University of Technology

Building and Construction Eng. Dept.

Final Exam- First Attempt – 2012/2013

Branch : All Branches — Class: 2nd year

Subject : Mathematics II — Time : 3 Hours

Examiner : Mathematics Committee — Date : 17/6/ 2013

Note: Answer Eight questions including Q1

Q1. IF $y = A\sin\frac{r\pi x}{b}.\sin\frac{r\pi ct}{b}$ where A, b, c and r are constants, prove that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$ (16 marks)

Q2. The directional derivative of a given function $W=f(x,y)$ at $P_0(0,2)$ in the direction toward $P_1(3,2)$ is 2 and in direction toward $P_2(3,5)$ is 4. Find the directional derivative of W in the direction of y - axis. (12 marks)

Q3. Solve the following differential equation: $2xy.dx = (x^2 + y^2)dy$ (12 marks)

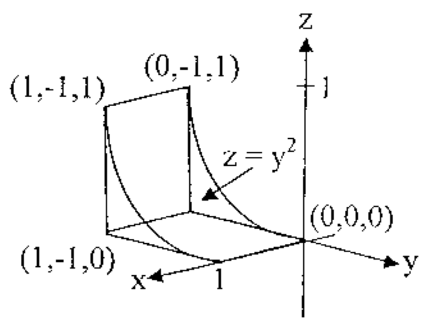
Q4. Solve the following differential equation using method of undetermined. (12 marks)

$$y'' - y' - 6y = e^{-x} - 7\cos x$$

Q5. The position of a particle at time is given by the equations $x = 2(\sin 2t + 1)$, $y = 2(\cos 2t - 1)$. Find the velocity vector, speed, tangent vector, normal vector and B-vector at $t = \frac{\pi}{2}$. (12 marks)

Q6. Find My for the region enclosed by the curve $y = 2\sqrt{x}$, the line $y = 4$ and the y – axis. (12 marks)

Q7. Write the five integral forms and solve one for the solid shown below. (12 marks)



Q8. Test the following series by ratio test. $\sum_{n=1}^{\infty}\frac{(3n)!}{n!(n+1)!(n+2)!}$ (12 marks)

Q9. Solve the following system of linear equations using matrix method. (12 marks)

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

Q10. For the complex function, show that: $\sin z.\cos z = \frac{\sin 2z}{2}$ (12 marks)

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Q. $y = A \sin \frac{2\pi x}{b} \cdot \sin \frac{2\pi c t}{b}$

Sol.

$$\frac{\partial y}{\partial x} = A \cos \frac{2\pi}{b}x \left(\frac{2\pi}{b}\right) \cdot \sin \frac{2\pi c t}{b}$$

$$\frac{\partial^2 y}{\partial x^2} = -A \sin\left(\frac{2\pi}{b}x\right)\left(\frac{2\pi}{b}\right)^2 \cdot \sin \frac{2\pi c t}{b}$$

$$= -\frac{A 2^2 \pi^2}{b^2} \sin \frac{2\pi x}{b} \cdot \sin \frac{2\pi c t}{b}$$

$$\frac{\partial y}{\partial t} = A \sin \frac{2\pi x}{b} \cdot \cos \frac{2\pi c t}{b} \cdot \left(\frac{2\pi c}{b}\right)$$

$$\frac{\partial^2 y}{\partial t^2} = A \sin \frac{2\pi x}{b}\left(-\sin \frac{2\pi}{b} c t\right)\left(\frac{2\pi c}{b}\right)^2$$

$$= -\frac{A 2^2 \pi^2 c^2}{b^2} \cdot \sin \frac{2\pi}{b} x \cdot \sin \frac{2\pi c t}{b}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t}$$

$$-\frac{\cancel{A}\, \cancel{2^2}\, \cancel{\pi^2}}{b^2} \sin \cancel{\frac{2\pi x}{b}} \cdot \sin \cancel{\frac{2\pi c t}{b}} =$$

