Subject : MathMatics

Department : Materal + physics + Applied chemistry

class : Third class

Main Articles :

partial differential equations , boundary value prollem, Some applications on partial differential equation

References :

Elements of partial differential equations

IAN N. SNEDDON

Simson proffesor of MathMatics

University of Glasgow

فرع الفيزياء + الكيمياء   فرع المواد

مدرسين المادة : رغد كاظم صالح + أثير جواد كاظم

محاضرة الفصل الاول

CHAPTER ONE

Partial differential equation

Introduction:- Apartial differential equation is an equation that involves apartial derivatives.

Order:- Is the order of the hights partial derivatives in the equation.

Ex:- write the order of the following equation

1- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (x+y)$ (first order)

2- $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 u}{\partial t^2}$ (Second order)

3- $\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 u}{\partial t^2}$ (Third order)

Remark:- The solution of apartial differential equation is any function which satisfies the equation

Ex:- Verify that $u = f(x,y) = e^x \sin y$ is asolution to the partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol: $\frac{\partial u}{\partial x} = e^x \sin y$, $\frac{\partial u}{\partial y} = e^x \cos y$

$\frac{\partial^2 u}{\partial x^2} = e^x \sin y$, $\frac{\partial^2 u}{\partial y^2} = -e^x \sin y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The separation of variables :-

Is the only Method that used in this chapter to solve this kind of equation in our study

Ex: Solve $\frac{\partial u}{\partial x} = y \frac{\partial u}{\partial y}$

Sol: let $u(x,y) = F(x)\,G(y)$

$\frac{\partial u}{\partial x} = F'G \quad , \quad \frac{\partial u}{\partial y} = FG'$

$\therefore F'G = yFG'$

$\frac{F'G}{FG} = \frac{yFG'}{FG}$ (divided both sides on FG)

$\frac{F'}{F} = y\frac{G'}{G} = K$ (K is a constant)

Now $\frac{F'}{F} = K \Rightarrow F' - KF = 0 \Rightarrow \frac{dF}{dx} - KF = 0$

$\int \frac{dF}{F} = \int K\,dx \Rightarrow \ln F = Kx + C_1$

$\Rightarrow F = e^{Kx} e^{C_1} \Rightarrow \boxed{F = A e^{Kx}}$

And $y\frac{G'}{G} = K \Rightarrow G' - \frac{KG}{y} = 0$

$\frac{dG}{dy} - \frac{KG}{y} = 0 \Rightarrow \int \frac{dG}{G} - \int \frac{K\,dy}{y} = 0$

$\ln G - K\ln y = C_2 \Rightarrow \boxed{G = B y^K}$

$\therefore u(x,y) = F(x)\,G(y)$

$\therefore u(x,y) = A e^{Kx} B y^K = C e^{Kx} y^K$ (where $C = AB$)

Ex: Solve $\frac{\partial u}{\partial x} = 4\frac{\partial^2 u}{\partial y^2}$

Sol: Let $u(x,y) = F(x)\,G(y)$

$$\frac{\partial u}{\partial x} = F'G \quad , \quad \frac{\partial^2 u}{\partial y^2} = FG''$$

$$F'G = 4FG'' = k$$

$$\frac{F'G}{FG} = \frac{4FG''}{FG} = k$$

$$\frac{F'}{F} = \frac{4G''}{G} = k$$

$$\frac{F'}{F} = k \Longrightarrow F' = kF \Longrightarrow \frac{dF}{dx} = kF$$

$$\int \frac{dF}{F} = \int k\,dx \Longrightarrow \ln F = kx + c_1$$

$$\Longrightarrow \boxed{F = A\,e^{kx}}$$

$$\frac{4G''}{G} = k \Longrightarrow 4G'' - kG = 0 \quad \text{———} \; (*)$$

where $(*)$ is the ordinary differential equation of the second order with constant coeffients. The solution of this kind is showing below

$$4r^2 - k = 0 \Longrightarrow r = \pm\sqrt{\frac{k}{4}}$$

Case ① if $k > 0$

$$\therefore G(y) = c_1 e^{r_1 y} + c_2 e^{r_2 y}$$

$$\therefore G(y) = c_1 e^{\frac{\sqrt{k}}{2}y} + c_2 e^{\frac{\sqrt{k}}{2}y}$$

$$\therefore u(x,y) = A\,e^{kx}\left(c_1 e^{\frac{\sqrt{k}}{2}y} + c_2 e^{\frac{\sqrt{k}}{2}y}\right)$$

4. Boundary value problem :-

The problem of finding the solution of a partial differential equation with condition is called a boundary value problem.

