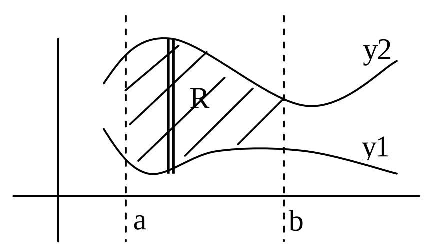
Double Integrals

Definition: let R be closed region in the x, y plane. If f is function of two variables that is define on the region R, then the double integrals of f on R is written by

$$\iint_R \mathbf{f(x,y)dA} \quad \textbf{or} \quad \iint_R \mathbf{f(x,y)dxdy}$$

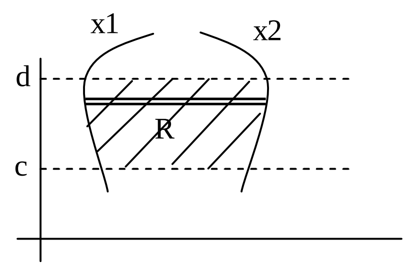
ملاحظة: اذا كانت المنحنيات بهذه الصيغة يؤخذ المقطع شاقولي dydx

$$\iint_R f(x,y)\ dA = \int_a^b \int_{y_2}^{y_1} f(x,y)\,dy\,dx$$



اما اذا كانت المنحنيات بالشكل التالي يؤخذ المقطع افقيا dxdy

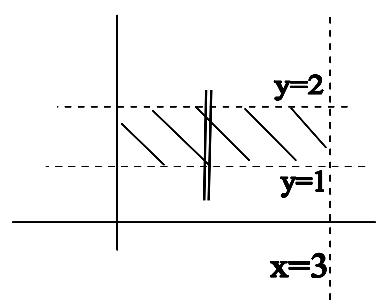
$$\iint_R f(x,y)\ dA = \int_c^d \int_{x_2}^{x_1} f(x,y)\,dxdy$$



EX.1: Evaluate $\int_0^3 \int_1^2 (1+8xy)dydx$

Since dydx $\Rightarrow$ vertical

$y=1$, $y=2$



$$\int_0^3 \int_1^2 (1+8xy)dydx = \int_0^3 (y+8x\frac{y^2}{2})\Big|_1^2 \mathbf{dx}$$

$$= \int_0^3 \{[2+4x(4)]-[1+4\mathbf{x}(1)]\}dx$$

$$= \int_0^3 \{[2+16x] - [1+4x]\}dx$$

$$= \int_0^3 \{1+12x\}dx$$

$$= (\mathbf{x}+12\frac{\mathbf{x}^2}{2})\Big|_0^3 = (3+6(9)) - (0) = (3+54) = \mathbf{57}$$

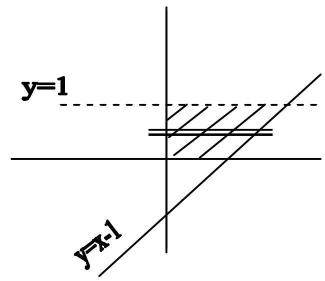
Ex.2: Evaluate $\int_0^1\int_0^{y+1}(2-3x+y)\,dx\,dy$

Solution: $\int_0^1\int_0^{y+1}(2-3x+y)\,dx\,dy = \int_0^1 \left(2x-\frac{3}{2}x^2+yx\right)\Big|_0^{y+1} dy$

$= \int_0^1 [2(y+1)-\frac{3}{2}(y+1)^2+y(y+1)]\,dy$

$= \int_0^1 (-\frac{1}{2}y^2+\frac{1}{2})\,dy$

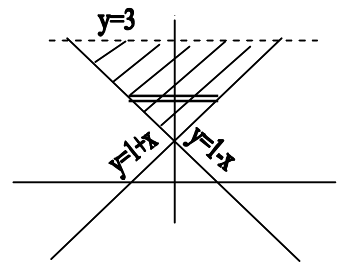
$= (-\frac{1}{2}\frac{y^3}{3}+\frac{1}{2}y)\Big|_0^1 = -\frac{1}{6}+\frac{1}{2}=\frac{1}{3}$



EX.3: Evaluate $\iint_R (2x-y^2)\,dA$ over the triangular region R enclosed by

$y=1-x$, $y=1+x$, $y=3$

Solution:



