Structural Analysis

Fundamental Principles

Structural analysis is the determination of forces and deformations of the structure due to applied loads. *Structural design* involves the arrangement and proportioning of structures and their components in such a way that the assembled structure is capable of supporting the designed loads within the allowable limit states. Ananalytical model is an idealization of the actual structure. The structural model should relate the actual behavior to material properties, structural details, and loading and boundary conditions as accurately as is practicable.

All structures that occur in practice are three-dimensional. For building structures that have regular layout and are rectangular in shape, it is possible to idealize them into two-dimensional frames arranged in orthogonal directions. *Joints* in a structure are those points where two or more members are connected. A truss is a structural system consisting of members that are designed to resist only axial forces. Axially loaded members are assumed to be pin-connected at their ends.

A structural system in which joints are capable of transferring end moments is called a *frame*. Members in this system are assumed to be capable of resisting bending moment axial force and shear force. A structure is said to be two dimensional or planar if all the members lie in the same plane. Beams are those members that are subjected to bending or flexure. They are usually thought of as being in horizontal positions and loaded with vertical loads. *Ties* are members that are subjected to axial tension only, while struts (columns or posts) are members subjected to axial compression only.

Boundary Conditions

A hinge represents a pin connection to a structural assembly and it does not allow translational movements (Figure 2.1a). It is assumed to be frictionless and to allow rotation of a member with respect to the others.

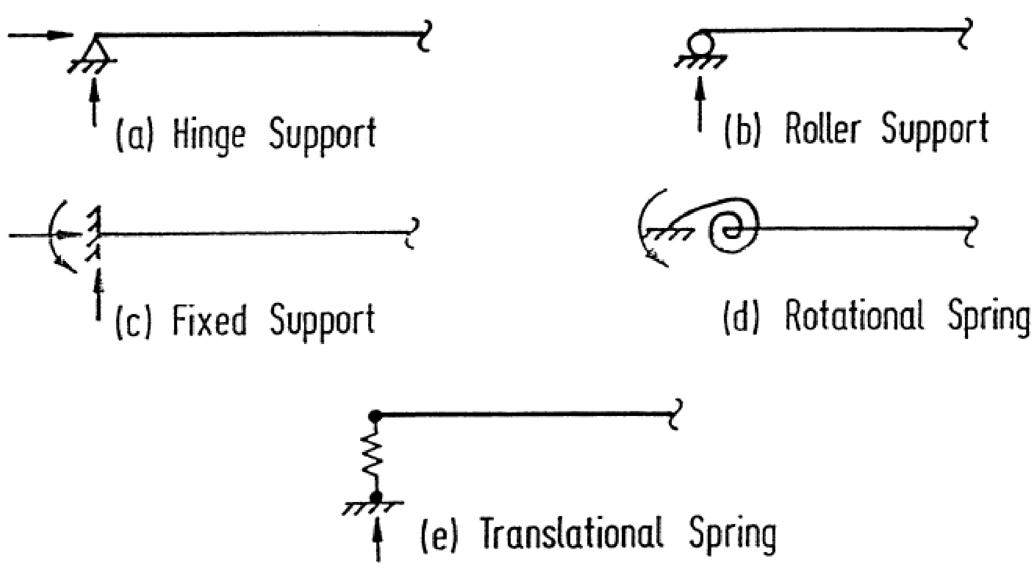

FIGURE 2.1: Various boundary conditions.

A *roller* represents a kind of support that permits the attached structural part to rotate freely with respect to the foundation and to translate freely in the direction parallel to the foundation surface (Figure 2.1b) No translational movement in any other direction is allowed. A *fixed support* (Figure 2.1c) does not allow rotation or translation in any direction. A *rotational spring* represents a support that provides some rotational restraint but does not provide any translational restraint (Figure 2.1d). A *translational spring* can provide partial restraints along the direction of deformation (Figure 2.1e).

Loads and Reactions

Loads may be broadly classified as *permanent loads* that are of constant magnitude and remain in one position and *variable loads* that may change in position and magnitude. Permanent loads are also referred to as *dead loads* which may include the self weight of the structure and other loads such as walls, floors, roof, plumbing, and fixtures that are permanently attached to the structure.

