University Of Technology		
Building and Construction Eng. Dept.		
Final Exam – 2013/2014		
Subject : Mathematics II	Class: 2^{nd}	
Branch : All branches	Time : 3 Hours	
Date : 4/6/ 2014		

Note: Answer only **eight** questions.

Q1: The discharge of water (Q) over a horizontal weir is given by $Q = C_d L H^{3/2}$, where H is the head over the weir, L is the length of the weir and assume Cd is constant . What is the percentage error in Q caused by an error in measuring H of ±0.2% and L of ±1% if the average value of H_o=2m and L_o =4m.

Q2: In which direction is the directional derivative of the function $w(x,y) = \frac{x^2-y^2}{x^2+y^2}$ at the point (1,1): a- equal to zero, b- Increase must rapidly.

Q3: Solve the following differential equation using undetermined coefficient method.
$y'' + 3y' + 2y = e^{-x} + e^{2x}$

Q4: Show that the vectors A=3i-2j+k, B=i-3j+5k and C=2i+j-4k form a right triangle.

Q5: The acceleration of a particle in the plane as a function of time (t) is a(t)=-3ti, when the particle's velocity and position are v(0)=2j and R(0)=4i. Find the position (x,y) of this particle when t=2sec.

Q6: Find the position $\bar{x}$ of the center of gravity for a thin plate bounded by the curve $y=x^2$ and x=2 and has a variable density δ =1+x+xy. At the first quadrant

Q7: Use the spherical coordinates to solve the following integral for the solid bounded above by the spherical surface $z = (4 - x^2 - y^2)^{1/2}$ and below by the xy-plane.

$$\iiint z^2.(x^2+y^2+z^2)^{\frac{1}{2}}.dv$$

Q8: Find the inverse of the matrix [A] given by $\begin{vmatrix} 6 & -1 & 5 \\ 2 & 1 & 3 \\ 4 & 0 & -2 \end{vmatrix}$ and show that: $[A]^{-1}.[A]=1$.

Q9: Find the nature of the following series:

a- $\sum_{n=1}^{n=\propto} \frac{n!x^{-n}}{(n+1)!}$

b- $\sum_{n=1}^{n=\propto} ln\frac{n}{(n+1)}$

Q10: A: Find $W=(i)^{\ln i}$

B: Show that $cosZ = cosx.coshy - isinx.sinhy$

Good Luck

Q_1 //

$$Q = C_d L H^{1.5} \qquad Q = f(L, H)$$

$$dQ = \frac{\partial Q}{\partial L} dL + \frac{\partial Q}{\partial H} dH$$

$$\frac{\partial Q}{\partial L} = C_d H^{1.5}$$

$$\frac{\partial Q}{\partial H} = 1.5 C_d L H^{0.5}$$

$$dL \simeq \Delta L = \mp \frac{1}{100}$$

$$dH \simeq \Delta H = \mp \frac{0.2}{100}$$

$$dQ = \mp C_d H^{1.5} \left(\frac{1}{100}\right) \mp 1.5 C_d L H^{0.5} \left(\frac{0.02}{100}\right)$$

for $H = 2\,m$ and $L = 4\,m$

$$dQ = \mp \frac{C_d (2)^{1.5}}{100} \mp 1.5\, C_d (4)(2)^{0.5} \left(\frac{0.02}{100}\right)$$

$$dQ = \mp 0.02828\, C_d \mp 0.00169\, C_d$$

$$dQ = \mp 0.03\, C_d = \Delta Q$$

$$Q_{\substack{H=2 \\ L=4}} = C_d (4)(2)^{1.5} = 11.3137\, C_d$$

$$\text{Percentage error} = \frac{\Delta Q}{Q} = \frac{\mp 0.003\, C_d}{11.3137\, C_d} = 0.027\%$$

Q2/

$$w = f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial w}{\partial x} = \frac{2x(x^2+y^2) - 2x(x^2-y^2)}{(x^2+y^2)^2} \Rightarrow \left.\frac{\partial w}{\partial x}\right|_{\substack{x=1\\y=1}} = \frac{2(1)(1^2+1^2) - 2(1)(1^2-1)}{(1^2+1^2)^2} = 1$$

$$\frac{\partial w}{\partial y} = \frac{-2y(x^2+y^2) - 2y(x^2-y^2)}{(x^2+y^2} \Rightarrow \left.\frac{\partial w}{\partial y}\right|_{\substack{x=1\\y=1}} = \frac{-2(1)(1^2+1^2) - 2(1)(1^2+1^2)}{(1^2+1^2)} = -1$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\therefore \frac{dw}{ds} = \cos\theta - \sin\theta$$

