



University of Technology
Building and Construction Eng. Dept.
Final Exam- First Attempt – 2012/2013

Branch : All Branches
Subject : Mathematics II
Examiner : Mathematics Committee

Class: 2nd year
Time : 3 Hours
Date : 17/6/ 2013



Note: Answer Eight questions including Q1

Q1. IF $y = A \sin \frac{r \pi x}{b} \cdot \sin \frac{r \pi ct}{b}$ where A, b, c and r are constants, prove that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{dt^2} \quad (16 \text{ marks})$$

Q2. The directional derivative of a given function $W = f(x, y)$ at $P_0(0, 2)$ in the direction toward $P_1(3, 2)$ is 2 and in direction toward $P_2(3, 5)$ is 4. Find the directional derivative of W in the direction of y - axis. (12 marks)

Q3. Solve the following differential equation: $2xy \cdot dx = (x^2 + y^2) dy$ (12 marks)

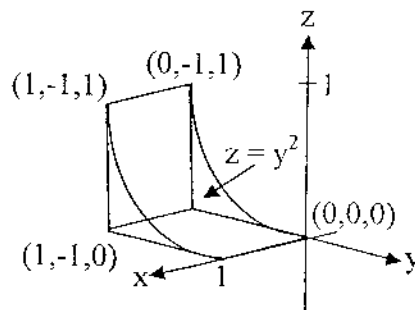
Q4. Solve the following differential equation using method of undetermined. (12 marks)

$$y'' - y' - 6y = e^{-x} - 7 \cos x$$

Q5. The position of a particle at time is given by the equations $x = 2(\sin 2t + 1)$, $y = 2(\cos 2t - 1)$. Find the velocity vector, speed, tangent vector, normal vector and B-vector at $t = \frac{\pi}{2}$. (12 marks)

Q6. Find My for the region enclosed by the curve $y = 2\sqrt{x}$, the line $y = 4$ and the y - axis. (12 marks)

Q7. Write the five integral forms and solve one for the solid shown below. (12 marks)



Q8. Test the following series by ratio test. $\sum_{n=1}^{\infty} \frac{(3n)!}{n!(n+1)!(n+2)!}$ (12 marks)

Q9. Solve the following system of linear equations using matrix method. (12 marks)

$$\begin{aligned} 10x_1 + 2x_2 - x_3 &= 27 \\ -3x_1 - 6x_2 + 2x_3 &= -61.5 \\ x_1 + x_2 + 5x_3 &= -21.5 \end{aligned}$$

Q10. For the complex function, show that: $\sin z \cdot \cos z = \frac{\sin 2z}{2}$ (12 marks)

Q. $y = A \sin \frac{x\pi X}{b} \cdot \sin \frac{x\pi Ct}{b}$

Sol.

$$\frac{dy}{dx} = A \cos \frac{x\pi X}{b} \left(\frac{x\pi}{b} \right) \cdot \sin \frac{x\pi Ct}{b}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -A \sin \left(\frac{x\pi X}{b} \right) \left(\frac{x\pi}{b} \right)^2 \cdot \sin \frac{x\pi Ct}{b} \\ &= -\frac{A x^2 \pi^2}{b^2} \sin \frac{x\pi X}{b} \cdot \sin \frac{x\pi Ct}{b} \end{aligned}$$

$$\frac{dy}{dt} = A \sin \frac{x\pi X}{b} \cdot \cos \frac{x\pi Ct}{b} \cdot \left(\frac{x\pi C}{b} \right)$$

$$\begin{aligned} \frac{d^2y}{dt^2} &= A \sin \frac{x\pi X}{b} \left(-\sin \frac{x\pi Ct}{b} \right) \cdot \left(\frac{x\pi C}{b} \right)^2 \\ &= -\frac{A x^2 \pi^2 C^2}{b^2} \cdot \sin \frac{x\pi X}{b} \cdot \sin \frac{x\pi Ct}{b} \end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{1}{C^2} \frac{d^2y}{dt^2}$$

$$-\frac{A x^2 \pi^2}{b^2} \sin \frac{x\pi X}{b} \cdot \sin \frac{x\pi Ct}{b} =$$

$$\frac{1}{C^2} \left(-\frac{A x^2 \pi^2 C^2}{b^2} \right) \sin \frac{x\pi X}{b} \cdot \sin \frac{x\pi Ct}{b}$$