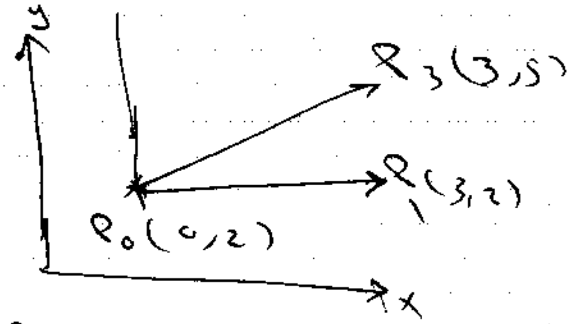
$$\cancel{\frac{1}{c^2}}\left(-\cancel{A}\, \frac{\cancel{2^2}\, \pi^2 \cancel{c^2}}{\cancel{b^2}}\right) \sin \cancel{\frac{2\pi x}{b}} \cdot \sin \cancel{\frac{2\pi c t}{b}}$$

R.H.S = L.H.S

o.k

Q2

$$\frac{dw}{ds} = \frac{\partial f}{\partial x}\cos\varphi + \frac{\partial f}{\partial y}\sin\varphi$$



$$P_0P_1 = (3-0)i + (2-2)j$$

$$= 3i$$

$$u = \frac{3i}{\sqrt{3^2+0^2}} = i \Rightarrow \varphi = 0$$

$$2 = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(0)$$

$$\therefore \frac{\partial f}{\partial x} = 2 \quad \text{--- (1)}$$

$$P_0P_2 = (3-0)i + (5-2)j$$

$$= 3i + 3j$$

$$u = \frac{3i+3j}{\sqrt{3^2+3^2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j \Rightarrow \varphi = 45$$

$$4 = \frac{\partial f}{\partial x}\left(\frac{1}{\sqrt{2}}\right) + \frac{\partial f}{\partial y}\frac{1}{\sqrt{2}} \quad \text{-- (2)}$$

Sub $\frac{\partial f}{\partial x} = 2$

$$4 = \frac{1}{\sqrt{2}}\left(\frac{\partial f}{\partial x}\right) + \frac{1}{\sqrt{2}}\frac{\partial f}{\partial y}$$

$$4 = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{\partial f}{\partial y} \Rightarrow \frac{\partial f}{\partial y} = 4\sqrt{2} - 2$$

$$\left.\frac{dw}{ds}\right\}_{max\ is} = 2\cos\varphi + (4\sqrt{2}-2)\sin\varphi$$

so... $\varphi = \frac{\pi}{2}$

$$\therefore \frac{dw}{ds} = 2\cos\frac{\pi}{2} + (4\sqrt{2}-2)\sin\frac{\pi}{2}$$

$$\therefore \frac{dw}{ds} = 4\sqrt{2} - 2$$

Q3:- $2xy \cdot dx = (x^2+y^2)\,dy$

Sol.

$$2xy\,dx - (x^2+y^2)\,dy = 0 \quad \big] \div 2xy\,dy$$

$$\frac{dx}{dy} - \frac{1}{2}\frac{x}{y} - \frac{1}{2}\frac{y}{x} = 0$$

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) = 0$$

let $v = \frac{x}{y} \Rightarrow f(v) = \frac{1}{2}\left(v + \frac{1}{v}\right)$

$$\frac{dy}{y} + \frac{dv}{v - f(v)} = 0$$

$$\frac{dy}{y} + \frac{dv}{v - \frac{v}{2} + \frac{1}{2v}} = 0$$

$$\frac{dy}{y} + \frac{dv}{\frac{v}{2} - \frac{1}{2v}} = 0$$

$$\frac{dy}{y} + \frac{dv}{\frac{2v^2-2}{4v}} = 0$$

$$\frac{dy}{y} + 2\int \frac{v \cdot dv}{v^2-1} = 0$$

$$\therefore \ln y + \ln(v^2-1) = c$$

$\ln[y \cdot (v^2-1)] = c$ $\quad e^{x}$

$$y(v^2-1) = e^c = c_1$$

$$y\left(\frac{x^2}{y^2} - 1\right) = c_1 \Rightarrow \boxed{\frac{x^2}{y} - y \pm c}$$