a- for $\frac{dw}{ds} = 0 \Rightarrow \cos\theta - \sin\theta = 0 \Rightarrow \tan\theta = 1$

$$\theta = \tan^{-1}(1) \Rightarrow \theta = \frac{\pi}{4}$$

b- for $\frac{dw}{ds} = max \Rightarrow \frac{d\left(\frac{dw}{ds}\right)}{d\theta} = 0$

$$\frac{d}{d\theta}\left(\frac{dw}{ds}\right) = -\sin\theta - \cos\theta = 0 \Rightarrow \tan\theta = -1$$

$$\theta = \tan^{-1}(-1) \Rightarrow \theta = -\frac{\pi}{4}$$

Q3// $y'' + 3y' + 2y = e^{-x} + e^{2x}$ --- ①

Sol/ $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r_1 = -1$ & $r_2 = -2$

$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, $r_1 \neq r_2$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

$y_p = Axe^{-x} + Be^{2x}$

$y'_p = Ae^{-x} - Axe^{-x} + 2Be^{2x}$

$y''_p = -Ae^{-x} - Ae^{-x} + Axe^{-x} + 4Be^{2x}$

$y''_p = -2Ae^{-x} + Axe^{-x} + 4Be^{2x}$

sub in (1)

$-2Ae^{-x} + Axe^{-x} + 4Be^{2x} + 3Ae^{-x} - 3Axe^{-x} + 6Be^{2x}$
$+ 2Axe^{-x} + 2Be^{2x} = e^{-x} + e^{2x}$

$Ae^{-x} + 12Be^{2x} = e^{-x} + e^{2x}$

$\therefore A = 1$ & $B = \frac{1}{12}$

$y_p = xe^{-x} + \frac{1}{2} e^{2x}$

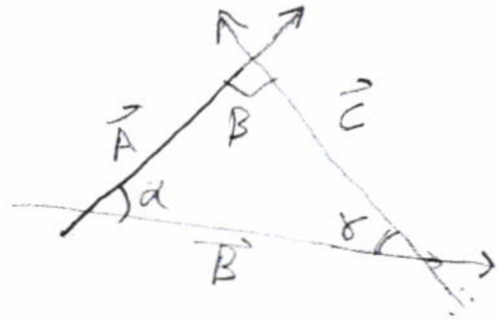
$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x} + \frac{1}{2} e^{2x}$

Q4

$\vec{A} = 3i - 2j + K$

$\vec{B} = i - 3j + 5K$

$\vec{C} = 2i + j - 4K$



$$\alpha = \cos^{-1} \frac{A \cdot B}{|A||B|}$$

$$\alpha = \cos^{-1} \frac{3(1) - 2(-3) + 1(5)}{\sqrt{3^2+2^2+1^2}\ \sqrt{1^2+3^2+5^2}} \Rightarrow \alpha = \cos^{-1}\left(\frac{14}{\sqrt{14}\ \sqrt{35}}\right)$$

$\alpha = 50.76^{\circ}$

$$B = \cos^{-1} \frac{A \cdot C}{|A||C|} = \cos^{-1} \frac{3(2) - 2(1) + 1(-4)}{\sqrt{14}\ \sqrt{2^2+1^2+4^2}} = \cos^{-1}\left(\frac{0}{\sqrt{14}\ \sqrt{21}}\right)$$

$B = 90^{\circ}$

$$\gamma = \cos^{-1} \frac{-B \cdot \vec{C}}{|B||C|} = \cos^{-1} \frac{-[1(2) - 3(1) + 5(-4)]}{\sqrt{35}\ \sqrt{21}} = \cos^{-1}\left(\frac{21}{\sqrt{35}\ \sqrt{21}}\right)$$

$\gamma = 39.32^{\circ}$

$$\alpha + B + \gamma = 50.76^{\circ} + 90^{\circ} + 39.32^{\circ} = 180^{\circ}$$

