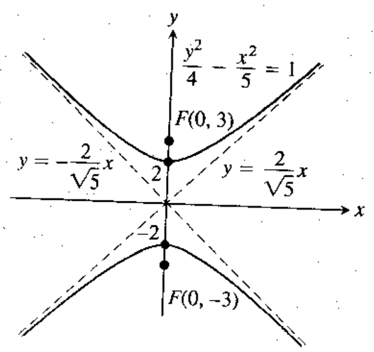
Center-to-focus distance : $c = \sqrt{a^2+b^2} = \sqrt{4+5} = 3$

Foci : $(0, \pm c) = (0, \pm 3)$

Vertices : $(0, \pm a) = (0, \pm 2)$

Center : $(0, 0)$

Asymptotes : $\frac{y^2}{4} - \frac{x^2}{5} = 0$ or $y = \pm \frac{2}{\sqrt{5}} x$ .




Eccentricity : الاختلاف المركزي (الشذوذ)

1. The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ is

$$\boxed{e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}} \qquad \boxed{e < 1}$$

2. The eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

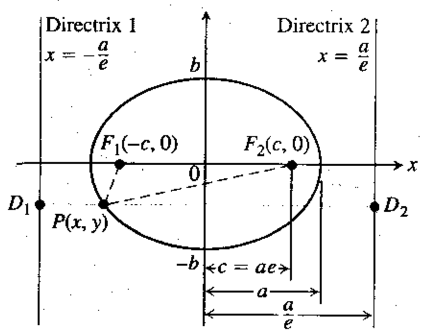
$$\boxed{e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}} \qquad \boxed{e > 1}$$

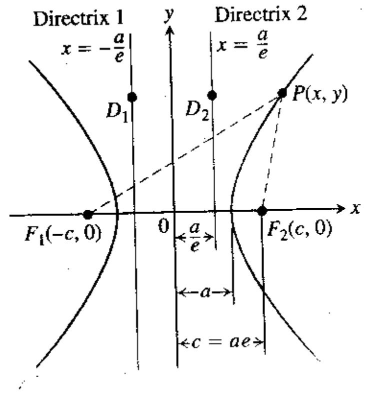
In both ellipse and hyperbola, the eccentricity is

$$\boxed{\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}}$$

3. The eccentricity of a parabola is

$$\boxed{e = 1}$$




Examples :

1. Locate the vertices of an ellipse of eccentricity 0.8 whose foci lie at the points $(0, \pm 7)$?

Solution :

$$e = c/a \quad , \quad a = c/e \Rightarrow a = 7/0.8 = 8.75$$

the vertices $(0, \pm a) = (0, \pm 8.75)$.

2. Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.

Solution :

$$9x^2 - 16y^2 = 144 \qquad \div 144$$

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1 \quad , \quad \frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$a^2 = 16 \;\&\; b^2 = 9 \quad (a = 4 \;\&\; b = 3)$$

$$c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$e = \frac{c}{a} = \frac{5}{4}.$$

Ex : Graph the following equations :

1. $x^2+y^2+4x-6y=12$

$$(x^2+4x+\quad)+(y^2-6y+\quad)=12$$

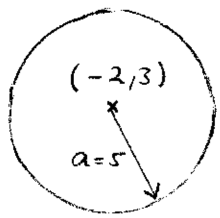
$$(x^2+4x+4)+(y^2-6y+9)=12+4+9=25$$

$$(x+2)^2+(y-3)^2=5^2$$

$$(x-h)^2+(y-k)^2=a^2$$

Circle $\Rightarrow$ center $(h,k)=(-2,3)$

radius $(a)=5$




2. $x^2-4x+3y^2+6y-2=0$

$$(x^2-4x+\quad)+3(y^2+2y+\quad)=2$$

$$(x^2-4x+4)+3(y^2+2y+1)=2+4+3=9$$

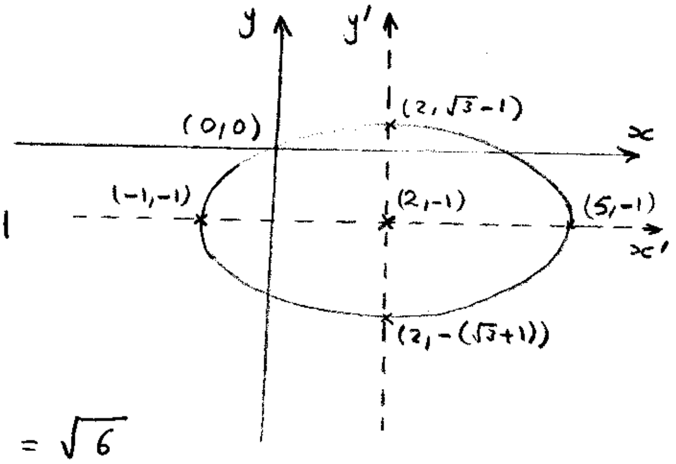
$$(x-2)^2+3(y+1)^2=9$$

$$\frac{(x-2)^2}{9}+\frac{(y+1)^2}{3}=1$$

$$x'=x-2\quad,\quad y'=y+1$$

$$\frac{x'^2}{3^2}+\frac{y'^2}{(\sqrt{3})^2}=1$$

$$a=3\quad,\quad b=\sqrt{3}\quad,\quad c=\sqrt{9-3}=\sqrt{6}$$




3. $4y^2+16y-9x^2+90x=245$

$$4(y^2+4y+\quad)-9(x^2-10x+\quad)=245$$

$$4(y^2+4y+4)-9(x^2-10x+25)=245+16-225$$

$$4(y+2)^2-9(x-5)^2=36$$

$$\frac{(y+2)^2}{9}-\frac{(x-5)^2}{4}=1$$

$$y'=y+2\quad,\quad x'=x-5$$

$$\frac{y'^2}{3^2}-\frac{x'^2}{2^2}=1$$

Asymptotes: $a = 3,\ b = 2$

$y + 2 = \pm \frac{3}{2}(x - 5)$, $\left(y = \pm \frac{b}{a}x\right)$

$2y_1 + 4 = 3x - 15 \Rightarrow 2y_1 = 3x - 19$

$y_1 = \frac{3}{2}x - \frac{19}{2}$

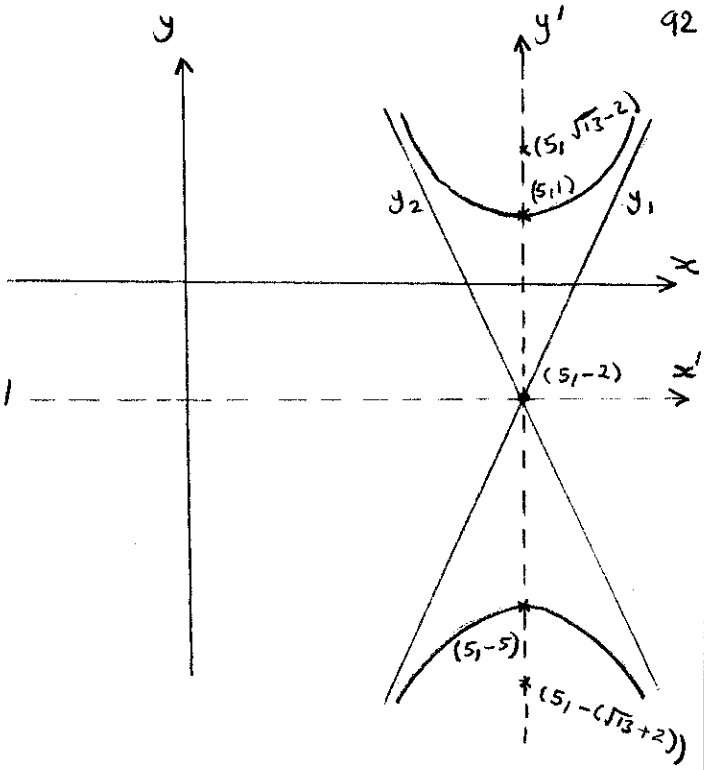
$2y_2 + 4 = -3x + 15 \Rightarrow 2y_2 = -3x + 11$

$y_2 = -\frac{3}{2}x + \frac{11}{2}$

$c = \sqrt{a^2 + b^2} = \sqrt{9+4} = \sqrt{13}$

Foci $= (0, \mp c)$

Vertices $= (0, \mp a)$




## Quadratic Equations and Rotations:

The Quadratic Equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The Cross Product Term:

$$Bxy$$

Equations for Rotating Coordinate Axes:

$$x = x'\cos\alpha - y'\sin\alpha$$
$$y = x'\sin\alpha + y'\cos\alpha$$

If we apply the rotation equations to the general quadratic equation, we obtain a new quadratic equation:

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

The new coefficients are related to the old ones by the equations:

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$
$$B' = B\cos 2\alpha + (C - A)\sin 2\alpha$$
$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$$
$$D' = D\cos\alpha + E\sin\alpha$$
$$E' = -D\sin\alpha + E\cos\alpha$$
$$F' = F.$$

Angle of Rotation:

$$\cot 2\alpha = \frac{A-C}{B} \quad , \quad \tan 2\alpha = \frac{B}{A-C}$$

Examples:

1. The x- and y-axes are rotated through an angle of $\pi/4$ radius about the origin. Find an equation for the hyperbola $2xy = 9$ in the new coordinates.

solution:

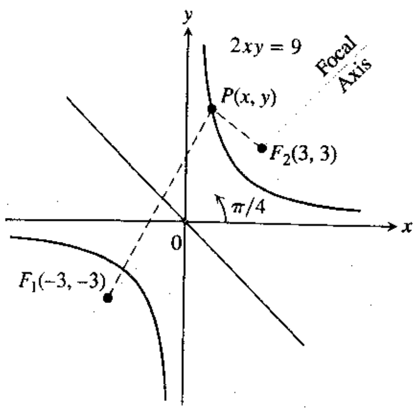
$$\cos(\pi/4) = 1/\sqrt{2} \quad , \quad \sin(\pi/4) = 1/\sqrt{2}$$