Ex: Solve the boundary value problem

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \text{with condition } u(0,y) = 8e^{-3y}$$

Sol: Let $u(x,y) = F(x)\,G(y)$

$$\frac{\partial u}{\partial x} = F'G \quad , \quad \frac{\partial u}{\partial y} = FG'$$

$$F'G = 4FG' \Longrightarrow \frac{F'}{F} = \frac{4G'}{G} = k$$

$$\frac{F'}{F} = k \Rightarrow F' = kF \Rightarrow \frac{dF}{dx} = kF \Rightarrow \int \frac{dF}{F} = \int k\,dx$$

$$\ln F = kx + c_1 \Longrightarrow \boxed{F = Ae^{kx}}$$

$$\frac{4G'}{G} = k \Rightarrow \frac{G'}{G} = \frac{k}{4} \Rightarrow G' = \frac{k}{4}G \Rightarrow \frac{dG}{dy} = \frac{k}{4}G$$

$$\int \frac{dG}{G} = \int \frac{k}{4}\,dy \Longrightarrow \ln G = \frac{k}{4}y + c_1$$

$$\boxed{G = Be^{\frac{k}{4}y}}$$

$$\therefore u(x,y) = Ae^{kx}\,Be^{\frac{k}{4}y} = ce^{k(x+\frac{y}{4})}$$

$$\because u(0,y) = ce^{k(0+\frac{y}{4})}$$

$$8e^{-3y} = ce^{\frac{k}{4}y}$$

$$\Longrightarrow \boxed{c = 8} \quad , \quad \frac{k}{4} = -3 \Longrightarrow \boxed{k = -12}$$

$$\therefore u(x,y) = 8e^{-12(x+\frac{y}{4})}$$

5.

Homework :-

1- Solve $y\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3u$, $u(0,y)=3e^{-y}-e^{-5y}$

2- Solve $y\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=0$, $u(x,0)=3e^{-4y}$

3- Solve $\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial y}+u$, $u(x,0)=3e^{-5x}+2e^{-3x}$

4- Solve $x\frac{\partial u}{\partial x}=u+y\frac{\partial u}{\partial y}$, $u(x,1)=3x^2$

6.

Some Applications of PDE

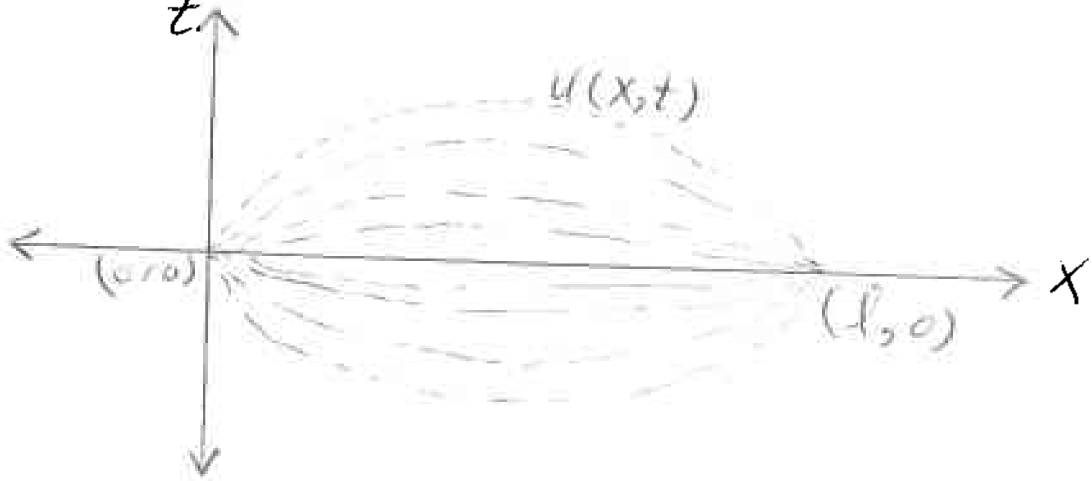
1- One dimensional wave equation

The form of this equation is

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with } C^2 = \frac{T}{M}$$

wher T: tension of the wire
M: the Mass of the Length unite of the wire

The following figure explains one dimensional wave equation



with conditions $u(0,t)=0$, $u(l,t)=0$ for all t

And $u(x,0)=f(x)$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

where $f(x)$ and $g(x)$ is any given functions

Now the solution of the wave equation is:

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad , \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

$$F\ddot{G} = C^2 F'' G \Longrightarrow \frac{\ddot{G}}{C^2 G} = \frac{F''}{F} = K$$

$$\frac{F''}{F} = K \Longrightarrow F'' - KF = 0 \text{ and } \ddot{G} - C^2 K G = 0$$

7

$u(0,t) = F(0)G(t) = 0$

$u(l,t) = F(l)G(t) = 0$

if $G \equiv 0$, then $u \equiv 0$, thus $G \not\equiv 0$

$\therefore F(0) = 0$ and $F(l) = 0$ ——— $(*)$

for $k = 0 \longrightarrow F(x) = ax + b \longrightarrow a = b = 0$

$\therefore F \equiv 0 \longrightarrow u \equiv 0$ But $u \not\equiv 0$

for $k = M^2 \longrightarrow F(x) = A e^{Mx} + B e^{-Mx}$

And from $(*)$ we obtain $F \equiv 0$

Now choose $k = -\lambda^2 \longrightarrow F'' + \lambda^2 F = 0$

$F(x) = A\cos\lambda x + B\sin\lambda x$

from $(*)$ we have

$F(0) = A \longrightarrow A = 0$

$F(l) = B\sin\lambda l \longrightarrow B \neq 0$

$\therefore \sin\lambda l = 0 \longrightarrow \lambda l = n\pi \longrightarrow \lambda = \frac{n\pi}{l}$

setting $B = 1$

$F_n(x) = \sin\left(\frac{n\pi x}{l}\right) \qquad n = 1, 2, 3, \ldots$

Now $k = -\lambda^2 = -\left(\frac{n\pi}{l}\right)^2$

$\therefore G'' - c^2 k G = 0 \longrightarrow G'' + \lambda_n^2 G = 0$

with $\lambda_n = \frac{cn\pi}{l}$

$\therefore G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$

$\therefore U_n(x,t) = F_n(x)\, G_n(t)$

$U_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)\left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right)$
for $n = 1, 2, 3, \ldots$

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t)\left(\sin\left(\frac{n\pi x}{l}\right)\right)$$

$$\therefore U(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) = f(x)$$

The coefficients (B_n) Must be chosen so that $U(x,0)$ becomes a half-range expansion of $f(x)$ namely the fourier of $f(x)$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Similarly

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}\left[(-B_n \lambda_n \sin \lambda_n t + \lambda_n B_n^* \cos \lambda_n t) \sin \frac{n\pi x}{l}\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{l} = g(x)$$

Thus B_n^* Must be chosen so that for $t=0$ $\frac{\partial u}{\partial t}$ becomes fourier sine series of $g(x)$

$$B_n^* \lambda_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

Since $\lambda_n = \frac{cn\pi}{l}$

$$\therefore B_n^* = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx$$

if $g(x) = 0$, then $B_n^* = 0$

$$\therefore U(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{l}$$
$$= \sum_{n=1}^{\infty} B_n \cos\left(\frac{cn\pi}{l} t\right) \sin\left(\frac{n\pi x}{l}\right)$$

2- one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

where c is positive value

$$\therefore F'' G = \frac{1}{c^2} F G'$$

$$\frac{F''}{F} = \frac{1}{c^2}\frac{G'}{G} = k$$

$$\frac{F''}{F} = k \longrightarrow F'' - kF = 0 \longrightarrow r^2 - k = 0 \longrightarrow r = \pm\sqrt{k}$$

Case 1 if $k > 0 \longrightarrow k = \lambda^2 \longrightarrow r = \pm\lambda$

$$\therefore F = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

$$\frac{G'}{G} = c^2 k \longrightarrow G' - c^2\lambda^2 G = 0 \longrightarrow r - c^2\lambda^2 = 0$$

$$\longrightarrow r = c^2\lambda^2 \longrightarrow G = c_3 e^{c^2\lambda^2 t}$$

$$\therefore u(x,t) = c_3 e^{c^2\lambda^2 t}\left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)$$

Case 2 if $k < 0 \longrightarrow k = -\lambda^2 \longrightarrow r = \pm\lambda i$

$$F = c_1 \cos\lambda x + c_2 \sin\lambda x$$

$$G = c_3 e^{-\lambda^2 c^2 t}$$

$$\therefore u(x,t) = c_3 e^{-\lambda^2 c^2 t}\left(c_1 \cos\lambda x + c_2 \sin\lambda x\right)$$

Case 3 if $k = 0$

$$F'' = 0 \longrightarrow F' = c_1 \longrightarrow F = c_1 x + c_2$$

$$G' = 0 \longrightarrow G = c_3$$

$$\therefore u(x,t) = c_3\left(c_1 x + c_2\right)$$