Q5//

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Q5//

$a(t) = -3t\,i = \frac{dv}{dt}$

$v(t) = \int a(t)\,dt \Rightarrow v(t) = \int(-3t\,i)\,dt + C_1$

$v(t) = \frac{-3}{2}t^2 i + C_1$

for $v(0) = 2j \Rightarrow 2j = -\frac{3}{2}(0)^2 i + C_1 \Rightarrow C_1 = 2j$

$\therefore V(t) = -\frac{3}{2}t^2 i + 2j = \frac{dR}{dt}$

$R(t) = \int v(t)\,dt$

$R(t) = \int(-\frac{3}{2}t^2 i + 2j)\,dt + C_2$

$R(t) = -\frac{3}{2}\frac{t^3}{3} i + 2jt + C_2$

$R(t) = -\frac{t^3}{2} i + 2jt + C_2$

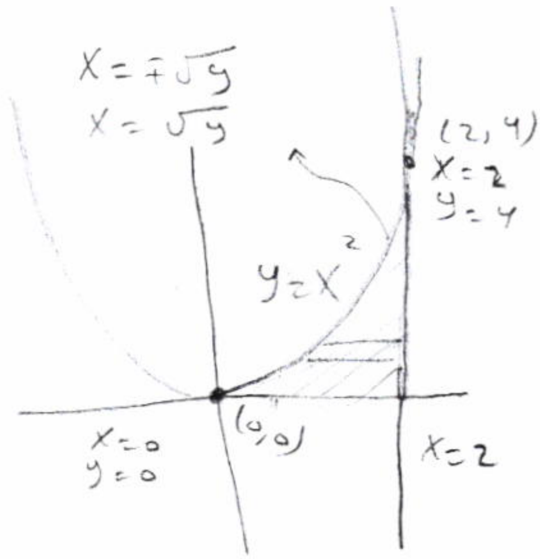
for $R(0) = 4i \Rightarrow 4i = \frac{-(0)^3}{2} i + 2j(0) + C_2 \Rightarrow C_2 = 4i$

$R(t) = -\frac{t^3}{2} i + 2tj + 4j \Rightarrow R(t) = (4 - \frac{t^3}{2})i + 2tj$

$\therefore R(2) = (4 - \frac{2^3}{2})i + 2(2)j \Rightarrow R(2) = (0)i + 4j$

$R(t) = Xi + yj \Rightarrow \boxed{X = 0\ ,\ y = 4}$

Q6 // $\bar{X} = \frac{My}{m}$



$* \quad m = \iint \delta(x,y)\, dx\, dy$

$m = \int_0^4 \int_{x=\sqrt{y}}^{2} (1 + x + xy)\, dx\, dy$

$m = \int_0^4 \left[x + \frac{x^2}{2} + \frac{x^2}{2} y \right]_{y^{\frac{1}{2}}}^{2} dy$

$m = \int_0^4 \left[\left(2 + \frac{2^2}{2} + \frac{2^2}{2} y\right) - \left(y^{\frac{1}{2}} + \frac{y}{2} + \frac{y^2}{2} \right) \right] dy$

$m = \int_0^4 \left(4 + \frac{3}{2} y - y^{\frac{1}{2}} - \frac{y^2}{2} \right) dy$

$m = \left[4y + \frac{3}{4} y^2 - \frac{2}{3} y^{\frac{3}{2}} - \frac{y^3}{6} \right]_0^4 \Rightarrow m = \left[4(4) + \frac{3(4)^2}{4} - \frac{2}{3} 4^{\frac{3}{2}} - \frac{4^3}{6} \right]$

$16 + 12 - 8$

$m = 12$

$* \quad My = \iint x\, \delta(x,y)\, dx\, dy$

$My = \int_0^4 \int_{\sqrt{y}}^{2} x(1 + x + xy)\, dx\, dy \Rightarrow My = \int_0^4 \int_{\sqrt{y}}^{2} (x + x^2 + x^2 y)\, dx\, dy$

$My = \int_0^4 \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{3} y \right]_{\sqrt{y}}^{2} dy$

$My = \int_0^4 \left[\left(\frac{2^2}{2} + \frac{2^3}{3} + \frac{2^3}{3} y \right) - \left(\frac{y}{2} + \frac{y^{\frac{3}{2}}}{3} + \frac{y^{\frac{3}{2}} \cdot y}{3} \right) \right] dy$

$My = \int_0^4 \left(2 + \frac{8}{3} + \frac{8}{3} y\right) - \left(\frac{y}{2} + \frac{y^{\frac{3}{2}}}{3} + \frac{y^{\frac{5}{2}}}{3} \right) dy$

$$M_y = \int_0^4 \left(4.667 + 2.6667y - \frac{y^{\frac{3}{2}}}{3} - \frac{y^{\frac{5}{2}}}{3}\right) dy$$

$$M_y = \left[4.667y + \frac{2.1667y^2}{2} - \frac{2}{15}y^{\frac{5}{2}} - \frac{2}{21}y^{\frac{7}{2}}\right]_0^4$$

$$M_y = \left[4.667(4) + \frac{2.667(4^2)}{2} - \frac{2}{15}(4)^{\frac{5}{2}} - \frac{2}{21}(4)^{\frac{7}{2}}\right) - 0$$

$$M_y = 23.54$$

$$\bar{X} = \frac{23.54}{12} \Longrightarrow \boxed{\bar{X} = 1.96 \text{ unit}}$$