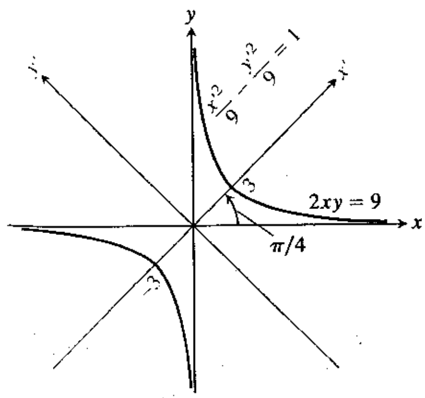
$$x = x'\cos\alpha - y'\cos\alpha \Rightarrow x = \frac{1}{\sqrt{2}}(x' - y').$$

$$y = x'\sin\alpha + y'\cos\alpha \Rightarrow y = \frac{1}{\sqrt{2}}(x' + y').$$

$$2xy = 9 \Rightarrow 2\left(\frac{1}{\sqrt{2}}(x'-y')\right)\left(\frac{1}{\sqrt{2}}(x'+y')\right) = 9$$

$$x'^2 - y'^2 = 9 \Rightarrow \frac{x'^2}{9} - \frac{y'^2}{9} = 1$$




2. The coordinate axes are to be rotated through an angle $\alpha$ to produce an equation for the curve

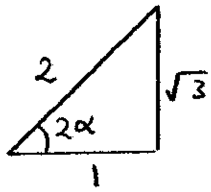
$$2x^2 + \sqrt{3}xy + y^2 - 10 = 0$$

that has no cross product term. Find $\alpha$ and the new equation. Identify the curve.

solution: $2x^2 + \sqrt{3}xy + y^2 - 10 = 0$

$A = 2$, $B = \sqrt{3}$, and $C = 1$

$D = 0$, $E = 0$, & $F = -10$




$$\cot 2\alpha = \frac{A-C}{B} = \frac{2-1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$2\alpha = \cot^{-1}(1/\sqrt{3}) = \frac{\pi}{3} \quad , \quad \alpha = \frac{\pi}{6}$$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$

$$A' = 2\cos^2\frac{\pi}{6} + \sqrt{3}\cos\frac{\pi}{6}\sin\frac{\pi}{6} + \sin^2\frac{\pi}{6} \Rightarrow A' = \frac{5}{2}$$

$$B' = B\cos 2\alpha + (C-A)\sin 2\alpha$$

$$B' = \sqrt{3}\cos\frac{\pi}{3} + (1-2)\sin\frac{\pi}{3} \Rightarrow B' = 0$$

$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$$

$$C' = 2\sin^2\frac{\pi}{6} - \sqrt{3}\sin\frac{\pi}{6}\cos\frac{\pi}{6} + \cos^2\frac{\pi}{6} \Rightarrow C' = \frac{1}{2}$$

$$D' = D\cos\alpha + E\sin\alpha$$

$$D' = 0\cos\frac{\pi}{6} + 0\sin\frac{\pi}{6} \Rightarrow D' = 0$$

$$E' = -D\sin\alpha + E\cos\alpha$$

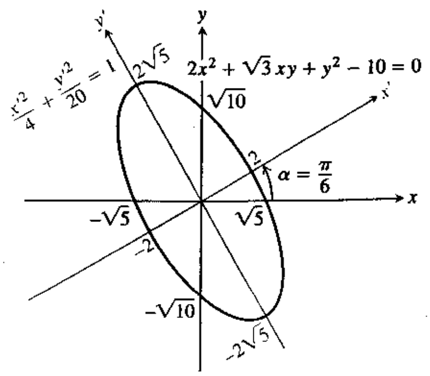
$$E' = 0\sin\frac{\pi}{6} + 0\cos\frac{\pi}{6} \Rightarrow E' = 0$$

$$F' = F \Rightarrow F' = -10$$

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

$$\frac{5}{2}x'^2 + \frac{1}{2}y'^2 - 10 = 0 \Rightarrow \frac{5}{2}x'^2 + \frac{1}{2}y'^2 = 10$$

$$\frac{x'^2}{4} + \frac{y'^2}{20} = 1 \Rightarrow \text{ellipse}$$




3. The coordinate axes are to be rotated through an angle $\alpha$ to produce an equation for the curve

$$x^2 + xy + y^2 - 6 = 0$$

that has no cross product term. Find $\alpha$ and the new equation. Identify the curve.

solution :

$A=1,\ B=1,\ C=1,\ D=0,\ E=0$, and $F=-6$.

$$\cot 2\alpha = \frac{A-C}{B} = \frac{1-1}{1} = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$

$$A' = 1\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right) + 1\left(\tfrac{1}{2}\right) \Rightarrow A' = \frac{3}{2}$$

$$B' = B\cos 2\alpha + (C-A)\sin 2\alpha$$

$$B' = 1(0) + (1-1)(1) \Rightarrow B' = 0$$

$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$$

$$C' = 1\left(\tfrac{1}{2}\right) - 1\left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{\sqrt{2}}\right) + 1\left(\tfrac{1}{2}\right) \Rightarrow C' = \frac{1}{2}$$

$$D' = D\cos\alpha + E\sin\alpha \Rightarrow D' = (0)\left(\tfrac{1}{\sqrt{2}}\right) + (0)\left(\tfrac{1}{\sqrt{2}}\right) \Rightarrow D' = 0$$

$$E' = -D\sin\alpha + E\cos\alpha \Rightarrow E' = -(0)\left(\tfrac{1}{\sqrt{2}}\right) + (0)\left(\tfrac{1}{\sqrt{2}}\right) \Rightarrow E' = 0$$

$$F' = F \Rightarrow F' = -6$$

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$$

$$\frac{3}{2}x'^2 + \frac{1}{2}y'^2 - 6 = 0 \Rightarrow \frac{x'^2}{4} + \frac{y'^2}{12} = 1 \Rightarrow \text{ellipse}$$

Another solution :

$$\alpha = \pi/4$$

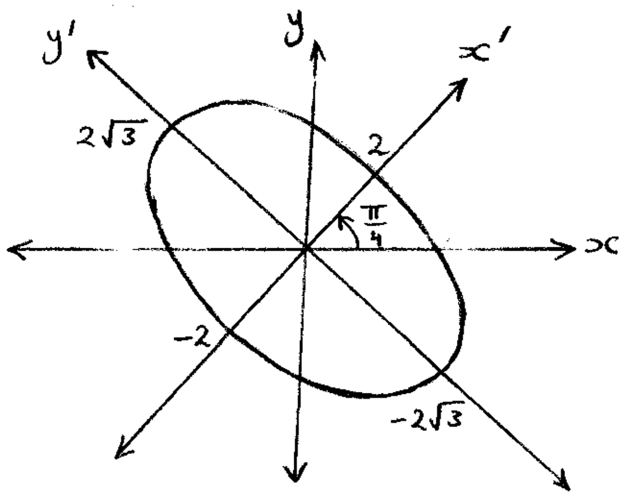
$$x = x'\cos\alpha - y'\sin\alpha \Rightarrow x = \frac{x'-y'}{\sqrt{2}}$$

$$y = x'\sin\alpha + y'\cos\alpha \Rightarrow y = \frac{x'+y'}{\sqrt{2}}$$

$$x^2 + xy + y^2 - 6 = 0$$

$$\frac{(x'-y')^2}{2} + \frac{(x'-y')(x'+y')}{2} + \frac{(x'+y')^2}{2} - 6 = 0$$

$$\frac{x'^2}{4} + \frac{y'^2}{12} = 1$$



## The Discriminant Test :

The quadratic curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The number $(B^2 - 4AC)$ is called the discriminant.

a) a parabola if $B^2 - 4AC = 0$

b) an ellipse if $B^2 - 4AC < 0$

c) a hyperbola if $B^2 - 4AC > 0$

Examples :

1. $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$

   represents a parabola because

   $B^2 - 4AC = (-6)^2 - 4.3.3 = 36 - 36 = 0$

2. $x^2 + xy + y^2 - 1 = 0$

   represents an ellipse because

   $B^2 - 4AC = (1)^2 - 4.1.1 = -3 < 0$

3. $xy - y^2 - 5y + 1 = 0$

   represents a hyperbola because

   $B^2 - 4AC = (1)^2 - 4(0)(-1) = 1 > 0$

# Polar Coordinates:

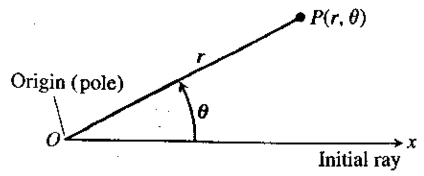
## Definition of Polar Coordinates:

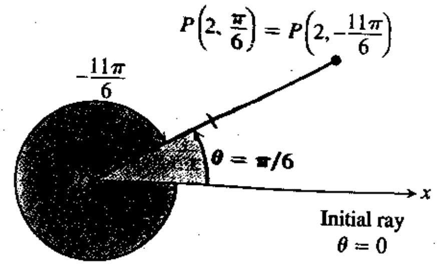
To define polar coordinates, we first fix an origin $O$ (called the pole) and an initial ray from $O$. Then each point $P$ can be located by assigning to it a polar coordinate pair $(r, \theta)$ in which $r$ gives the directed distance from $O$ to $P$ and $\theta$ gives the directed angle from the initial ray to ray $OP$.