Variable loads are commonly referred to as live or imposed loads which may include those caused by construction operations, wind, rain, earthquakes, snow, blasts, and temperature changes in addition to those that are movable, such as furniture and warehouse materials.

Ponding load is due to water or snow on a flat roof which accumulates faster than it runs off. *Wind loads* act as pressures on windward surfaces and pressures or suctions on leeward surfaces. *Impact loads* are caused by suddenly applied loads or by the vibration of moving or movable loads. They are usually taken as a fraction of the live loads. *Earthquake loads* are those forces caused by the acceleration of the ground surface during an earthquake.

A structure that is initially at rest and remains at rest when acted upon by applied loads is said to be in a state of *equilibrium.* The resultant of the external loads on the body and the supporting forces or reactions is zero. If a structure or part thereof is to be in equilibrium under the action of a system of loads, it must satisfy the six static equilibrium equations, such as

$$\sum Fx = 0;\ \sum Fy = 0;\ \sum Fz = 0$$
$$\sum Mx = 0;\ \sum My = 0;\ \sum Mz = 0 \qquad (2.1)$$

The summation in these equations is for all the components of the forces *(F)* and of the moments (*M)* about each of the three axes *x, y*, and *z*. If a structure is subjected to forces that lie in one plane, say *x*-*y*, the above equations are reduced to:

$\sum Fx = 0;\ \sum Fy = 0;\ \sum Mz = 0$ (2.2)

Consider, for example, a beam shown in Figure 2.2a under the action of the loads shown.

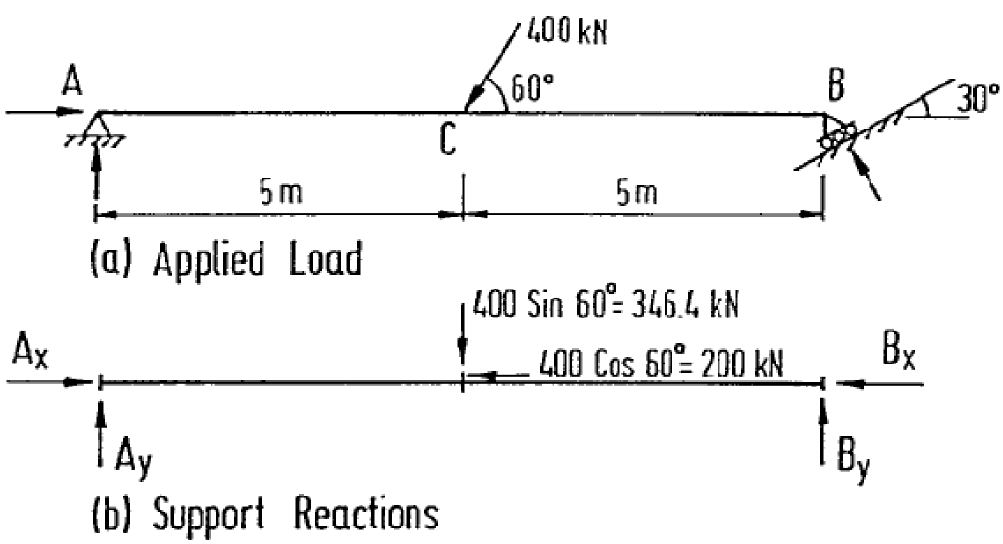

FIGURE 2.2: Beam in equilibrium.

The reaction at support *B* must act perpendicular to the surface on which the rollers are constrained to roll upon. The support reactions and the applied loads, which are resolved in vertical and horizontal directions, are shown in Figure 2.2b.

From geometry, it can be calculated that $By = \sqrt{3}Bx$. Equation 2.2 can be used to determine the magnitude of the support reactions. Taking moment about B gives

$10Ay - 346.4x5 = 0$

From which

$$Ay = 173.2 \text{ kN}.$$

Equating the sum of vertical forces, $\sum Fy$ to zero gives

$$173.2 + By - 346.4 = 0$$

and, hence, we get

$$By = 173.2 \text{ kN}.$$

Therefore,

$$Bx = By/\sqrt{3} = 100 \text{ kN}.$$

Equilibrium in the horizontal direction, $\sum Fx = 0$ gives,

$Ax - 200 - 100 = 0$

and, hence,

$Ax = 300$ kN.