Q4 $\quad y'' - y' - 6y = e^{-x} - 7\cos x$

$$y'' - y' - 6y = 0$$

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0 \qquad r_1 = 3 \qquad r_2 = -2$$

$$\therefore y_h = C_1 \cdot e^{3x} + C_2 \cdot e^{-2x}$$

$$y_p = y_{p_1} + y_{p_2}$$

$\underline{y_{p_1}}$: $F_1(x) = e^{-x}$ $\quad r = -1 \neq r_1 \neq r_2$

$F_2(x) = -7\cos x$ $\quad k = 1$ and real roots

$$\therefore y_p = Ae^{-x} + B\sin x + C\cos x$$

$$y'_p = -Ae^{-x} + B\cos x - C\sin x$$

$$y''_p = Ae^{-x} - B\sin x - C\cdot\cos x$$

$$\underline{Ae^{-x}} - B\sin x - C\cos x + \underline{Ae^{-x}} - B\cos x + C\sin x$$

$$-\underline{6Ae^{-x}} - 6B\sin x - 6C\cos x$$

$$= \underline{e^{-x}} - 7\cos x$$

equ ① : $A + A - 6A = 1 \Rightarrow A = \frac{-1}{4}$

equ ② : $-B + C - 6B = 0 \Rightarrow C = 7B$

equ ③ : $-C - B - 6C = -7$

$$-7(7B) - B = -7 \qquad B = \frac{-7}{-50} = \frac{7}{50}$$

$$\therefore C = 7 * \frac{7}{50} = \frac{49}{50}$$

Qu// $\therefore$ $y_p = \frac{-1}{4} e^{-x} + \frac{7}{50} \sin x + \frac{49}{50} \cos x$

then $y = y_h + y_p$

$$y = c_1 \cdot e^{3x} + c_2 e^{-2x} - \frac{1}{4} e^{-x} + \frac{7}{50} \sin x + \frac{49}{50} \cos x$$

Q5 $x = 2(\sin 2t + 1)$

$y = 2(\cos 2t - 1)$

Sol.

$\vec{R} = xi + yj$

$= (2\sin 2t + 1)i + 2(\cos 2t - 1)j$

$\vec{v} = 4\cos 2t \, i \quad 4\sin 2t \, j$

$\vec{v} \ @ \ \frac{\pi}{2} = -4i$

$\text{speed} = |\vec{v}| = \sqrt{(4\cos 2t)^2 + (-4\sin 2t)^2}$

$= 2$

$T = \frac{\vec{v}}{|\vec{v}|} = 2\cos 2t \, i - 2\sin 2t \, j$

$\vec{T} \ @ \ \frac{\pi}{2} = -2i$

$\vec{N} = 2\sin 2t \, i + 2\cos 2t \, j$

$\vec{N} \ @ \ \frac{\pi}{2} = -2j$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{vmatrix}$$

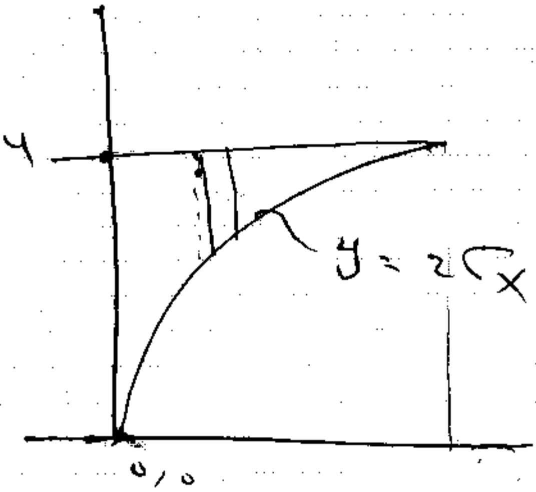
$= (0)i + (0)j + 4k$

$\vec{B} = 4k$

Q6

$$M_y = \iint x\, \delta(x,y)\, dA$$

$$M_y = \int_0^4 \int_{y=2\sqrt{x}}^{y=4} x \cdot dy \cdot dx$$



$$= \int_0^4 x[y]_{2\sqrt{x}}^{4} \cdot dx$$

$$= \int_0^4 x(4-2\sqrt{x})\, dx$$

$4 = 2\sqrt{x}$

$\sqrt{x} = 2$

$x = 4$

$$M_y = \left[\frac{4}{2}x^2 - 2\,\frac{x^{5/2}}{\frac{5}{2}}\right]_0^4$$

$$= 2 \times 16 - \frac{4}{5}(4)^{5/2}$$

$$= 6.4$$

Q7:

① $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz \cdot dy \cdot dx$

$= \int_0^1 \int_{-1}^0 [z]_0^{y^2} \cdot dy \cdot dx = \int_0^1 \int_{-1}^0 y^2 \cdot dy \cdot dx$