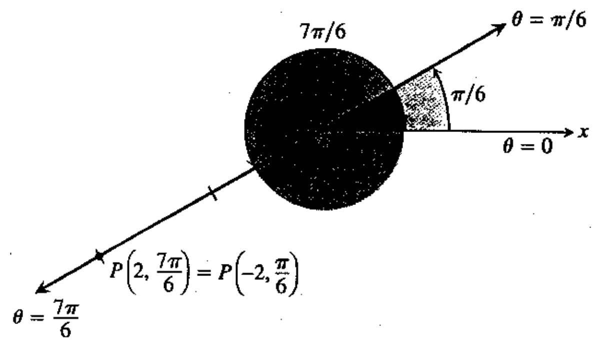
As in trigonometry, $\theta$ is positive when measured counter-clockwise and negative when measured clockwise.

$$P(r, \theta)$$

Directed distance from $O$ to $P$. — Directed angle from initial ray to $OP$.




Example: Find all the polar coordinates of the point $P(2, \pi/6)$

Solution:

For $r = 2$

$$\frac{\pi}{6},\ \frac{\pi}{6} \pm 2\pi,\ \frac{\pi}{6} \pm 4\pi,\ \frac{\pi}{6} \pm 6\pi,\ \ldots$$

For $r=-2$ $\quad \left(\frac{\pi}{6}-\pi = \frac{\pi-6\pi}{6} = \frac{-5\pi}{6}\right)$

$$-\frac{5\pi}{6},\ -\frac{5\pi}{6}\pm 2\pi,\ -\frac{5\pi}{6}\pm 4\pi,\ -\frac{5\pi}{6}\pm 6\pi,\ \ldots$$

The corresponding coordinate pairs of $P$ are

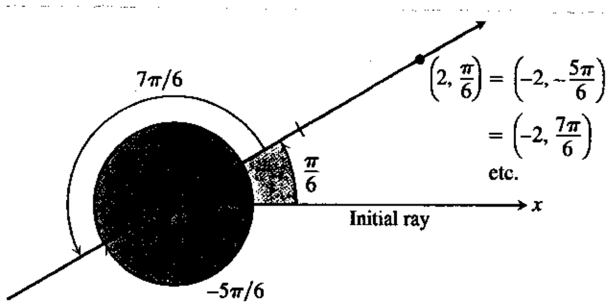
$$\left(2, \frac{\pi}{6}+2n\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

and

$$\left(-2, -\frac{5\pi}{6}+2n\pi\right), \quad n=0,\pm1,\pm2,\ldots$$

$n=0 \Rightarrow \left(2,\frac{\pi}{6}\right)$ & $\left(-2,-\frac{5\pi}{6}\right)$.

$n=1 \Rightarrow \left(2,\frac{13\pi}{6}\right)$ & $\left(-2,\frac{7\pi}{6}\right)$.




## Polar Equations and Graphs:

| Equation | Graph |
|---|---|
| $r=a$ | Circle radius $\lvert a\rvert$ centered at $O$. |
| $\theta=\theta_0$ | Line through $O$ making an angle $\theta_0$ with the initial ray. |

Example: Graph the sets of points whose polar coordinates satisfy the following conditions

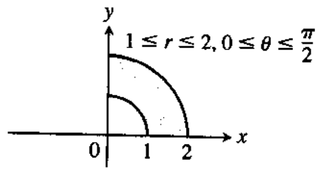
a) $1\leqslant r\leqslant 2$ & $0\leqslant\theta\leqslant\frac{\pi}{2}$

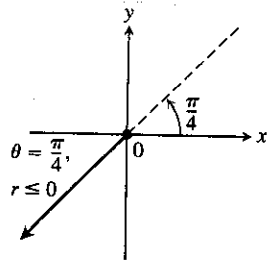
b) $-3\leqslant r\leqslant 2$ & $\theta=\frac{\pi}{4}$

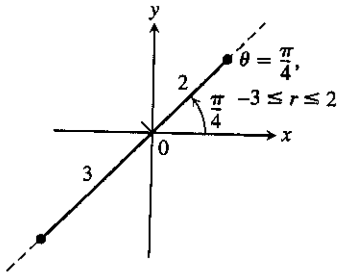
c) $r\leqslant 0$ & $\theta=\frac{\pi}{4}$

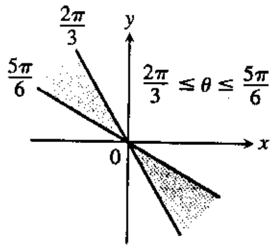
d) $\frac{2\pi}{3}\leqslant\theta\leqslant\frac{5\pi}{6}$ (no restriction on $r$).

Solution:



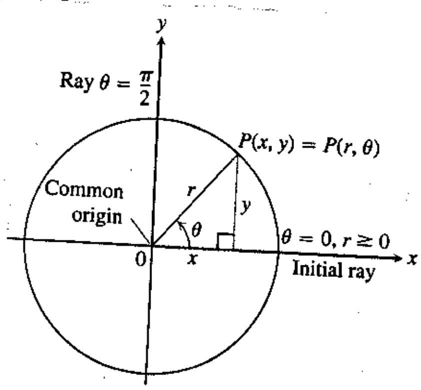




## Relating Polar and Cartesian Coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$




| Polar equation | Cartesian equivalent |
|---|---|
| $r\cos\theta = 2$ | $x = 2$ |
| $r^2\cos\theta\sin\theta = 4$ | $xy = 4$ |
| $r^2\cos^2\theta - r^2\sin^2\theta = 1$ | $x^2 - y^2 = 1$ |
| $r = 1 + 2r\cos\theta$ | $y^2 - 3x^2 - 4x - 1 = 0$ |
| $r = 1 - \cos\theta$ | $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$ |

* $r = 1 + 2r\cos\theta \Rightarrow r^2 = (1 + 2r\cos\theta)^2 \Rightarrow x^2 + y^2 = (1 + 2x)^2$
$x^2 + y^2 = 1 + 4x + 4x^2 \Rightarrow x^2 + y^2 - 4x^2 - 4x - 1 = 0 \Rightarrow$
$y^2 - 3x^2 - 4x - 1 = 0$.

* $r = 1 - \cos\theta \Rightarrow r^2 = r - r\cos\theta \Rightarrow x^2 + y^2 = \sqrt{x^2 + y^2} - x$
$(x^2 + y^2) + x = \sqrt{x^2 + y^2} \Rightarrow ((x^2 + y^2) + x)^2 = (\sqrt{x^2 + y^2})^2$
$(x^2 + y^2)^2 + 2(x^2 + y^2)x + \cancel{x^2} = \cancel{x^2} + y^2$
$x^4 + 2x^2y^2 + y^4 + 2x^3 + 2y^2x - y^2 = 0$

Example: Find a polar equation for the circle $x^2+(y-3)^2=9$

Solution:

$$x^2+(y-3)^2=9 \Rightarrow x^2+y^2-6y+9=9$$
$$x^2+y^2-6y=0 \Rightarrow r^2-6r\sin\theta=0$$
$$r=0 \quad \text{or} \quad r-6\sin\theta=0 \Rightarrow r=6\sin\theta.$$

Example: Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

a) $r\cos\theta=-4$

$$r\cos\theta=x \Rightarrow x=-4$$

The graph: Vertical line through $x=-4$ on the $x$-axis.

b) $r^2=4r\cos\theta$

$$x^2+y^2=4x \Rightarrow x^2-4x+y^2=0 \Rightarrow x^2-4x+4+y^2=4$$
$$(x-2)^2+y^2=4$$

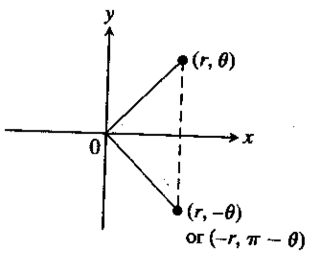
The graph: Circle, radius 2, center $(h,k)=(2,0)$.

c) $r=\dfrac{4}{2\cos\theta-\sin\theta}$

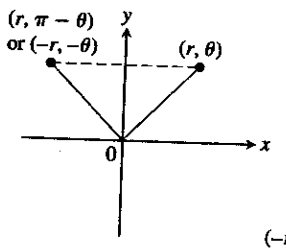
$$r(2\cos\theta-\sin\theta)=4 \Rightarrow 2r\cos\theta-r\sin\theta=4$$
$$2x-y=4 \Rightarrow y=2x-4$$

The graph: Line, slope $m=2$, $y$-intercept $b=-4$.

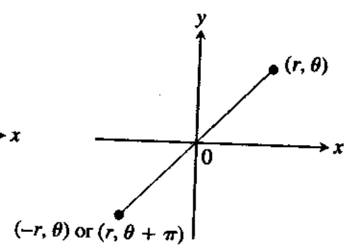
## Graphing in Polar Coordinates:



(a) About the x-axis



(b) About the y-axis



(c) About the origin

**Symmetry Tests for Polar Graphs**

1. *Symmetry about the x-axis:* If the point $(r, \theta)$ lies on the graph, the point $(r, -\theta)$ or $(-r, \pi-\theta)$ lies on the graph (Figure 10.43a).
2. *Symmetry about the y-axis:* If the point $(r, \theta)$ lies on the graph, the point $(r, \pi-\theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
3. *Symmetry about the origin:* If the point $(r, \theta)$ lies on the graph, the point $(-r, \theta)$ or $(r, \theta+\pi)$ lies on the graph (Figure 10.43c).

## Slope :

The slope of polar curve $r = f(\theta)$ is given by $dy/dx$, not by $r' = df/d\theta$.