Trusses

A structure that is composed of a number of bars pin connected at their ends to form a stable framework is called a truss. If all the bars lie in a plane, the structure is a planar truss. It is generally assumed that loads and reactions are applied to the truss only at the joints. The centroidal axis of each member is straight, coincides with the line connecting the joint centers at each end of the member, and lies in a plane that also contains the lines of action of all the loads and reactions. Many truss structures are three dimensional in nature and a complete analysis would require consideration of the full spatial interconnection of the members. However, in many cases, such as bridge structures and simple roof systems, the three-dimensional framework can be subdivided into planar components for analysis as planar trusses without seriously compromising the accuracy of the results.

Figure 2.17 shows some typical idealized planar truss structures.

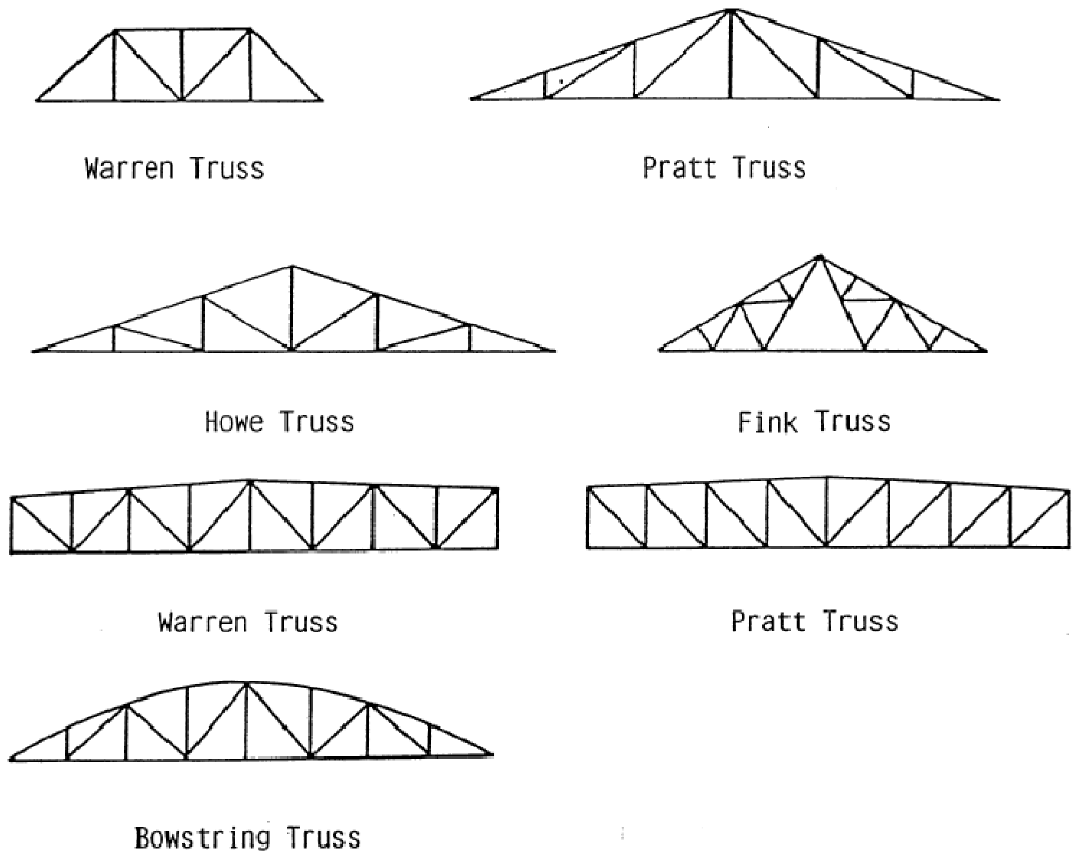

FIGURE 2.17: Typical planar trusses.

There exists a relation between the number of members, m, number of joints, j , and reaction components, r. The expression is

$$m = 2j - r \qquad (2.20)$$

Which must be satisfied if it is to be statically determinate internally. The least number of reaction components required for external stability is r. If m exceeds $(2j - r)$, then the excess members are called redundant members and the truss is said to be statically indeterminate.