R.H.S = L.H.S

∴

Q2

$$\frac{dw}{ds} = \frac{df}{dx} \cos \phi + \frac{df}{dy} \sin \phi$$

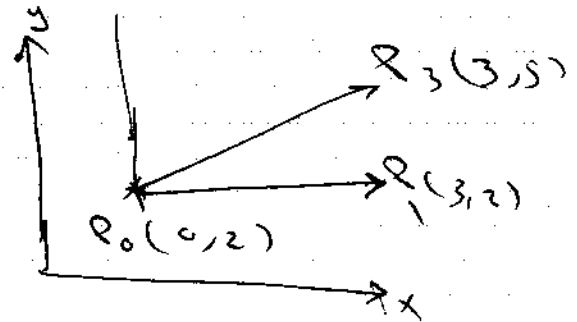
$$P_0 P_1 = (3-0)i + (2-2)j$$

$$= 3i$$

$$u = \frac{3i}{\sqrt{3^2+0^2}} = i \Rightarrow \phi = 0$$

$$2 = \frac{df}{dx}(1) + \frac{df}{dy}(0)$$

$$\therefore \frac{df}{dx} = 2 \quad \text{--- (1)}$$



$$P_0 P_2 = (3-0)i + (5-2)j$$

$$= 3i + 3j$$

$$u = \frac{3i + 3j}{\sqrt{3^2+3^2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \Rightarrow \phi = 45^\circ$$

$$4 = \frac{df}{dx} \left(\frac{1}{\sqrt{2}}\right) + \frac{df}{dy} \frac{1}{\sqrt{2}} \quad \text{--- (2)}$$

Sub. $\frac{df}{dx} = 2$

$$4 = \frac{1}{\sqrt{2}} \left(\frac{df}{dx}\right) + \frac{1}{\sqrt{2}} \frac{df}{dy}$$

$$4 = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{df}{dy} \Rightarrow \frac{df}{dy} = 4\sqrt{2} - 2$$

$$\frac{dw}{ds} = 2 \cos \phi + (4\sqrt{2} - 2) \sin \phi$$

So $\phi = \frac{\pi}{2}$

$$\therefore \frac{dw}{ds} = 2 \cos \frac{\pi}{2} + (4\sqrt{2} - 2) \sin \frac{\pi}{2}$$

$$\therefore \frac{dw}{ds} = 4\sqrt{2} - 2$$

Q3: $2xy \cdot dx = (x^2 + y^2) dy$

Sol.

$$2xy dx - (x^2 + y^2) dy = 0 \quad \int \div 2xy dy$$

$$\frac{dx}{dy} = \frac{1}{2} \frac{x}{y} - \frac{1}{2} \frac{y}{x} = 0$$

$$\frac{dx}{dy} = \frac{1}{2} \left(\frac{x}{y} + \frac{y}{x} \right) = 0$$

let $v = \frac{x}{y} \Rightarrow f(v) = \frac{1}{2} \left(v + \frac{1}{v} \right)$

$$\frac{dy}{y} + \frac{dv}{v - f(v)} = 0$$

$$\frac{dy}{y} + \frac{dv}{v - \left(\frac{v}{2} + \frac{1}{2v} \right)} = 0$$

$$\frac{dy}{y} + \frac{dv}{\frac{v}{2} - \frac{1}{2v}} = 0$$

$$\frac{dy}{y} + \frac{dv}{\frac{v^2 - 1}{2v}} = 0$$

$$\frac{dy}{y} + 2 \int \frac{v \cdot dv}{v^2 - 1} = 0$$

$$\ln y + \ln(v^2 - 1) = c$$

$$\ln[y \cdot (v^2 - 1)] = c \quad \text{exp.}$$

$$y(v^2 - 1) = e^c = c_1$$

$$y \left(\frac{x^2}{y^2} - 1 \right) = c_1 \Rightarrow \boxed{\frac{x^2}{y} - y = c}$$

$$\text{Q4} \quad y'' - y' - 6y = e^{-x} - 7 \cos x$$

$$y'' - y' - 6y = 0$$

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0 \quad r_1 = 3 \quad r_2 = -2$$

$$\therefore y_h = C_1 \cdot e^{3x} + C_2 \cdot e^{-2x}$$

$$y_p = y_{p1} + y_{p2}$$

$$\underline{y_{p1}}: F_1(x) = e^{-x} \quad r = -1 \neq r_1 \neq r_2$$

$$F_2(x) = -7 \cos x \quad k=1 \text{ and real roots}$$

$$\therefore y_p = A e^{-x} + B \sin x + C \cos x$$

$$y'_p = -A e^{-x} + B \cos x - C \sin x$$

$$y''_p = A e^{-x} - B \sin x - C \cos x$$

$$\underline{A e^{-x}} - B \sin x - C \cos x + \underline{A e^{-x}} - B \cos x + C \sin x$$