$x = r\cos\theta = f(\theta)\cos\theta$ , $y = r\sin\theta = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(f(\theta)\cdot\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cdot\cos\theta)} = \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

$$\boxed{\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}} \quad \frac{dx}{d\theta} \neq 0 \text{ at } (r,\theta).$$

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$ and the slope equation gives :

$$\boxed{\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0}$$

Examples :

1. Graph the curve $r = 1 - \cos\theta$

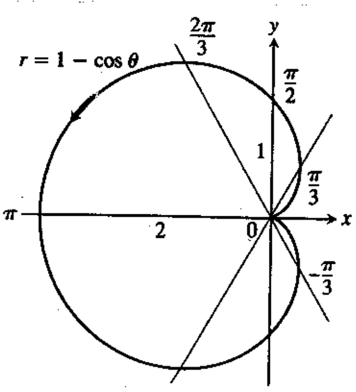
solution: The curve is symmetric about the $x$-axis because

$(r,\theta)$ on the graph $\Rightarrow r = 1 - \cos\theta$

$\Rightarrow r = 1 - \cos(-\theta)$ , $\cos\theta = \cos(-\theta)$

$\Rightarrow (r,-\theta)$ on the graph.

| $\theta$ | 0 | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ | $4\pi/3$ | $3\pi/2$ | $5\pi/3$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $r = 1 - \cos\theta$ | 0 | 1/2 | 1 | 3/2 | 2 | 3/2 | 1 | 1/2 | 0 |



Notes:

a. The cosine has period $2\pi$.

b. The curve leaves the origin with slope $\tan(0)=0$ and returns to the origin with slope $\tan(2\pi)=0$.

c. The curve is called a cardioid because of its heart shape.

2. Graph the curve $r^2 = 4\cos\theta$

solution:

The equation $r^2=4\cos\theta$ requires $\cos\theta \geqslant 0$, so we get the entire graph by running $\theta$ from $-\pi/2$ to $\pi/2$.

The curve is symmetric about the $x$-axis because

$$\begin{aligned}(r,\theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \quad , \cos\theta = \cos(-\theta) \\ &\Rightarrow (r,-\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

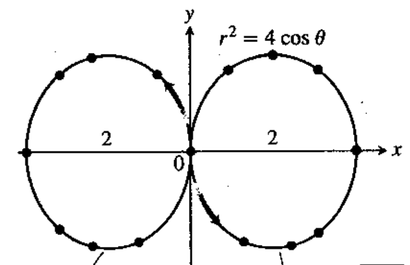
$$\begin{aligned}(r,\theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r,\theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the $y$-axis.

The curve passes through the origin when $\theta=-\pi/2$ and $\theta=\pi/2$.

It has a vertical tangent times because $\tan\theta$ is infinite.

| $\theta$ | $\cos\theta$ | $r = \pm 2\sqrt{\cos\theta}$ |
|---|---|---|
| 0 | 1 | $\pm 2$ |
| $\pm\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\approx \pm 1.9$ |
| $\pm\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\approx \pm 1.7$ |
| $\pm\frac{\pi}{3}$ | $\frac{1}{2}$ | $\approx \pm 1.4$ |
| $\pm\frac{\pi}{2}$ | 0 | 0 |



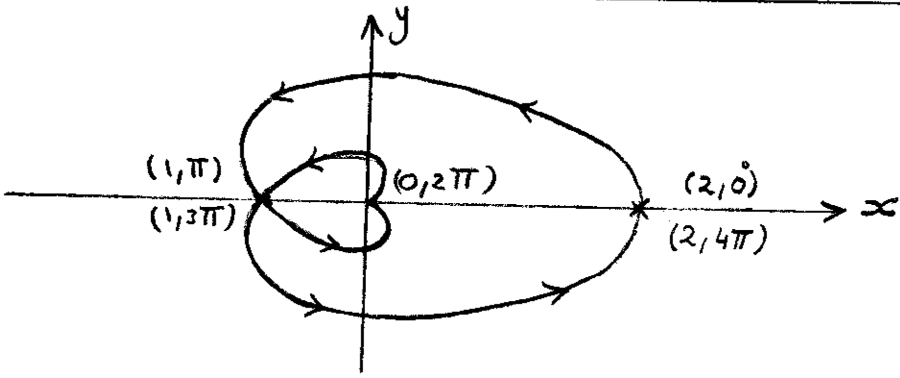

Loop for $r = -2\sqrt{\cos\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ Loop for $r = 2\sqrt{\cos\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

3. Graph the curve $r = 1 + \cos\frac{\theta}{2}$

Solution: Since the cosine has period $2\pi$, we must let $\theta$ run from $0$ to $4\pi$ to produce the entire graph.

The curve is symmetric about the x-axis.

| $\theta$ | $0$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ | $4\pi/3$ | $3\pi/2$ | $5\pi/4$ | $2\pi$ | $5\pi/2$ | $3\pi$ | $7\pi/2$ | $4\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1+\cos\frac{\theta}{2}$ | 2 | 1.86 | 1.7 | 1.5 | 1 | 0.5 | 0.3 | 0.13 | 0 | 0.3 | 1 | 1.7 | 2 |




4. Graph the curve $r^2 = \sin 2\theta$

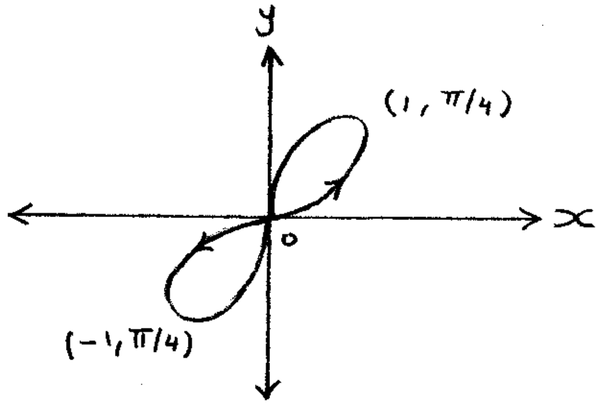
Solution: The curve is symmetric about the origin

$r = \pm\sqrt{\sin 2\theta}$

$\theta$ from $0$ to $\pi/2$ & from $\pi$ to $3\pi/2$

the equation $r^2 = \sin 2\theta$ requires $\sin 2\theta \geqslant 0$

| $\theta$ | $0$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $\pi$ | $5\pi/4$ | $4\pi/3$ | $3\pi/2$ |
|---|---|---|---|---|---|---|---|---|
| $r = \pm\sqrt{\sin 2\theta}$ | 0 | ±1 | ±0.9 | 0 | 0 | ±1 | ±0.9 | 0 |

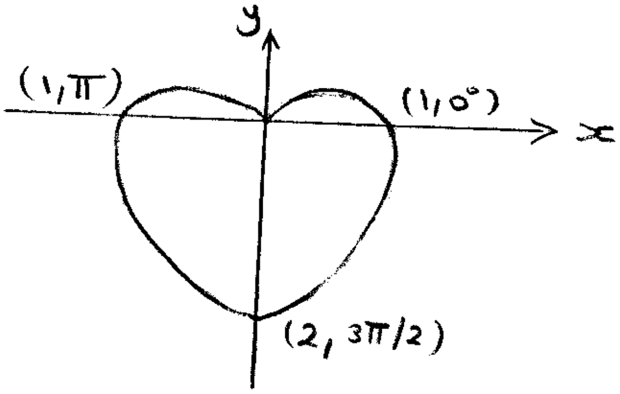



5. Graph the curve $r = 1 - \sin\theta$

Solution: The curve is symmetric about the y-axis.

$\sin\theta = \sin(-\theta)$

| $\theta$ | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $5\pi/6$ | $\pi$ | $4\pi/3$ | $3\pi/2$ | $5\pi/3$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r = 1 - \sin\theta$ | 1 | 0.5 | 0.3 | 0.13 | 0 | 0.13 | 0.5 | 1 | 1.86 | 2 | 1.86 | 1 |




6. Show that the point $(2, \pi/2)$ lies on the curve $r = 2\cos 2\theta$.

Solution: $r = 2\cos 2\theta$ , $(2, \pi/2)$

$$2 = 2\cos 2\left(\frac{\pi}{2}\right) = 2\cos\pi = -2$$

$$2 \neq -2$$

The point $(2, \pi/2)$ does not lie on the curve. The magnitude is right, but the sign is wrong.

7. Find the points of intersection of the curves

$$r^2 = 4\cos\theta \quad \& \quad r = 1 - \cos\theta.$$

solution:

$$r = 1 - \cos\theta \quad \& \quad \cos\theta = r^2/4$$

$$r = 1 - \cos\theta = 1 - \frac{r^2}{4} \Rightarrow 4r = 4 - r^2$$

$$r^2 + 4r - 4 = 0 \Rightarrow r = -2 \pm 2\sqrt{2}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The value of $\theta$ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\theta = \cos^{-1}(1 - r) \Rightarrow \theta = \cos^{-1}(1 - (2\sqrt{2} - 2))$$

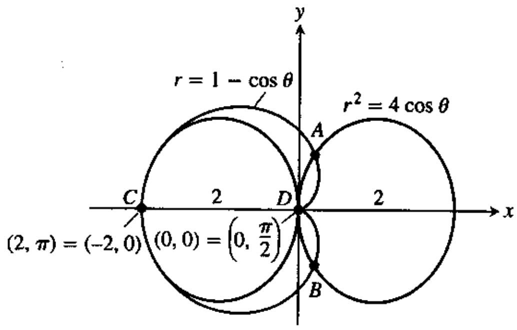
$$\theta = \cos^{-1}(3 - 2\sqrt{2}) \Rightarrow \theta = \pm 80°.$$

We have thus identified two intersection points:

$$(r, \theta) = (2\sqrt{2} - 2, \pm 80).$$

* If we graph the equations $r^2 = 4\cos\theta$ and $r = 1-\cos\theta$ together, we see that the curves also intersect at the point $(2, \pi)$ and the origin.