Truss analysis gives the bar forces in a truss; for a statically determinate truss, these bar forces can be found by employing the laws of statics to assure internal equilibrium of the structure. The process requires repeated use of free-body diagrams from which individual bar forces are determined. The *method of joints* is a technique of truss analysis in which the bar forces are determined by the sequential isolation of joints—the unknown bar forces at one joint are solved and become known bar forces at subsequent joints. The other method is known as *method of sections* in which equilibrium of a part of the truss is considered.

Method of Joints

An imaginary section may be completely passed around a joint in a truss. The joint has become a free body in equilibrium under the forces applied to it. The equations $\sum H = 0$ and $\sum V = 0$ may be applied to the joint to determine the unknown forces in members meeting there. It is evident that no more than two unknowns can be determined at a joint with these two equations.

Example:

A truss shown in Figure 2.18 is symmetrically loaded, and it is sufficient to solve half the truss by considering the joints 1 through 5. At Joint 1, there are two unknown forces.

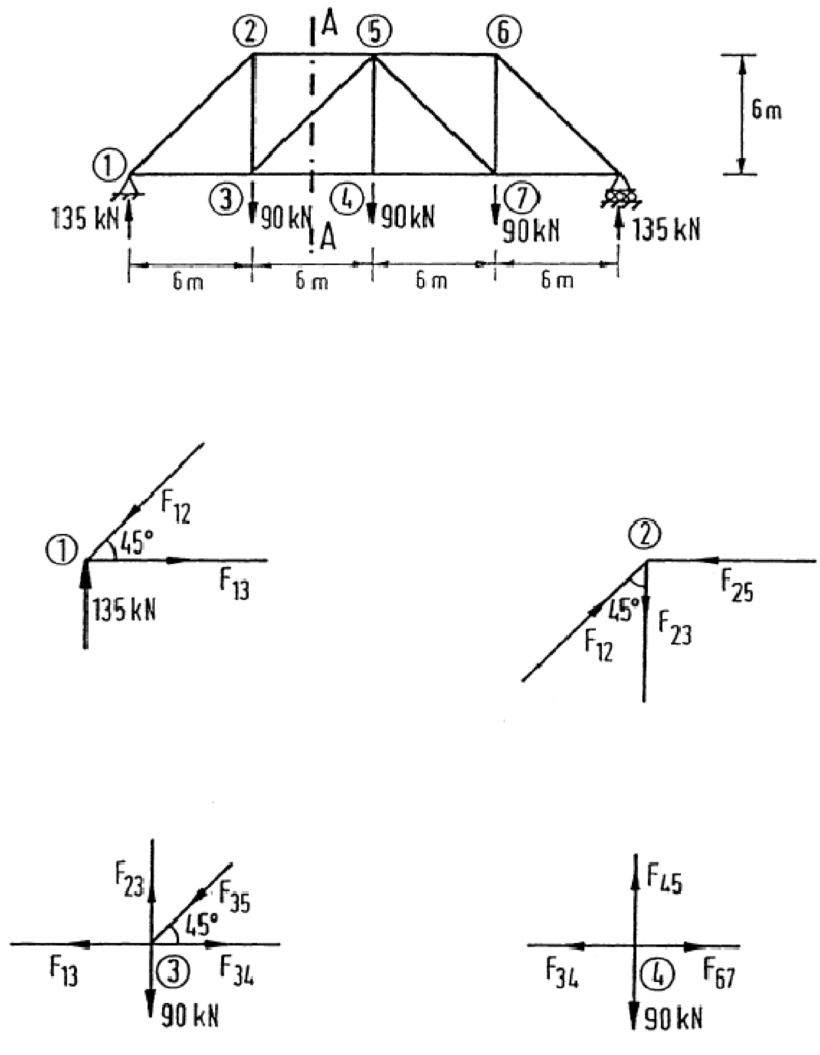

FIGURE 2.18: Example—methods of joints, planar truss.

Summation of the vertical components of all forces at Joint 1 gives vertical components of all forces at Joint 1 gives

$135 - F12 \sin 45 = 0$

which in turn gives the force in the member 1-2, $F12 = 190.0$ kN (compressive). Similarly, summation of the horizontal components gives
$F13 - F12 \cos 45 = 0$
Substituting for $F12$ gives the force in the member 1-3 as
$F13 = 135$ kN (tensile).