Q7//

$$Z=(4-x^2-y^2)^{\frac{1}{2}}$$

$$Z^2=4-x^2-y^2 \Rightarrow x^2+y^2+Z^2=4 \Rightarrow \rho^2=4 \Rightarrow \rho=2$$

$$Z=\rho\cos\phi \quad \& \quad \rho=(x^2+y^2+Z^2)^{\frac{1}{2}}$$

$$\iiint Z^2(x^2+y^2+Z^2)^{\frac{1}{2}}dV=\iiint \rho^2\cos^2\phi\cdot\rho\, dV$$

$$=\iiint \rho^3\cos^2\phi\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\int_0^{\pi}\int_0^{\pi}\int_0^{2}\rho^5\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\int_0^{\pi}\int_0^{\pi}\left[\frac{\rho^6}{6}\right]_0^2\cos^2\phi\sin\phi\, d\phi\, d\theta=\frac{64}{6}\int_0^{\pi}\int_0^{\pi}\cos^2\phi\sin\phi\, d\phi\, d\theta$$

$$=\frac{64}{6}\int_0^{\pi}\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi}d\theta=\frac{64}{18}\int_0^{\pi}\left[-\cos^3(\pi)-(-\cos^3(0))\right]d\theta$$

$$=\frac{64}{18}\int_0^{\pi}2\, d\theta=\frac{64}{9}\int_0^{\pi}\theta$$

$$\boxed{=\frac{64}{9}\pi}.$$

Q8/ $|A| = \begin{vmatrix} 6 & -1 & 5 \\ 2 & 1 & 3 \\ 4 & 0 & -2 \end{vmatrix}$

$|A| = 6\left[-2-0\right] - (-1)\left[-4-12\right] + 5\left[0-4\right]$

$|A| = -12 - 16 - 20 = -48$

$\text{Cof. } A = \begin{bmatrix} +A_{11} & -A_{12} & +A_{13} \\ -A_{21} & +A_{22} & -A_{23} \\ +A_{31} & -A_{32} & +A_{33} \end{bmatrix}$

$A_{11} = +\begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} = -2$ $\quad A_{12} = -\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -(-4-12)$
$= +16$

$A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = -4$. $\quad A_{21} = -\begin{vmatrix} -1 & 5 \\ 0 & -2 \end{vmatrix} = -2$

$A_{22} = \begin{vmatrix} 6 & 5 \\ 4 & -2 \end{vmatrix} = -12 - 20 = -32$

$A_{23} = -\begin{vmatrix} 6 & -1 \\ 4 & 0 \end{vmatrix} = -(4) = -4$ $\quad A_{31} = \begin{vmatrix} -1 & 5 \\ 1 & 3 \end{vmatrix} = -8$

$A_{32} = -\begin{vmatrix} 6 & 5 \\ 2 & 3 \end{vmatrix} = -(18-10) = -8$

$A_{33} = \begin{vmatrix} 6 & -1 \\ 2 & 1 \end{vmatrix} = 6 + 2 = 8$

$\text{Cof. } A = \begin{bmatrix} -2 & 16 & -4 \\ -2 & -32 & -4 \\ -8 & -8 & 8 \end{bmatrix}$

$\text{adj. } A = (\text{Cof. } A)^T = \begin{bmatrix} -2 & -2 & -8 \\ 16 & -32 & -8 \\ -4 & -4 & 8 \end{bmatrix}$

$$A^{-1} = \frac{adj.A}{|A|} = \frac{1}{-48} \cdot \begin{bmatrix} -2 & -2 & -8 \\ 16 & -32 & -8 \\ -4 & -4 & 8 \end{bmatrix}$$

$$\therefore A^{-1} \cdot A = \frac{1}{-48} \begin{bmatrix} -2 & -2 & -8 \\ 16 & -32 & -8 \\ -4 & -4 & 8 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 & 5 \\ 2 & 1 & 3 \\ 4 & 0 & -2 \end{bmatrix}$$

$$= \frac{1}{-48} \begin{bmatrix} (-12-4-32) & (2-2) & (-10-6+16) \\ (96-64-32) & (-16-32) & (80-96+16) \\ (-24-8+32) & (4-4) & (-20-12-16) \end{bmatrix}$$

$$= \frac{1}{-48} \begin{bmatrix} -48 & 0 & 0 \\ 0 & -48 & 0 \\ 0 & 0 & -48 \end{bmatrix}$$

$$\therefore [A]^{-1} \cdot [A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [I]$$