- On the curve $r = 1-\cos\theta$, the point $(2,\pi)$ is reached when $\theta = \pi$.
- On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, by the coordinates $(-2, 0)$.
- The curve $r = 1-\cos\theta$ reaches the origin when $\theta = 0$, $(0,0)$.
- The curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$, $(0, \pi/2)$.




The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$
Only $A$ and $B$ were found by simultaneous solution.
The other two were disclosed by graphing.

# Determinates :

A rectangular array of numbers like

$$A = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{vmatrix}$$

is called a matrix of order 2 by 3 because it has 2 rows and 3 columns.

The matrix $A$ has

$a_{11} = 2$ , $a_{12} = 1$ , $a_{13} = 3$

$a_{21} = 1$ , $a_{22} = 0$ , $a_{23} = -2$

**Note**: A matrix with the same number of rows and columns is called a square matrix.

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$ matrix of order 2 (square matrix).

$$\det(A) = \det \begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{array} = a_{11}a_{22} - a_{21}a_{12}$$

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ matrix of order 3 (square matrix).

$$\det(A) = \det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array}$$

$$= a_{11} \begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{array} - a_{12} \begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{array} + a_{13} \begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{array}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Example: Evaluate the determinate of the matrix

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} \end{array}$$

Solution:

$$\det(A) = 2\begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 2(-1+6) - 1(3+4) + 3(9+2)$$
$$= 10 - 7 + 33$$
$$= 36 .$$

## Properties of Determinates:

1. The value of the determinate is unchanged if all the corresponding rows and columns are interchanged.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \quad (\text{Transport matrix}).$$

Ex:

$$A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Transport matrix $A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix}$

$$\det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 1 & 2 \end{vmatrix} \end{array} = 1\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 5 .$$

$$\det \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix} \end{array} = 1\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 5.$$

2. If all the elements of a row or a column of a determinate are multiplied by the same number k, then the value of the determinate is multiplied by k.

$$\begin{vmatrix} 3 & 9 & 5 \\ 4 & 6 & 0 \\ -1 & -3 & 0 \end{vmatrix} = 3 \begin{vmatrix} 3 & 3 & 5 \\ 4 & 2 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 3 \cdot 2 \begin{vmatrix} 3 & 3 & 5 \\ 2 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix}$$

3. If any two rows or any two columns of a determinate are interchanged then the sign of the value of the determinate is changed.

Ex:

$$A = \begin{vmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\det(A) = 2 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$\therefore \det(A) = 36$ .

$$\det(B) = 1 \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} -1 & -2 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}$$

$\therefore \det(B) = -36$ .

4. The determinate is zero if

a. Two rows or columns of a matrix is identical.

Ex.:

$$A = \begin{vmatrix} 2 & 2 & 1 \\ 3 & 3 & 4 \\ 1 & 1 & 3 \end{vmatrix}$$

$$\det(A) = 2 \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} = 0$$

b. Every element in any row or any column is zero.

Ex:

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 7 & 2 \end{vmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 0 & 0 \\ 7 & 2 \end{vmatrix} - 2 \begin{vmatrix} 0 & 0 \\ 5 & 2 \end{vmatrix} + 3 \begin{vmatrix} 0 & 0 \\ 5 & 7 \end{vmatrix} = 0$$

Ex:

$$A = \begin{vmatrix} 1 & 0 & 3 \\ 6 & 0 & 2 \\ 3 & 0 & 5 \end{vmatrix}$$

$$\det(A) = 1 \begin{vmatrix} 0 & 2 \\ 0 & 5 \end{vmatrix} - 0 \begin{vmatrix} 6 & 2 \\ 3 & 5 \end{vmatrix} + 3 \begin{vmatrix} 6 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

## Cramer's Rule:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

$$\det(A) = \det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

if $D \neq 0$

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

Example: Solve the system

$$3x - y = 9$$
$$x + 2y = -4$$

Solution:

$$D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 6 + 1 = 7$$

$$x = \frac{\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix}}{7} = \frac{18-4}{7} = \frac{14}{7} = 2 .$$

$$y = \frac{\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix}}{7} = \frac{-12-9}{7} = \frac{-21}{7} = -3 .$$

Systems of three equations in three unknowns work :

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$x = \frac{1}{D}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$y = \frac{1}{D}\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$z = \frac{1}{D}\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Example: Solve the equations by using Cramer's rule.

$$2x_1 + 4x_2 + 2x_3 = 16$$
$$2x_1 - x_2 - 2x_3 = -6$$
$$4x_1 + x_2 - 2x_3 = 0$$

Solution:

$$D = \begin{vmatrix} 2 & 4 & 2 \\ 2 & -1 & -2 \\ 4 & 1 & -2 \end{vmatrix}$$

$$D = 2\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 2 & -2 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} \Rightarrow D = 4$$

$$x_1 = \frac{1}{4}\begin{vmatrix} 16 & 4 & 2 \\ -6 & -1 & -2 \\ 0 & 1 & -2 \end{vmatrix}$$

$$x_1 = \frac{1}{4}\left[16\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} -6 & -2 \\ 0 & -2 \end{vmatrix} + 2\begin{vmatrix} -6 & -1 \\ 0 & 1 \end{vmatrix}\right]$$

$$x_1 = 1$$

$$x_2 = \frac{1}{4}\begin{vmatrix} 2 & 16 & 2 \\ 2 & -6 & -2 \\ 4 & 0 & -2 \end{vmatrix}$$

$$x_2 = \frac{1}{4}\left[2\begin{vmatrix} -6 & -2 \\ 0 & -2 \end{vmatrix} - 16\begin{vmatrix} 2 & -2 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 4 & 0 \end{vmatrix}\right]$$

$$x_2 = 2$$

$$x_3 = \frac{1}{4}\begin{vmatrix} 2 & 4 & 16 \\ 2 & -1 & -6 \\ 4 & 1 & 0 \end{vmatrix}$$

$$x_3 = \frac{1}{4}\left[2\begin{vmatrix} -1 & -6 \\ 1 & 0 \end{vmatrix} - 4\begin{vmatrix} 2 & -6 \\ 4 & 0 \end{vmatrix} + 16\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}\right]$$

$$x_3 = 3$$

<u>Note</u> :

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Examples :

1.

$$\begin{vmatrix} 8 & 0 & 2 & 0 \\ 5 & 1 & -3 & 0 \\ -4 & 3 & 7 & -3 \\ 4 & 0 & 6 & 0 \end{vmatrix} = -(-3)\begin{vmatrix} 8 & 0 & 2 \\ 5 & 1 & -3 \\ 4 & 0 & 6 \end{vmatrix} = 3(1)\begin{vmatrix} 8 & 2 \\ 4 & 6 \end{vmatrix}$$

$$= 3(2)(2)\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 12(2)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 24(6-1) = 120$$

2.

$$\begin{vmatrix} 1 & 4 & 3 & 1 \\ 2 & 8 & 2 & 5 \\ 4 & -4 & -1 & -3 \\ 2 & 5 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 4+(-4)(1) & 3+(-3)(1) & 1+(-1)(1) \\ 2 & 8+(-4)(2) & 2+(-3)(2) & 5+(-1)(2) \\ 4 & -4+(-4)(4) & -1+(-3)(4) & -3+(-1)(4) \\ 2 & 5+(-4)(2) & 3+(-3)(2) & 3+(-1)(2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -4 & 3 \\ 4 & -20 & -13 & -7 \\ 2 & -3 & -3 & 1 \end{vmatrix} = (1)\begin{vmatrix} 0 & -4 & 3 \\ -20 & -13 & -7 \\ -3 & -3 & 1 \end{vmatrix}$$

$$= -(-4)\begin{vmatrix} -20 & -7 \\ -3 & 1 \end{vmatrix} + 3\begin{vmatrix} -20 & -13 \\ -3 & -3 \end{vmatrix} =$$

3.

$$\begin{vmatrix} 50 & 2 & -9 \\ 250 & -10 & 45 \\ -150 & 6 & 27 \end{vmatrix} = (50)(2)(9)\begin{vmatrix} 1 & 1 & -1 \\ 5 & -5 & 5 \\ -3 & 3 & 3 \end{vmatrix}$$

$$900(5)(3)\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 13500\begin{vmatrix} 1 & 1 & -1+(1)(1) \\ 1 & -1 & 1+(1)(1) \\ -1 & 1 & 1+(1)(-1) \end{vmatrix}$$

$$13500\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 13500(-2)\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -27000(1+1) = -54000.$$

# Partial Derivatives:

The calculus of several variables is basically single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a partial derivative.

## Partial Derivative with Respect to $x$:

The partial derivative of $f(x,y)$ with respect to $x$ at the point $(x_0, y_0)$ is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exists.

## Partial Derivative with Respect to $y$:

The partial derivative of $f(x,y)$ with respect to $y$ at the point $(x_0, y_0)$ is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

provided the limit exists.

$$\frac{\partial f}{\partial x} = f_x \quad , \quad \frac{\partial f}{\partial y} = f_y \quad , \quad \frac{\partial z}{\partial x} = z_x$$

$$\underbrace{\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0)} \quad , \quad \underbrace{\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0)}$$

Partial derivative of $f$ with respect to $x$ at $(x_0, y_0)$ or $f$ sub $x$ at $(x_0, y_0)$.

Partial derivative of $f$ with respect to $y$ at $(x_0, y_0)$ or $f$ sub $y$ at $(x_0, y_0)$.