Now, Joint 2 is cut completely and it is found that there are two unknown forces $F25$ and $F23$.
Summation of the vertical components gives
$F12 \cos 45 - F23 = 0.$
Therefore,
$F23 = 135$ kN (tensile).
Summation of the horizontal components gives
$F12 \sin 45 - F25 = 0$
and hence
$F25 = 135$ kN (compressive).
After solving for Joints 1 and 2, one proceeds to take a section around Joint 3 at which there are now two unknown forces, namely, $F34$ and $F35$. Summation of the vertical components at Joint 3 gives
$F23 - F35 \sin 45 - 90 = 0$
Substituting for $F23$, one obtains $F35 = 63.6$ kN (compressive). Summing the horizontal components and substituting for $F13$ one gets
$-135 - 45 + F34 = 0$
Therefore,
$F34 = 180$ kN (tensile).
The next joint involving two unknowns is Joint 4. When we consider a section around it, the summation of the vertical components at Joint 4 gives
$F45 = 90$ kN (tensile).
Now, the forces in all the members on the left half of the truss are known and by symmetry the forces in the remaining members can be determined. The forces in all the members of a truss can also be determined by making use of the method of section.

Frames

Frames are statically indeterminate in general; special methods are required for their analysis. Slope deflection and moment distribution methods are two such methods commonly employed. Slope deflection is a method that takes into account the flexural displacements such as rotations and deflections and involves solutions of simultaneous equations. Moment distribution on the other hand involves successive cycles of computation, each cycle drawing closer to the "exact" answers.

The method is more labor intensive but yields accuracy equivalent to that obtained from the "exact" methods. This method, however, remains the most important hand-calculation method for the analysis of frames.

Slope Deflection Method

This method is a special case of the stiffness method of analysis, and it is convenient for hand analysis of small structures. Moments at the ends of frame members are expressed in terms of the rotations and deflections of the joints. Members are assumed to be of constant section between each pair of supports. It is further assumed that the joints in a structure may rotate or deflect, but the angles between the members meeting at a joint remain unchanged.

The member force-displacement equations that are needed for the slope deflection method are written for a member *AB* in a frame. This member, which has its undeformed position along the *x* axis is deformed into the configuration shown in Figure 2.21. The positive axes, along with the positive member-end force components and displacement components, are shown in the figure.

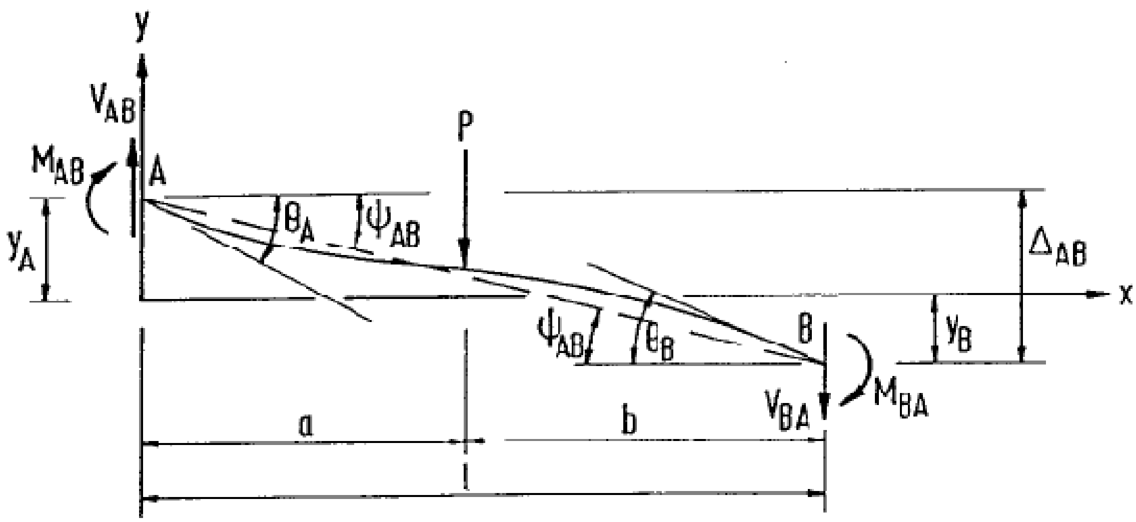

FIGURE 2.21: Deformed configuration of a beam.