Qg 11

a- $\sum_{n=1}^{n=\infty} \frac{n!\,x^{-n}}{(n+1)!}$

$$a_n = \frac{n!\,x^{-n}}{(n+1)n!} \Rightarrow a_n = \frac{x^{-n}}{(n+1)}$$

$$a_{n+1} = \frac{x^{-(n+1)}}{((n+1)+1)} \Rightarrow a_{n+1} = \frac{x^{-n} \cdot x^{-1}}{n+2}$$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{\cancel{x^{-n}} \cdot x^{-1}}{n+2}}{\frac{\cancel{x^{-n}}}{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{(n+2)x} = \lim_{n\to\infty} \frac{\frac{n}{n}+\frac{1}{n}}{(\frac{n}{n}+\frac{2}{n})x}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{(1+\frac{2}{n})x} \Rightarrow \rho = \frac{1+\frac{1}{\infty}}{(1+\frac{2}{\infty})x} \Rightarrow \rho = \frac{1}{x}$$

$\therefore$ the series is convergence if $x > 1$

Q9//

b. $\sum_{n=1}^{n=\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{n=\infty} [\ln n - \ln(n+1)]$

$S_n = [\ln(1) - \cancel{\ln(2)}] + [\cancel{\ln(2)} - \cancel{\ln(3)}] + [\cancel{\ln 3} - \ln 4] + \dots$

$+ [\ln(\cancel{n-1}) - \cancel{\ln(n)}] + [\cancel{\ln(n)} - \ln(n+1)]$

$= \ln(1) - \ln(n+1) = -\ln(n+1)$

nth term test

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1)$

$= -\ln(\infty) = -\infty$

divergence

$Q_{10}/ \quad w = i^{\ln i}$

$w = a^z \qquad a = i,$
$\qquad\qquad z = \ln i$

$$w = a^z = \exp\left[z\left(\ln|a| + i(\theta_a + 2\pi K)\right)\right]$$

$a = i \Rightarrow |a| = \sqrt{1^2} \Rightarrow |a| = 1$

$\theta = \cos^{-1}\frac{0}{1} \Rightarrow \theta = \frac{\pi}{2}$

$\therefore w = \exp\left[z\left(\ln(1) + i\left(\frac{\pi}{2} + 2\pi K\right)\right)\right]$

$w = \exp\left[zi\left(\frac{\pi}{2} + 2\pi K\right)\right]$

$z = w_2 = \ln i = \ln z$

$w_2 = \ln z = \ln|z| + i(\theta + 2\pi K)$

$z = i \Rightarrow |z| = \sqrt{1^2} \Rightarrow |z| = 1$

$\theta = \cos^{-1}\frac{0}{1} \Rightarrow \theta = \frac{\pi}{2}$

$w_2 = \ln(1) + i\left(\frac{\pi}{2} + 2\pi K\right)$

$w_2 = 0 + i\left(\frac{\pi}{2} + 2\pi K\right)$

$w_2 = i\left(\frac{\pi}{2} + 2\pi K\right)$

$w = \exp\left[i\left(\frac{\pi}{2} + 2\pi K\right) i\left(\frac{\pi}{2} + 2\pi K\right)\right]$

$w = \exp\left(-\left(\frac{\pi}{2} + 2\pi K\right)^2\right)$

Q10/B

$\cos Z = \cos x \cdot \cosh y - i \sin x \cdot \sinh y$

$$\cos Z = \cos(x+yi) = \frac{e^{(x+yi)i} + e^{-(x+yi)i}}{2}$$

$$= \frac{e^{xi-y} + e^{-x+yi}}{2} = \frac{e^{xi} \cdot e^{-y} + e^{-xi} \cdot e^{y}}{2}$$

$$= \frac{(\cos x + i\sin x)\cdot e^{-y} + (\cos x - i\sin x)\cdot e^{y}}{2}$$

$$= \frac{\cos x e^{-y} + i\sin x e^{-y} + \cos x e^{y} - i\sin x e^{y}}{2}$$

$$= \cos x \frac{(e^{-y}+e^{y})}{2} + i\sin x \frac{(e^{-y}-e^{y})}{2}$$

$$= \cos x \frac{(e^{y}+e^{-y})}{2} + i\sin x(-1)\frac{(e^{y}-e^{-y})}{2}$$

$$= \cos x \cosh y + i \sin x \sinh y = R.H.S$$