Examples :

1. Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4,-5)$ if
$$f(x,y)=x^2+3xy+y-1$$

Solution :
$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+3xy+y-1)=2x+3y\cdot 1+0-0$$
$$\frac{\partial f}{\partial x}=2x+3y \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(4,-5)}=2(4)+3(-5)=-7.$$
$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+3xy+y-1)=0+3x\cdot 1+1-0$$
$$\frac{\partial f}{\partial y}=3x+1 \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(4,-5)}=3(4)+1=13.$$

2. Find $\partial f/dy$ if $f(x,y)=y\sin xy$

Solution:
$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(y\sin xy)=y\frac{\partial}{\partial y}(\sin xy)+\sin xy\frac{\partial}{\partial y}(y)$$
$$=y\cos xy\ .\ x+\sin xy\ .\ 1$$
$$=xy\cos xy+\sin xy$$

3. Find $f_x$ and $f_y$ if
$$f(x,y)=\frac{2y}{y+\cos x}$$

Solution :
$$f_x=\frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial x}(2y)-2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$
$$=\frac{(y+\cos x)(0)-2y(-\sin x)}{(y+\cos x)^2}=\frac{2y\sin x}{(y+\cos x)^2}$$
$$f_y=\frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial y}(2y)-2y\frac{\partial}{\partial y}(y+\cos x)^2}{(y+\cos x)^2}$$
$$=\frac{(y+\cos x)(2)-2y(1)}{(y+\cos x)^2}=\frac{2\cos x}{(y+\cos x)^2}$$

4. Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines $z$ as a function of the two independent variables $x$ & $y$ and the partial derivative exists.

solution: We differentiate both sides of the equation with respect to $x$, holding $y$ constant and treating $z$ as a differentiable function of $x$:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1 \implies \frac{\partial z}{\partial x} = \frac{z}{yz - 1}$$

## Functions of More Than Two Variables:

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

Examples:

1. If $x, y$, and $z$ are independent variables and

$$f(x, y, z) = x\sin(y + 3z)$$

Find $\partial f/\partial x$, $\partial f/\partial y$, & $\partial f/\partial z$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\sin(y+3z)) = \sin(y+3z)\cdot 1$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\sin(y+3z)) = x\cos(y+3z)\cdot 1$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x\sin(y+3z)) = x\cos(y+3z)\cdot 3$$

$$= 3x\cos(y+3z).$$

2. If resistors of $R_1$, $R_2$, & $R_3$ ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Find the value of $\partial R/\partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution:

To find $\partial R/\partial R_2$, we treat $R_1$ and $R_3$ as constants and, using implicit differentiation, differentiate both sides of the equation with respect to $R_2$:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2^2} = \left(\frac{R}{R_2}\right)^2$$

when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3+2+1}{90} = \frac{6}{90} = \frac{1}{15}.$$

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

<u>Second-Order Partial Derivatives</u>:

$\frac{\partial^2 f}{\partial x^2}$ "d squared f dx squared" or $f_{xx}$ "f sub xx".

$\frac{\partial^2 f}{\partial y^2}$ "d squared f dy squared" or $f_{yy}$ "f sub yy"

$\frac{\partial^2 f}{\partial x \partial y}$ "d squared f dx dy" or $f_{yx}$ "f sub yx"

$\frac{\partial^2 f}{\partial y \partial x}$ "d squared f dy dx" or $f_{xy}$ "f sub xy"

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad , \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\frac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to $y$, then with respect to $x$.

$f_{yx} = (f_y)_x$ Means the same thing.

Example: If $f(x,y) = x\cos y + y e^x$, find

$\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + y e^x) = \cos y + y e^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + y e^x) = -x\sin y + e^x$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x \end{aligned}\right\} \text{equal}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = y e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

<u>The Mixed Derivative Theorem</u>:

If $f(x,y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are defined throughout an open region containing a point $(a,b)$ and are all continuous at $(a,b)$, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example: Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2+1}$$

Solution: The symbol $\partial^2 w/\partial x \partial y$ tell us to differentiate first with respect to $y$ and then with respect to $x$. If we postpone the differentiation with respect to $y$ and differentiate first with respect to $x$, however, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1$$

If we differentiate first with respect to $y$, we obtain

$$\frac{\partial^2 w}{\partial x \partial y} = 1$$ as well. (Mixed Derivative Theorem).

## Partial Derivatives of Still Higher Order:

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \quad , \quad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx} \; .$$

Examples:

1. Find $f_{yxyz}$ if $f(x,y,z) = 1 - 2xy^2z + x^2y$.

Solution: We first differentiate with respect to the variable $y$, then $x$, the $y$ again, and finally with respect to $z$.

$$f_y = -4xyz + x^2$$
$$f_{yx} = -4yz + 2x$$
$$f_{yxy} = -4z$$
$$f_{yxyz} = -4 \, .$$

2. Find $Z_{yyxx}$ if $Z = x^2y^3 + 2x$

Solution:

$$Z_y = 3y^2x^2 \qquad \frac{\partial z}{\partial y}$$
$$Z_{yy} = 6yx^2 \qquad \frac{\partial^2 z}{\partial y^2}$$
$$Z_{yyx} = 12xy \qquad \frac{\partial^3 z}{\partial x \partial y^2}$$
$$Z_{yyxx} = 12y \qquad \frac{\partial^4 z}{\partial x^2 \partial y^2}$$

# The Chain Rule :

## Chain Rule for Functions of Two Independent Variables :

If $w = f(x, y)$ has continuous partial derivatives $f_x$ and $f_y$ and if $x = x(t)$, $y = y(t)$ are differentiable functions of $t$, then the composite $w = f(x(t), y(t))$ is a differentiable function of $t$ and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Example : Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to $t$ along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$ ?

Solution :

$$\frac{\partial w}{\partial x} = y = \sin t \quad , \quad \frac{\partial w}{\partial y} = x = \cos t$$

$$\frac{dx}{dt} = -\sin t \quad , \quad \frac{dy}{dt} = \cos t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t$$

$$\frac{dw}{dt} = \cos 2t$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2 \cdot \frac{\pi}{2}\right) = \cos \pi = -1$$

## Chain Rule for Functions of Three Independent Variables :

If $w = f(x, y, z)$ is differentiable and $x$, $y$, and $z$ are differentiable functions of $t$, then $w$ is a differentiable function of $t$ and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} .$$

Example: Find $dw/dt$ if

$w = xy + z$ , $x = \cos t$ , $y = \sin t$ , $z = t$ .

What is the derivative's value at $t=0$ ?

Solution:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} .$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$\frac{dw}{dt} = \cos 2t + 1$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \cos 2t + 1 = \cos(0) + 1 = 1+1 = 2 .$$

## Chain Rule for Two Independent Variables and Three Intermediate Variables :

Suppose that $w = f(x,y,z)$ , $x = g(r,s)$, $y = h(r,s)$, and $z = k(r,s)$. If all four functions are differentiable, then $w$ has partial derivatives with respect to $r$ and $s$, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Example: Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of $r$ and $s$ if

$w = x + 2y + z^2$, $x = \frac{r}{s}$ , $y = r^2 + \ln s$ , $z = 2r$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + 4z$$

$$= \frac{1}{s} + 4r + 4(2r)$$

$$\frac{\partial w}{\partial r} = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

$$\frac{\partial w}{\partial s} = \frac{2}{s} - \frac{r}{s^2}$$

* If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} \quad , \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Example: Express $\partial w/\partial r$ and $\partial w/\partial s$ in term of r and s if
$w = x^2 + y^2$ , $x = r - s$ , $y = r + s$

Solution:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$= (2x)(1) + (2y)(1)$$

$$= 2(r-s) + 2(r+s)$$

$$\frac{\partial w}{\partial r} = 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= (2x)(-1) + (2y)(1)$$

$$= -2(r-s) + 2(r+s)$$

$$\frac{\partial w}{\partial s} = 4s$$

* If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial r} \quad , \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial s}$$

Example: Express $\partial w/\partial r$ and $\partial w/\partial s$ in term of r and s if
$w = x^2$ , $x = r - s$

Solution:

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial r} = (2x)(1) = 2x = 2(r-s)$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial s} = (2x)(-1) = -2x = -2(r-s)$$

## Implicit Differentiation :

Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y)=0$ defines $y$ as a differentiable function of $x$. Then at any point where $F_y \neq 0$,

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

Example : Find $dy/dx$ if $y^2 - x^2 - \sin xy = 0$

Solution : Take $F(x,y) = y^2 - x^2 - \sin xy$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$= \frac{2x + y\cos xy}{2y - x\cos xy}.$$

# Complex Numbers :

Complex numbers are numbers of the form $a+ib$ where $a$ and $b$ are real numbers and $i=\sqrt{-1}$.

$$a + ib$$

Real Part          Imaginary Part

## Properties of Complex Number :

1. Addition and Substraction :

$$(a+ib)+(c+id) = (a+c)+i(b+d).$$
$$(a+ib)-(c+id) = (a-c)+i(b-d).$$

Examples :

1. $(2+3i)+(4-5i) = 6-2i$
2. $-3i-(2+3i) = -2-6i$

2. Multiplication :

$$\begin{aligned}(a+ib)(c+id) &= ac+iad+ibc+i^2bd \\ &= ac+iad+ibc-bd \\ &= (ac-bd)+i(ad+bc)\end{aligned}$$

Examples :

1. $(5-2i)(3+4i) = 15+20i-6i-8i^2$
$= 15+14i+8$
$= 23+14i.$
2. $(2+3i)(6-2i) = 12-4i+18i-6i^2$
$= 12+14i+6$
$= 18+14i$

3. Division

$$\frac{c+id}{a+ib} = \frac{c+id}{a+ib} \times \frac{a-ib}{a-ib} = \frac{ac-ibc+iad-i^2bd}{a^2+b^2}$$

$$= \frac{ac+bd}{a^2+b^2} + \frac{(ad-bc)i}{a^2+b^2}$$

Examples:

1. $\dfrac{2+3i}{6-2i} = \dfrac{2+3i}{6-2i} \times \dfrac{6+2i}{6+2i} = \dfrac{12+4i+18i+6i^2}{36+12i-12i-4i^2}$

$= \dfrac{6+22i}{40} = \dfrac{3}{20} + \dfrac{11}{20} i$

2. $\dfrac{3}{2+3i} \times \dfrac{2-3i}{2-3i} = \dfrac{6-9i}{13} = \dfrac{6}{13} - \dfrac{9}{13} i$

## 4. Equality:

$a+ib = c+id$

if and only if $a=c$ and $b=d$.