The equations for end moments are written as

$$M_{AB} = \frac{2EI}{l}(2\theta_A + \theta_B - 3\psi_{AB}) + M_{FAB}$$

$$M_{BA} = \frac{2EI}{l}(2\theta_B + \theta_A - 3\psi_{AB}) + M_{FBA} \quad (2.21)$$

in which *MFAB* and *MFBA* are fixed-end moments at supports *A* and *B*, respectively, due to the applied load. *AB* is the rotation as a result of the relative displacement between the member ends
A and *B* given as

$$\psi_{AB} = \frac{\Delta_{AB}}{l} = \frac{y_A + y_B}{l} \quad (2.22)$$

where M_{AB} is the relative deflection of the beam ends. *ΔA* and *ΔB* are the vertical displacements at ends *A* and *B*. Fixed-end moments for some loading cases may be obtained from Figure 2.8.

The slope deflection equations in Equation 2.21 show that the moment at the end of a member is dependent on member properties *EI*, dimension *l*, and displacement quantities. The fixed-end moments reflect the transverse loading on the member.

Example:

Consider the frame shown in Figure 2.22, subjected to sidesway Δ to the right of the frame. Equation 2.21 can be applied to each of the members of the frame as follows:

Member *AB*:

$$M_{AB} = \frac{2EI}{20}\left(2\theta_A + \theta_B - \frac{3\Delta}{20}\right) + M_{FAB}$$

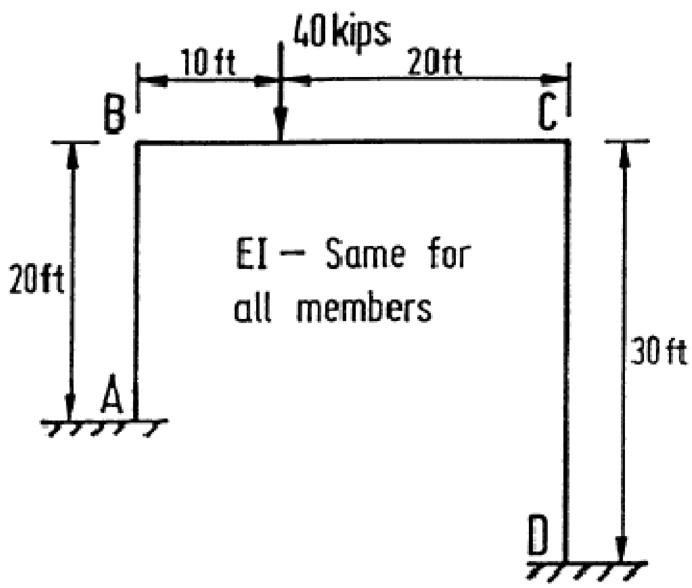

FIGURE 2.22: Example—slope deflection method.

$$\begin{aligned} M_{BA} &= \frac{2EI}{20}\left(2\theta_B + \theta_A - \frac{3\Delta}{20}\right) + M_{FBA} \\ \theta_A = 0, \qquad M_{FAB} &= M_{FBA} = 0 \end{aligned}$$

Hence,

$$M_{AB} = \frac{2EI}{20}(\theta_B - 3\psi) \tag{2.23}$$

$$M_{BA} = \frac{2EI}{20}(2\theta_B - 3\psi) \tag{2.24}$$

in which

$$\psi = \frac{\Delta}{20}$$

Member BC:

$$\begin{aligned} M_{BC} &= \frac{2EI}{30}(2\theta_B + \theta_C - 3 \times 0) + M_{FBC} \\ M_{CB} &= \frac{2EI}{30}(2\theta_C + \theta_B - 3 \times 0) + M_{FCB} \\ M_{FBC} &= -\frac{40 \times 10 \times 20^2}{30^2} = -178 \text{ ft-kips} \\ M_{FCB} &= -\frac{40 \times 10^2 \times 20}{30^2} = 89 \text{ ft-kips} \end{aligned}$$