$\iint_R (2x-y^2)\,dA = \int_1^3\int_{1-y}^{y-1}(2x-y^2)\,dx\,dy$

$= \int_1^3 (x^2-y^2x)\Big|_{1-y}^{y-1} dy = \int_1^3 \{[(y-1)^2-y^2(y-1)]-[(1-y)^2-y^2(1-y)]\}\,dy$

$= \int_1^3 \{y^2-2y+1-y^3+y^2-1+2y-y^2+y^2-y^3\}\,dy$

$= \int_1^3 (-2y^3+2y^2)\,dy = (-2\frac{y^4}{4}+2\frac{y^3}{3})\Big|_1^3 = \frac{-18}{2}+18-(\frac{-1}{2}+\frac{2}{3})$

$= 18-\frac{81}{2}+\frac{1}{2}-\frac{2}{3} = \left|18-\frac{244}{6}\right| = \frac{-68}{3}$

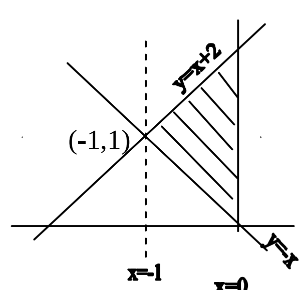
Reversing the Order of Integration

Ex.1: Write an equivalent double of integration reversed $\int_{-1}^{0}\int_{-x}^{x+2}(x^2+y^2)\,dy\,dx$

Solution:

$-x \le y \le x+2 \Rightarrow y=-x, y=x+2$

$-1 \le x \le 0 \Rightarrow x=-1, x=0$



$0 \le \mathbf{y} \le 1, \quad 1 \le \mathbf{y} \le 2$

$\mathbf{y} \le \mathbf{x} \le 0, \quad \mathbf{y} - 2 \le \mathbf{x} \le 0$

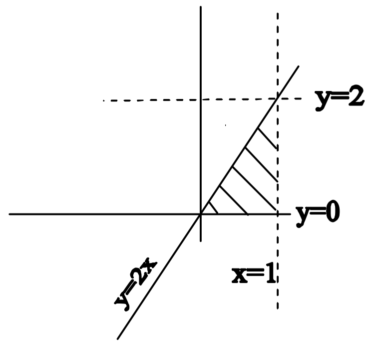
$$\int_{-1}^{0}\int_{-\mathbf{x}}^{\mathbf{x}+2} (x^2 + y^2)dy\ dx$$

$$= \int_{0}^{1}\int_{-\mathbf{y}}^{0} (\mathbf{x}^2 + \mathbf{y}^2)\mathbf{dxdy} + \int_{1}^{2}\int_{\mathbf{y}-2}^{0} (\mathbf{x}^2 + \mathbf{y}^2)\mathbf{dxdy}$$

Ex.2: Evaluate by reversing the order of integration $\int_0^2\int_{\frac{y}{2}}^{1} e^{x^2}\ dx\,dy$

$x = 1, \quad x = \frac{y}{2} \quad \Rightarrow \quad y = 2x$

$y \quad \text{from} \quad 0 \to 2$



$$\int_0^2\int_{\frac{y}{2}}^{1} e^{x^2}\ dx\,dy = \int_0^1\int_0^{2x} e^{x^2}\ dy\,dx = \int_0^1 e^{x^2} y\Big|_0^{2x} dx$$

$$= \int_0^1 e^{x^2}(2x - 0)dx$$

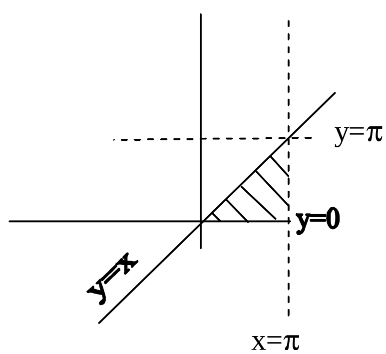
$$= e^{x^2}\Big|_0^1 = e^1 - e^0 = \mathbf{e - 1}$$

Ex.3: Evaluate $\int_0^p\int_y^p \frac{\sin x}{x}\ dx\,dy$

Solution: By reversing the order

$\mathbf{x} = \mathbf{y}, \mathbf{x} = \pi,\ 0 \le \mathbf{y} \le \pi \Rightarrow$