Example:

1. $2+3i = x+yi$

$\therefore \; x=2$ & $y=3$

2. $x+(x+y)i = 2+5i$

$x=2$ & $x+y=5 \Rightarrow y = 5-2 = 3$.

Note:

$i=\sqrt{-1}$ , $i^2=-1$ , $i^3 = i^2 . i = -1\, i = -i$

$i^4 = i^2 . i^2 = 1$ , $i^5 = i^4 . i = 1\, i = i$

$i^6 = i^5 . i = i . i = i^2$.

Examples:

1. $(3+5i)-(2+5i) = 1$

2. $(3+5i)-(3+2i) = 3i$

3. $\dfrac{2-3i}{2+i} \Rightarrow \dfrac{2-3i}{2+i} = a+bi \Rightarrow$

$2-3i = (a+bi)(2+i) \Rightarrow 2-3i = 2a+ai+2bi+bi^2$

$2-3i = 2a+(a+2b)i + b(-1)$

$2-3i = (2a-b)+(a+2b)i$

$2 = 2a-b \qquad \times 2$

$-3 = a+2b$

$$4 = 4a - 2b$$
$$-3 = a + 2b$$
$$1 = 5a \Rightarrow a = 1/5 \qquad b = -8/5$$

4. $\sqrt{8-6i} \Rightarrow \sqrt{8-6i} = a+bi$

$$8-6i = (a+bi)^2 \Rightarrow 8-6i = a^2+2abi+b^2i^2$$
$$8-6i = a^2+2abi-b^2 \Rightarrow 8-6i = (a^2-b^2)+2abi$$
$$8 = a^2-b^2$$
$$6 = 2ab \Rightarrow b = 3/a$$
$$8 = a^2 - \frac{9}{a^2} \Rightarrow 8 = \frac{a^4-9}{a^2} \Rightarrow 8a^2 = a^4-9$$
$$a^4-8a^2-9=0 \Rightarrow (a^2-9)(a^2+1)=0$$
$$a^2 = 9 \Rightarrow a = \mp 3$$
$$a^2 = -1 \Rightarrow a = \mp\sqrt{-1} \quad \text{or}$$
$$\therefore b = \frac{3}{a} = \frac{3}{\mp 3} = \mp 1$$

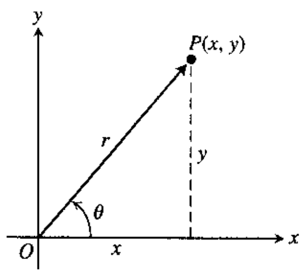
$$\sqrt{8+6i} = \mp 3 \mp i$$

## Argand Diagrams:

There are two geometric representations of the complex number $z = x+iy$:

1. as the point $P(x,y)$ in the $xy$-plane.
2. as the vector $\overrightarrow{OP}$ from the origin to $P$.

In each representation, the $x$-axis is called the real axis and the $y$-axis is the imaginary axis. Both representations are Argand diagrams for $x+iy$.



In terms of the polar coordinates of $x$ and $y$, we have

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad ,$$

and

$$z = x + iy = r\,(\cos\theta + i\sin\theta) \,.$$

We define the absolute value of a complex number $x+iy$ to be the length $r$ of a vector $\overrightarrow{OP}$ from the origin to $P(x, y)$.

$$|z| = |x+iy| = \sqrt{x^2+y^2} \,.$$

$$\therefore \quad r = |x+iy| \quad \Rightarrow \quad |z| = r$$

The polar angle $\theta$ is called the argument of $z$ and is written $\theta = \arg z$. $\quad \therefore \theta = \arg z = \tan^{-1}(y/x)$

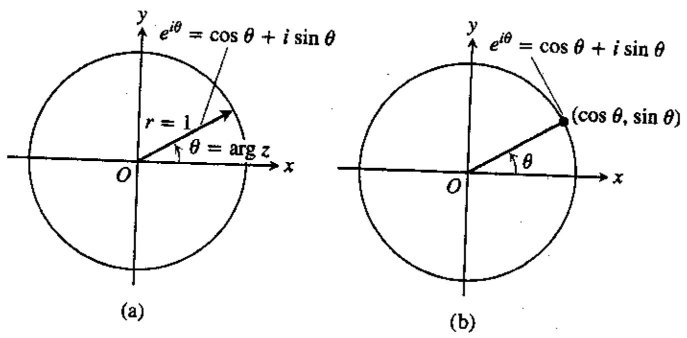
The following equation gives a useful formula connecting a complex number $z$, its conjugate $\bar{z}$, and its absolute value $|z|$, namely,

$$z \cdot \bar{z} = |z|^2$$

Euler's Formula :

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\therefore \quad z = r\,e^{i\theta}$$




Argand diagrams for $e^{i\theta} = \cos\theta + i\sin\theta$ (a) as a vector and (b) as a point.

Products : (Multiply two complex numbers)

$$z_1 = r_1 e^{i\theta_1} \quad , \quad z_2 = r_2 e^{i\theta_2}$$

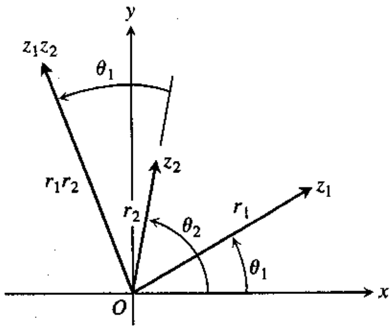
$$|z_1| = r_1 \quad \arg z_1 = \theta_1 \quad ; \quad |z_2| = r_2 \quad , \quad \arg z_2 = \theta_2$$

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 \, e^{i(\theta_1+\theta_2)}.$$

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2 .$$

The product of two complex numbers is represented by a vector whose length is the product of the lengths of two factors and whose argument is the sum of their arguments




When $z_1$ and $z_2$ are multiplied, $|z_1 z_2| = r_1 \cdot r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.

## Quotients :

Suppose $r_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2 .$$

## Powers :

If $n$ is a positive integer

$$z^n = z \cdot z \cdot \;\ldots\; \cdot z. \qquad n \text{ factors}$$

$$z = r e^{i\theta}$$

$$z^n = (r e^{i\theta})^n = r^n e^{i(\theta+\theta+\cdots+\theta)}$$

$$= r^n e^{in\theta} .$$

If we take $r=1$, we obtain De Moivre's Theorem.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Examples:

1. Find the absolute value and the argument of $z = 1-\sqrt{3}\,i$

solution:

$$|z| = r = \sqrt{x^2+y^2}$$

$$z = x+iy \Rightarrow x=1 \;\&\; y=-\sqrt{3}$$

$$\therefore |z| = \sqrt{x^2+y^2} = \sqrt{(1)^2+(-\sqrt{3})^2} = \sqrt{4} = 2$$

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -60$$

2. Express $1-i$ in polar form

solution:

$$z = x+iy \quad , \quad z = 1-i \Rightarrow x=1 \;\&\; y=-1$$

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{(1)^2+(-1)^2} = \sqrt{2}\,.$$

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-1) = -45\,.$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = \sqrt{2}\,(\cos(-45) + i\sin(-45)) \Rightarrow z = 0.9998 - 0.9998\,i$$

3. Let $z_1 = 1+i$ , $z_2 = \sqrt{3}-i$ . find $z_1 \cdot z_2$ , $\arg(z_1 z_2)$, the absolute value (modulus) of $z_1 z_2$ and $\frac{z_1}{z_2}$.