Hence,

$$M_{BC} = \frac{2EI}{30}(2\theta_B + \theta_C) - 178 \tag{2.25}$$

$$M_{CB} = \frac{2EI}{30}(2\theta_C + \theta_B) + 89 \tag{2.26}$$

Member CD:

$$\begin{aligned} M_{CD} &= \frac{2EI}{30}\left(2\theta_C + \theta_D - \frac{3\Delta}{30}\right) + M_{FCD} \\ M_{DC} &= \frac{2EI}{30}\left(2\theta_D + \theta_C - \frac{3\Delta}{30}\right) + M_{FDC} \\ M_{FCD} &= M_{FDC} = 0 \end{aligned}$$

Hence,

$$M_{DC} = \frac{2EI}{30}\left(\theta_C - 3\times\frac{2}{3}\psi\right) = \frac{2EI}{30}(2\theta_C - 2\psi) \quad (2.27)$$

$$M_{DC} = \frac{2EI}{30}\left(\theta_C - 3\times\frac{2}{3}\psi\right) = \frac{2EI}{30}(\theta_C - 2\psi) \quad (2.28)$$

Considering moment equilibrium at Joint B

$$\sum M_B = M_{BA} + M_{BC} = 0$$

Substituting for M_{BA} and M_{BC}, one obtains

$$\frac{EI}{30}(10\theta_B + 2\theta_C - 9\psi) = 178$$

or

$$10\theta_B + 2\theta_C - 9\psi = \frac{267}{K} \quad (2.29)$$

where $K = \frac{EI}{20}$.
Considering moment equilibrium at Joint C

$$\sum M_C = M_{CB} + M_{CD} = 0$$

Substituting for M_{CB} and M_{CD} we get

$$\frac{2EI}{30}(4\theta_C + \theta_B - 2\psi) = -89$$

or

$$\theta_B + 4\theta_C - 2\psi = -\frac{66.75}{K} \quad (2.30)$$

Summation of base shear equals to zero, we have

$$\sum H = H_A + H_D = 0$$

or

$$\frac{M_{AB} + M_{BA}}{1_{AB}} + \frac{M_{CD} + M_{DC}}{1_{CD}} = 0$$

Substituting for M_{AB}, M_{BA}, M_{CD}, and M_{DC} and simplifying

$$2\theta_B + 12\theta_C - 70\psi = 0 \quad (2.31)$$

Solution of Equations 2.29 to 2.31 results in

$$\theta_B = \frac{42.45}{K}$$

$$\theta_C = \frac{20.9}{K}$$

and

$$\psi = \frac{12.8}{K} \quad (2.32)$$

Substituting for θB; θC, and ψ from Equations 2.32 into Equations 2.23 to 2.28 we get,

MAB = 10.10 ft-kips

MBA = 93 ft-kips

MBC = −93 ft-kips

MCB = 90 ft-kips

MCD = −90 ft-kips

MDC = −62 ft-kips

Moment Distribution Method

The moment distribution method involves successive cycles of computation, each cycle drawing closer to the "exact" answers. The calculations may be stopped after two or three cycles, giving a very good approximate analysis, or they may be carried on to whatever degree of accuracy is desired. Moment distribution remains the most important hand-calculation method for the analysis of continuous beams and frames and it may be solely used for the analysis of small structures. Unlike the slope deflection method, this method does require the solution to simultaneous equations.

The terms constantly used in moment distribution are fixed-end moments, unbalanced moment, distributed moments, and carry-over moments. When all of the joints of a structure are clamped to prevent any joint rotation, the external loads produce certain moments at the ends of the members to which they are applied. These moments are referred to as *fixed-end moments.* Initially the joints in a structure are considered to be clamped. When the joint is released, it rotates if the sum of the fixed-end moments at the joint is not zero. The difference between zero and the actual sum of the end moments is the *unbalanced moment.* The unbalanced moment causes the joint to rotate. The rotation twists the ends of the members at the joint and changes their moments. In other words, rotation of the joint is resisted by the members and resisting moments are built up in the members as they are twisted. Rotation continues until equilibrium is reached—when the resisting moments equal the unbalanced moment—at which time the sum of the moments at the joint is equal to zero. The moments developed in the members resisting rotation are the *distributed moments.* The distributed moments in the ends of the member cause moments in the other ends, which are assumed fixed, and these are the *carry-over moments.*