CHAPTER ONE Partial differential equations

Introduction; a partial differential equation is an equation that involves a partial derivatives

Order: is the order of the highest partial derivatives in the equation

EX WRITE THE ORDER OF

1- x $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = (x+y)$

2- $\frac{\partial y}{\partial x} = 4\frac{\partial u}{\partial t}$

3- $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$

The solution of a partial differential equation is any function which satisfies the equation

EX verify that $u = e^x \sin y$ is a solution to the equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

Solution

$\frac{\partial u}{\partial x} = e^x \sin y$ $\frac{\partial u}{\partial x} = e^x \sin y$

$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = e^x \sin y$ $\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = 0$

THE SEPARATION OF VARIABLES

EX solve $\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$

Solution

Let $u(x,y) = F(x)G(y)$

$\frac{\partial y}{\partial x} = \frac{dF}{dx} G$ $\frac{\partial u}{\partial y} = \frac{dG}{dy}F$

$\frac{dF}{dx} G = y \frac{dG}{dy}F$

$\frac{\frac{dF}{dx}}{F} = y\frac{\frac{dG}{dy}}{G} = k$

$\frac{\frac{dF}{dx}}{F} = k \xrightarrow{yields} \frac{dF}{dx} + kF = 0 \xrightarrow{yields}$ F= ae^{kx}

$y\frac{dG}{dy} = k \xrightarrow{yields}$ G= by^k

$U(x, y) = by^k \, ae^{kx}$ $= Ce^{kx} \, y^k$

EX solve $\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial y}\right\} - u = 0$

Solution

Let U$(x, y) = F\,G$

$\frac{\partial U}{\partial x} = \frac{dF}{dx}\,G$ AND $\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right) = \frac{dF}{dx}\frac{dG}{dy}$

$\frac{\frac{dF}{dx}}{F} = k$

$\frac{dF}{dx} + kF = 0$

$F = Ae^{kx}$

$\frac{G}{\frac{dG}{dy}} = k \xrightarrow{yields} \frac{\frac{dG}{dy}}{G} = \frac{1}{k} \xrightarrow{yields} \frac{dG}{dy} + \frac{G}{k} = 0$

G= $Be^{y/k}$

U$(x, y) =$ $Be^{y/k} \, Ae^{kx}$

BOUNDARY VALUE PROBLEM

the problem of finding the solution of a partial differential equation with condition is called a boundary value problem

EX SOLVE B.V.P. $\frac{\partial U}{\partial x} = 4\frac{\partial U}{\partial y}$; U$(0, y) = 8e^{-3y}$

Solution

Let U$(x, y) = F\,G$

$\frac{\partial U}{\partial x} = \frac{dF}{dx}\,G$ and $\frac{\partial U}{\partial y} = \frac{dG}{dy}F$

$\frac{dF}{dx}G = 4F\frac{dG}{dy} \xrightarrow{yields} \frac{\frac{dF}{dx}}{F} = 4\frac{\frac{dG}{dy}}{G} = k$

F= Ae^{kx}

$4\frac{\frac{dG}{dy}}{G} = k \xrightarrow{yields} \frac{\frac{dG}{dy}}{G} = \frac{k}{4}$

G= $Be^{k/4\,y}$

U$(x, y) = Be^{k/4\,y}Ae^{kx} = Ce^{k\left(x+\frac{y}{4}\right)}$

U$(0, y) = Ce^{k\left(0+\frac{y}{4}\right)}$

8$e^{-3y} = Ce^{k/4\,y}$

C= 8 , $\frac{k}{4} = -3 \xrightarrow{yields} k = -12$

EX solve $\frac{\partial U}{\partial y} = 4\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial x}\right)$,U$(0, y) = U(\pi, y) = 0$

$U(x, 0) = 2\sin x - 3\sin 3x$

Solve

Let U$(x, y) = F(x)G(y)$

$\frac{\partial U}{\partial x} = F\frac{dG}{dy}$ and $\frac{\partial U}{\partial y} = G\frac{dF}{dx} \xrightarrow{yields} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = \frac{d}{dx}\left(\frac{dF}{dx}\right)G$