Solution.

$$z_1 = x+iy \Rightarrow z_1 = 1+i \Rightarrow x=1 \;\&\; y=1$$

$$|z_1| = r_1 = \sqrt{x^2+y^2} = \sqrt{2}$$

$$\theta_1 = \arg z_1 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = 45\,.$$

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1\, e^{\theta_1 i}$$

$$z_1 = \sqrt{2}\,(\cos(45) + i\sin(45)) = \sqrt{2}\, e^{\frac{\pi}{4} i}$$

$$z_2 = x+iy \Rightarrow z_2 = \sqrt{3}-i \Rightarrow x=\sqrt{3} \;\&\; y=-1$$

$$|z_2| = r_2 = \sqrt{x^2+y^2} = 2$$

$$\theta_2 = \arg z_2 = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30 .$$

$$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2) = r_2 e^{\theta_2 i}$$

$$z_2 = 2(\cos(-30) + i\sin(-30)) = 2e^{-\frac{\pi}{6}i}$$

$$z_1 . z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$z_1 z_2 = 2\sqrt{2}\, e^{i\left(\frac{\pi}{4}-\frac{\pi}{6}\right)} = 2\sqrt{2}\, e^{\frac{\pi}{12}i} \quad \text{Euler form.}$$

$$z_1 . z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$$z_1 . z_2 = 2\sqrt{2}\,(\cos(\pi/12) + i\sin(\pi/12)) \quad \text{Polar form.}$$

$$z_1 . z_2 = 2.73 + 0.73i$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2$$

$$\arg(z_1 z_2) = \frac{\pi}{4} + \left(-\frac{\pi}{6}\right) = \frac{\pi}{12}$$

$$|z_1 z_2| = r_1 r_2 = 2\sqrt{2} \quad \text{(modulus).}$$

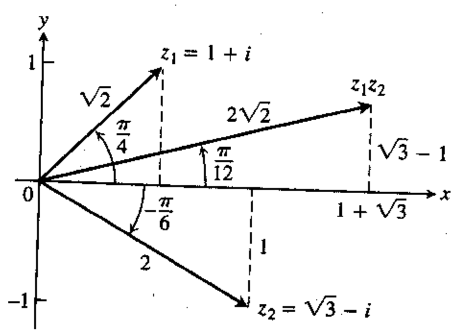
$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\pi/4}}{2\, e^{-i\pi/6}} = \frac{\sqrt{2}}{2} e^{\frac{5\pi}{12}i} \quad \text{Euler form.}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}(\cos(5\pi/12) + i\sin(5\pi/12)) \quad \text{Polar form.}$$

$$\frac{z_1}{z_2} = 0.183 + 0.683i$$

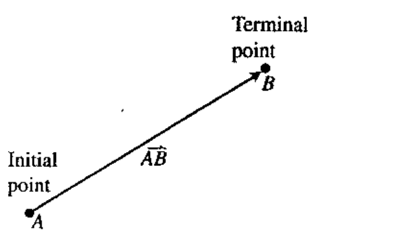


# Vector Analysis:

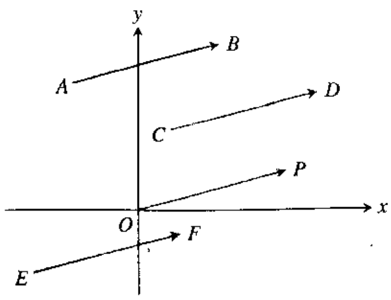
- Scalar: Quantities that are completely known or determined from their magnitude only like length and weight.
- Vector: Quantities that have magnitude and direction like velocity and force.

## Component Form:

A vector is directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point $A$ and terminal point $B$; its length is denoted by $|\overrightarrow{AB}|$. Two vectors are equal if they have the same length and direction.




The directed line segment $\overrightarrow{AB}$.




The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

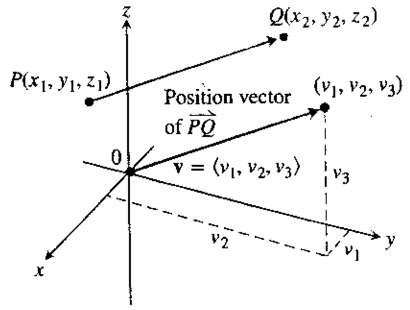
If $v$ is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1, v_2)$, then the component form of $v$ is

$$v = (v_1, v_2).$$
$$v = (x_2 - x_1, y_2 - y_1)$$

If $v$ is a three-dimensional vector equal to the vector with initial point at the origin and terminal point $(v_1, v_2, v_3)$, then the component form of $v$ is

$$v = (v_1, v_2, v_3).$$

$$v = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$




A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and v are parallel and have the same length.

* Two vectors are equal if and only if their standard position vectors are identical. Thus $(u_1, u_2, u_3)$ and $(v_1, v_2, v_3)$ are equal if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

* The magnitude or length of the vector $v = \overrightarrow{PQ}$ is the nonnegative number

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

* The only vector with length 0 is the zero vector $0 = (0, 0)$ or $0 = (0, 0, 0)$. This vector is also the only vector with no specific direction.

Example: Find the (a) component form and (b) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution:

(a) $v_1 = x_2 - x_1 = -5 - (-3) = -2$

$v_2 = y_2 - y_1 = 2 - 4 = -2$

$v_3 = z_2 - z_1 = 2 - 1 = 1$

The component form of $\overrightarrow{PQ}$ is $v = (-2, -2, 1)$

(b) The length or magnitude of $v = \overrightarrow{PQ}$ is

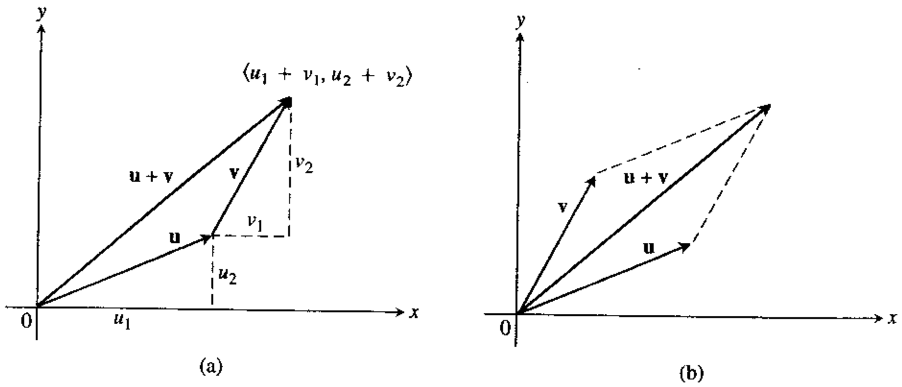
$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = 3.$$

# Vector Algebra Operations :

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors with $k$ a scalar

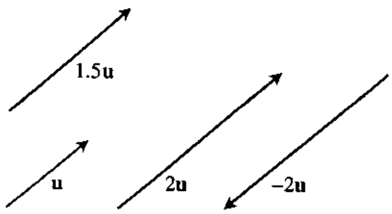
Addition : $\quad u+v = (u_1+v_1 , u_2+v_2 , u_3+v_3)$

Scalar multiplication : $\quad ku = (ku_1, ku_2, ku_3)$




(a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.

if $k > 0 \Rightarrow ku$ has the same direction as $u$.
if $k < 0 \Rightarrow$ the direction of $ku$ is opposite to that of $u$.




$$|ku| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)}$$

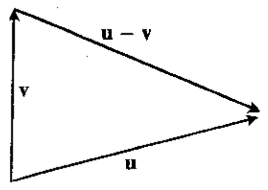
$$= \sqrt{k^2}\,\sqrt{u_1^2 + u_2^2 + u_3^2} = |k|\,|u|$$

$(-1)u = -u$ has the same length as $u$ but points in the opposite direction.

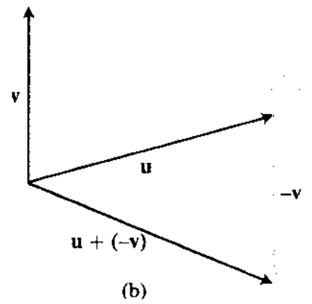
$u - v = u + (-v)$ difference of two vectors.

if $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

$$u - v = (u_1 - v_1 , u_2 - v_2 , u_3 - v_3)$$




(a)




(b)

The vector $u-v$ when added to $v$, gives $u$.

$u-v = u+(-v)$.

Example: Let $u=(-1,3,1)$ and $v=(4,7,0)$. Find
a) $2u+3v$ b) $u-v$ c) $\left|\frac{1}{2}u\right|$

Solution:

a) $2u+3v = 2(-1,3,1)+3(4,7,0) = (-2,6,2)+(12,21,0)$
$= (10,27,2)$

b) $u-v = (-1,3,1)-(4,7,0) = (-1-4,3-7,1-0)$
$= (-5,-4,1)$

c) $\left|\frac{1}{2}u\right| = \left|\left(-\frac{1}{2},\frac{3}{2},\frac{1}{2}\right)\right| = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}$

**Properties of Vector Operations**

Let **u**, **v**, **w** be vectors and $a$, $b$ be scalars.

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $0\mathbf{u}=\mathbf{0}$
6. $1\mathbf{u}=\mathbf{u}$
7. $a(b\mathbf{u})=(ab)\mathbf{u}$
8. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$
9. $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$

## Unit Vectors:

A vector $v$ of length 1 is called a unit vector. The standard unit vectors are:

$i=(1,0,0)$, $j=(0,1,0)$, and $k=(0,0,1)$

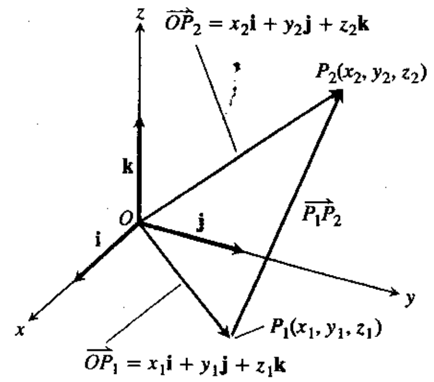
Any vector $v=(v_1,v_2,v_3)$ can be written as a linear combination of the standard unit vectors as follows:

$$v = (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3)$$
$$= v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1)$$
$$= v_1 i + v_2 j + v_3 k$$

We call the scalar (or number) $v_1$ the $i$-component of the vector $v$, $v_2$ the $j$-component, and $v_3$ the $k$-component.

$$P_1(x_1, y_1, z_1) \quad , \quad P_2(x_2, y_2, z_2)$$

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$




The vector from $P_1$ to $P_2$ is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

$$\left| \frac{1}{|v|} v \right| = \frac{1}{|v|} |v| = 1$$

$\dfrac{v}{|v|}$ is a unit vector in the direction of $v$, called the direction of the nonzero vector $v$.

Examples:

1. Find a unit vector $u$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Solution:

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$
$$= (3 - 1)i + (2 - 0)j + (0 - 1)k$$
$$= 2i + 2j - k$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3 .$$

$$u = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2i + 2j - k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

The unit vector $u$ is the direction of $\overrightarrow{P_1P_2}$.