$$= \underline{2A e^{-x}} - 6B \sin x - 6C \cos x \\ = \underline{e^{-x}} - 7 \cos x$$

$$\text{equ. ①: } A + A - 6A = 1 \Rightarrow A = \frac{-1}{4}$$

$$\text{equ. ②: } -B + C - 6B = 0 \Rightarrow C = 7B$$

$$\text{equ. ③: } -C - B - 6C = -7$$

$$-7(7B) - B = -7 \quad B = \frac{-7}{-50} = \frac{7}{50}$$

$$\therefore C = 7 \times \frac{7}{50} = \frac{49}{50}$$

$$\text{Qull so } y_p = -\frac{1}{4}e^{-x} + \frac{7}{50}\sin x + \frac{49}{50}\cos x$$

then $y = y_h + y_p$

$$y = C_1 e^{3x} + C_2 e^{-2x} - \frac{1}{4}e^{-x} + \frac{7}{50}\sin x + \frac{49}{50}\cos x$$

Q5

$$x = 2(\sin 2t + 1)$$

$$y = 2(\cos 2t - 1)$$

Sol.

$$\vec{r} = x\mathbf{i} + y\mathbf{j}$$

$$= (2\sin 2t + 1)\mathbf{i} + 2(\cos 2t - 1)\mathbf{j}$$

$$\frac{d\vec{r}}{dt} = 4\cos 2t\mathbf{i} - 4\sin 2t\mathbf{j}$$

$$\frac{d^2\vec{r}}{dt^2} = -4\mathbf{j}$$

$$\text{Speed} = |\dot{\vec{r}}| = \sqrt{(4\cos 2t)^2 + (-4\sin 2t)^2}$$
$$= 2$$

$$\vec{T} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} = 2\cos 2t\mathbf{i} - 2\sin 2t\mathbf{j}$$

$$\frac{d\vec{T}}{dt} = -2\mathbf{j}$$

$$\vec{N} = 2\sin 2t\mathbf{i} + 2\cos 2t\mathbf{j}$$

$$\frac{d\vec{N}}{dt} = -2\mathbf{j}$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos 2t & -2\sin 2t & 0 \\ 2\sin 2t & 2\cos 2t & 0 \end{vmatrix}$$

$$= (0)\mathbf{i} + (0)\mathbf{j} + 4\mathbf{k}$$

$$\vec{B} = 4\mathbf{k}$$

Q6

$$M_y = \iint x \delta(x, y) dA$$

$$M_y = \int_0^4 \int_{y=2\sqrt{x}}^{y=4} x \cdot dy \cdot dx$$

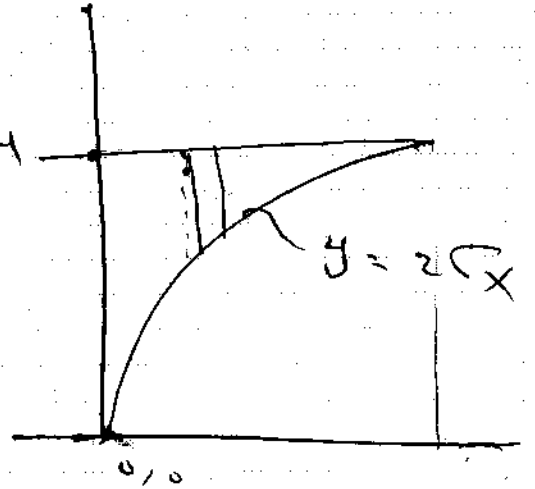
$$= \int_0^4 x [y]_{2\sqrt{x}}^4 \cdot dx$$

$$= \int_0^4 x (4 - 2\sqrt{x}) dx$$

$$M_y = \left[\frac{4}{2} x^2 - 2 \frac{x^{5/2}}{5/2} \right]_0^4$$

$$= 2 \times 16 - \frac{4}{5} (4)^{5/2}$$

$$= 6.4$$



$$\begin{aligned} 4 &= 2\sqrt{x} \\ \sqrt{x} &= 2 \\ x &= 4 \end{aligned}$$

Q7:

$$\textcircled{1} \int_0^1 \int_{-1}^0 \int_0^{y^2} dz \cdot dy \cdot dx$$

$$= \int_0^1 \int_{-1}^0 [z]_0^{y^2} \cdot dy \cdot dx = \int_0^1 \int_{-1}^0 y^2 \cdot dy \cdot dx$$

$$= \int_0^1 \left[\frac{y^3}{3} \right]_{-1}^0 \cdot dx = \frac{1}{3} \int_0^1 dx = \frac{1}{3} [x]_0^1 = \frac{1}{3} = 0.333$$

$$\textcircled{2} \int_{-1}^0 \int_0^1 \int_0^{y^2} dz \cdot dx \cdot dy$$

$$\textcircled{3} \int_0^1 \int_{-1}^0 \int_0^{y^2} dx \cdot dy \cdot dz$$

$$\textcircled{4} \int_{-1}^0 \int_0^1 \int_0^{y^2} dx \cdot dz \cdot dy$$

$$\textcircled{5} \int_0^1 \int_0^0 \int_{-1}^0 dy \cdot dx \cdot dz$$

Ex 8:

$$u_n = \frac{3^n!}{n! (n+1)! (n+2)!}$$

$$u_{n+1} = \frac{(3^{n+3})!}{(n+1)! (n+2)! (n+3)!}$$

$$u_{n+1} = \frac{(3n+3)(3n+2)(3n+1)3n!}{(n+1)n! \cdot (n+2)(n+1)n! \cdot (n+3)(n+2)(n+1)n!}$$

$$u_n = \frac{3n!}{n! (n+1)! n! (n+2)(n+1)n!}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} =$$

$$\lim_{n \rightarrow \infty} \frac{(3n+3)(3n+2)(3n+1)3n!}{(n+1)n! \cdot (n+2)(n+1)n! \cdot (n+3)(n+2)(n+1)n!}$$

$$\frac{3n!}{n! (n+1)! n! (n+2)(n+1)n!}$$

$$= \lim_{n \rightarrow \infty} \frac{3(n+1)(3n+2)(3n+1)}{(n+3)(n+2)(n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{3(3 + \frac{2}{n})(3 + \frac{1}{n})}{(1 + \frac{3}{n})(1 + \frac{2}{n})}$$

$$= \frac{3 \times 3 \times 3}{1 \times 1} = 27 > 1$$

divergent series

$$\begin{aligned} \underline{Q9} : \quad & 10x_1 + 2x_2 - x_3 = 27 \\ & -3x_1 - 6x_2 + 2x_3 = -61.5 \\ & x_1 + x_2 + 5x_3 = -21.5 \end{aligned}$$

$$|A| = \begin{vmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{vmatrix} = 10(-30-2) - 2(-15-2) - 1(-3+6) \\ = -289$$

$$[A] = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\text{cof. } A = \begin{bmatrix} -32 & 17 & 3 \\ -11 & 51 & -8 \\ -2 & -17 & -54 \end{bmatrix}$$

$$[\text{cof. } A]^T = \begin{bmatrix} -32 & -11 & -2 \\ 17 & 51 & -17 \\ 3 & -8 & -54 \end{bmatrix}$$

$$[A]^{-1} = \frac{[\text{cof. } A]^T}{|A|} = \begin{bmatrix} 0.111 & 0.038 & 6.92 \times 10^{-3} \\ -0.059 & -0.176 & 0.059 \\ -0.01 & 0.028 & 0.187 \end{bmatrix}$$

$$[x] = [A]^{-1} \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 8.0 \\ -6.0 \end{bmatrix} \begin{matrix} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{matrix}$$

$$Q_{10} \quad \sin z \cdot \cos z = \frac{\sin 2z}{2}$$

Sol.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{--- (1)}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{--- (2)}$$

$$\begin{aligned} \therefore \sin z \cdot \cos z &= \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iz} + e^{-iz}}{2} \\ &= \frac{1}{4i} (e^{2iz} - e^{-2iz}) \end{aligned}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i2z} = \cos 2z + i \sin 2z$$

$$e^{-i2z} = \cos 2z - i \sin 2z$$

$$e^{2iz} - e^{-2iz} = \cos 2z + i \sin 2z - (\cos 2z - i \sin 2z)$$

$$\frac{e^{2iz} - e^{-2iz}}{4i} = \frac{\cancel{\cos 2z} + i \sin 2z - (\cancel{\cos 2z} - i \sin 2z)}{4i}$$

$$= \frac{2i \sin 2z}{4i}$$

$$= \frac{\sin 2z}{2}$$

$$L.H.S = R.H.S$$

∴ Proved