F$\frac{dG}{dy} = 4\frac{d}{dx}\left(\frac{dF}{dx}\right)G$

$\frac{\frac{dG}{dy}}{4G} = \frac{\frac{d}{dx}\left(\frac{dF}{dx}\right)}{F} = -k^2$

$\frac{dG}{dy} + 4k^2$G= 0

$G = Ce^{-k^2y}$

$\frac{\frac{d}{dx}\left(\frac{dF}{dx}\right)}{F} = -k^2$

$\frac{\partial}{\partial x}\frac{dF}{dx} + k^2$F = 0

F= $A\cos kx + \mathrm{B}\sin kx$

U$(x, y) = A\cos kx + \text{B}\sin kx\, Ce^{-k^2 y}$

U$(x, y) = D\cos kx + \text{E}\sin kx\, e^{-k^2 y}$

U$(\pi, y) = Ee^{-4k^2 y}\sin \pi k \quad = 0$

$sin\ \ k\pi = 0$

k $\pi = n\pi \xrightarrow{yields} k = n$

U$(x, y) = Ee^{-4n^2 y}\sin nx$

U$(x, 0) = E_1 e^{-4n_1^2}\sin n_1 x + E_2 e^{-4n_2^2}\sin n_2\, x$

2$\sin x - 3\sin 3x = E_1\sin n_1 x + E_2\sin n_2\, x$

$E_{1=2}$, $E_2 = -3$, $n_1 = 1$, n_2 =3

U$(x, y) = 2e^{-4y}\sin x - 3e^{-36y}\sin 3x$

<u>EXERCIES</u>

1- solve 4$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = 3U$;U$(0, y) = 3e^{-y} - e^{-5y}$
2- solve y$\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 0$; $U(x, 0) = 3e^{-4x}$
3- $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial y} + U$; $U(x, 0) = 3e^{-5x} - 2e^{-3x}$
4- x$\frac{\partial U}{\partial x} = U + y\frac{\partial U}{\partial y}$; $U(x, 1) = 3x^2$

CHAPTER TWO SOME SEPECIAL FUNCTIONS

2-1 GAMMA FUNCTION

WE can define gamma function by

$\gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \quad , n > 0$

we must note that the variable (x) is the integration variable and the integration depend on the value of (n)

Now we have some important rules which we need in this chapter

1- when (n) is positive integer we can use the relation $\gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \quad , n > 0$

or $\gamma(n+1) = n\gamma(n)$ or $n + 1 == n!$

2-BY apply $(n = 1)$ in the definition above we have $\gamma(1) = 1$

3-we must note that $\gamma\left(\frac{1}{2}\right) = \sqrt[2]{\pi}$

4- IF $\left(n = -p - \frac{1}{2}\right)$, where p ≥ 0 then we used the relation
$\gamma\left(-p-\frac{1}{2}\right) = (-1)^{p+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\ldots\ldots..\left(\frac{2}{2p+1}\right)\sqrt{\pi}$

5- When $0 <$ n< 1 then $\gamma(n)\gamma(1-n) = \frac{\pi}{\sin(n\pi)}$

NOW WE TAKE SOME EXAMPLE

<u>EX .</u> Compute the following

1- $\gamma(3) = 2! = 2$

2- $\gamma(3.5) = 2.5\gamma(2.5) = 2.5 \times 1.5\gamma(1.5) = 2.5 \times 1.5 \times 0.5\gamma(0.5) = 2.5 \times 1.5 \times 0.5 \times \sqrt{\pi}$

3- $\gamma(-2.5) = (-1)^{2+1}\left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{2}{5}\right)\sqrt{\pi}$

4- $\gamma\left(\frac{1}{3},\frac{2}{3}\right) = \frac{\pi}{\sin\left(\frac{\pi}{3}\right)}$

<u>EX</u> Evalute by using special function

1- $\int_0^\infty x^{-1/3}\ e^{-8x}\,dx$

solution

LET y= $-8x \xrightarrow{yields} x = \frac{-y}{8} \xrightarrow{yields} dx = \frac{-dy}{8}$

$$\int_0^\infty \frac{-y^{-1/3}}{8^{-1/3}}\ e^{-y}\ \frac{-dy}{8} = \frac{1}{8^{-2/3}}\int_0^\infty y^{-1/3}e^{-y}\,dy$$