$= \int_0^1 \left[\frac{y^3}{3}\right]_{-1}^0 \cdot dx = \frac{1}{3}\int_0^1 dx = \frac{1}{3}[x]_0^1 = \frac{1}{3} = 0.333$

② $\int_{-1}^0 \int_0^1 \int_0^{y^2} dz \cdot dx \cdot dy$

③ $\int_0^{y^2} \int_{-1}^0 \int_0^1 dx \cdot dy \cdot dz$

④ $\int_{-1}^0 \int_0^{y^2} \int_0^1 dx \cdot dz \cdot dy$

⑤ $\int_0^{y^2} \int_0^1 \int_{-1}^0 dy \cdot dx \cdot dz$

Q8: $u_n = \dfrac{3n!}{n!\,(n+1)!\,(n+2)!}$

$$u_{n+1} = \frac{(3n+3)!}{(n+1)!\,(n+2)!\,(n+3)!}$$

$$u_{n+1} = \frac{(3n+3)(3n+2)(3n+1)\,3n!}{(n+1)n! \cdot (n+2)(n+1)n! \cdot (n+3)(n+2)(n+1)n!}$$

$$u_n = \frac{3n!}{n!\,(n+1)\cdot n!\,(n+2)(n+1)\,n!}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} =$$

$$= \lim_{n\to\infty} \frac{\dfrac{(3n+3)(3n+2)(3n+1)\,\cancel{3n!}}{(n+\cancel{1})\,\cancel{n!}\cdot(n+\cancel{2})(n+\cancel{1})\,\cancel{n!}\cdot(n+3)(n+2)(n+1)\,\cancel{n!}}}{\dfrac{\cancel{3n!}}{\cancel{n!}\,(n+\cancel{1})\cdot\cancel{n!}\,(n+\cancel{2})(n+\cancel{1})\,\cancel{n!}}}$$

$$= \lim_{n\to\infty} \frac{3(n+\cancel{1})(3n+2)(3n+1)}{(n+3)(n+2)\cancel{(n+1)}}$$

$$= \lim_{n\to\infty} \frac{3\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)}{\left(1+\frac{3}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \frac{3 * 3 * 3}{1 * 1} = 27 > 1$$

divergent series

Q9: $10x_1 + 2x_2 - x_3 = 27$

$-3x_1 - 6x_2 + 2x_3 = -61.5$

$x_1 + x_2 + 5x_3 = -21.5$

$$|A| = \begin{vmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{vmatrix} = 10(-30-2) - 2(-15-2) - 1(-3+6)$$

$$= -289$$

$$[A] = \begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\text{Cof. } A = \begin{bmatrix} -32 & 17 & 3 \\ -11 & 51 & -8 \\ -2 & -17 & -54 \end{bmatrix}$$

$$[\text{Cof. } A]^T = \begin{bmatrix} -32 & -11 & -2 \\ 17 & 51 & -17 \\ 3 & -8 & -54 \end{bmatrix}$$

$$[A]^{-1} = \frac{[\text{Cof. } A]^T}{|A|} = \begin{bmatrix} 0.111 & 0.038 & 6.92*10^{-3} \\ -0.059 & -0.176 & 0.059 \\ -0.01 & 0.028 & 0.187 \end{bmatrix}$$

$$[x] = [A]^{-1} \begin{bmatrix} 27 \\ -61.5 \\ -21.5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 8.0 \\ -6.0 \end{bmatrix} \begin{matrix} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{matrix}$$

Q10 $\sin z \cdot \cos z = \dfrac{\sin 2z}{2}$

Sol.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad -(1)$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad -(2)$$

$$\therefore \sin z \cdot \cos z = \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{1}{4i}\left(e^{2iz} - e^{-2iz}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\therefore e^{i2z} = \cos 2z + i\sin 2z$$

$$e^{-i2z} = \cos 2z - i\sin 2z$$

$$\frac{e^{2iz} - e^{-2iz}}{4i} = \frac{\cancel{\cos 2z} + i\sin 2z - (\cancel{\cos 2z} - i\sin 2z)}{4i}$$

$$= \frac{2\cancel{i}\sin 2z}{\cancel{4i}_{2}}$$

$$= \frac{\sin 2z}{2}$$

$$L.H.S = R.H.S$$