$= \frac{1}{8^{-2/3}}\ \gamma\left(\frac{2}{3}\right)$

2- $\int_{-\infty}^{\infty} e^{-9x^2}\,dx = 2\int_0^\infty e^{-9x^2}$dx

LET y= $9x^2 \xrightarrow{yields} x = \frac{\sqrt{y}}{3} \xrightarrow{yields}$ dx = $\frac{y^{-1/2}dy}{6}$

$= \frac{2}{6}\int_0^\infty e^{-y}y^{-1/2}$ dy

$= \frac{1}{3}\gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}$

TRY TO SOLVE THE FOLLOWING EXAMPLES

3- $\gamma(-7.5)$
4- $\int_0^\infty 4x^4\,e^{-x^2}$dx
5- $\int_0^\infty \frac{x^k}{k^x}\,dx$

2.2- BETA FUNCTION

The beta function is defined by

$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$,m>0 ,$n>0$

<u>REMARK</u> ; $\beta(m,n) = \beta(n,m)$

<u>ANOTHER</u> FORMS OF BETA FUNCTION

$1 -\quad \beta(m,n) = 2\int_0^{\pi/2}(\sin\theta)^{2m-1}\,(\cos\theta)^{2m-1}d\theta$

$2 -\quad \beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$

2.3-RELATIONSHIP BETWEEN GAMMA AND BETA FUNCTIONS

$\beta(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$

EXAMPLES

1- EVALUTE $\int_0^1 x^4(1-x)^3\,dx$

$= \beta(5,4) = \frac{\gamma(5)\gamma(4)}{\gamma(9)} = \frac{4!3!}{8!}$

2$-\int_0^2 \frac{x^2}{\sqrt{2-x}}dx$

LET $\quad x = 2y \xrightarrow{yields} y = \frac{x}{2} \xrightarrow{yields} dx = 2dy$

$\int_0^1 \frac{4y^2 2dy}{\sqrt{2-2y}} = \frac{8}{\sqrt{2}}\int_0^1 y^2(1-y)^{-1/2}dy = \frac{8}{\sqrt{2}}\beta\left(3,\frac{1}{2}\right) = \frac{8}{\sqrt{2}}\frac{\gamma(3)\gamma\left(\frac{1}{2}\right)}{\gamma\left(\frac{7}{2}\right)}$

3 $-\int_0^\infty \frac{dy}{1+y^4}$

$LET\ \ x = y^4 \xrightarrow{yields} y = x^{1/4} \xrightarrow{yields} dx = 4x^3dy$

$\frac{1}{4}\int_0^\infty \frac{x^{-3/4}}{1+x}dx = \frac{1}{4}\frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\pi\sqrt{2}}{4}$

EXCERCISES

1- $\int_0^{\pi/2}(\sin\theta)^4(\cos\theta)^4 d\theta$

2- $\int_0^{\pi/2}(\sin\theta)^5 d\theta$

3- $\int_0^{\pi/2}\sqrt{\tan\theta}\,\mathrm{d}\theta$

4- $\int_0^\infty \frac{xdx}{1+x^6}$

5- $\int_0^\infty \frac{x^2}{1+x^4}dx$

6- $\int_0^\infty e^{-x^3}dx$

7- $\int_0^\infty \sqrt[4]{x}\,e^{-\sqrt{x}}dx$

8- $\int_0^\infty y^3 e^{-2y^5}dy$

9- $\int_0^\infty y^3 e^{-2y^5}dy$

10- $\int_0^\infty \frac{e^{-at}}{\sqrt{t}}dt$

11- $\gamma(n) = \int_0^1\left(\ln\frac{1}{x}\right)^{n-1}dx$

12- $\int_0^1(\ln x)^4 dx$

13- $\int_0^\infty (x\ln x)^3 dx$

14-$\int_0^\infty \sqrt[3]{\ln\left(\frac{1}{x}\right)}\,dx$