$0 \le \mathbf{x} \le \pi, \quad 0 \le \mathbf{y} \le \mathbf{x}$



$$\Rightarrow \int_0^{\pi}\int_y^{\pi} \frac{\sin \mathbf{x}}{\mathbf{x}} \, d\mathbf{x}\, dy = \int_0^{\pi}\int_0^{x} \frac{\sin \mathbf{x}}{\mathbf{x}} \, dy\, d\mathbf{x}$$

$$= \int_0^{\pi} \frac{\sin \mathbf{x}}{\mathbf{x}} \cdot \mathbf{y}\Big|_0^{x} \mathbf{dx}$$

$$= \int_0^{\pi} \frac{\sin \mathbf{x}}{\mathbf{x}} \cdot \mathbf{x} \;\; \mathbf{dx}$$

$$= \int_0^{\pi} \sin \mathbf{x} \;\; \mathbf{dx} \; = \; -\cos \mathbf{x}\Big|_0^{\pi} \; = \; -(-1-1) \; = \; \mathbf{2}$$

The Area

Definition: The area of the region R defined by $\mathbf{A} = \iint_{\mathbf{R}} \mathbf{dA}$.

Ex.1: Draw the region bounded by $y=e^x$, y=sinx, x=π, x=-π and evaluate its area.

Solution:

$$\mathbf{A} = \int_{-\pi}^{\pi}\int_{\sin x}^{e^x} \mathbf{dydx}$$

$$= \int_{-\pi}^{\pi} \mathbf{y}\Big|_{\sin \mathbf{x}}^{\mathbf{e^x}}$$

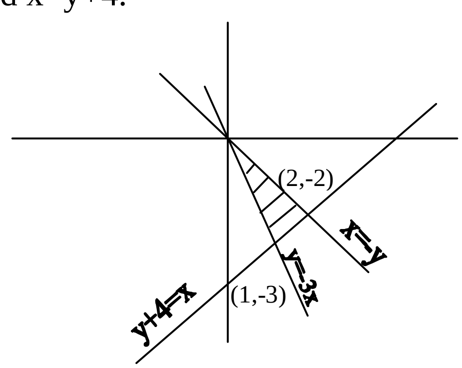
$$= \int_{-\pi}^{\pi} (\mathbf{e^x} - \sin \mathbf{x})\mathbf{dx}$$

$$= \mathbf{e^x} + \cos \mathbf{x}\Big|_{-\pi}^{\pi}$$

$$\mathbf{e}^{\pi} - \mathbf{e}^{-\pi} + \cos\pi - \cos(-\pi) = \mathbf{e}^{\pi} - \mathbf{e}^{-\pi}$$

Ex.2: find the area bounded by y=-x, y=-3x and x=y+4.

Solution:



$$\mathbf{x} = -\mathbf{x} + 4 \Rightarrow \mathbf{x} = 2,\ \mathbf{y} = -2$$

$$\mathbf{x} = -3\mathbf{x} + 4 \Rightarrow \mathbf{x} = 1,\ \mathbf{y} = -3$$

$$-\mathbf{x} = -3\mathbf{x} \Rightarrow \mathbf{x} = 0,\ \mathbf{y} = 0$$

$$-3 \le \mathbf{y} \le -2,\ -2 \le \mathbf{y} \le 0$$

$$\frac{-\mathbf{y}}{3} \le \mathbf{x} \le \mathbf{y} + 4,\ \frac{-\mathbf{y}}{3} \le \mathbf{x} \le -\mathbf{y}$$

$$\mathbf{A}=\int_{-3}^{-2}\int_{\frac{-y}{3}}^{y+4}\mathbf{dxdy}+\int_{-2}^{0}\int_{\frac{-y}{3}}^{-y}\mathbf{dxdy}$$

$$=\int_{-3}^{-2}\mathbf{x}\Big|_{\frac{-4}{3}}^{y+4}\mathbf{dy}+\int_{-2}^{0}\mathbf{x}\Big|_{\frac{-4}{3}}^{-y}\mathbf{dy}$$

$$=\int_{-3}^{-2}(\mathbf{y}+4+\frac{4}{3})\mathbf{dy}+\int_{-2}^{0}(-\mathbf{y}+\frac{\mathbf{y}}{3})\mathbf{dy}$$

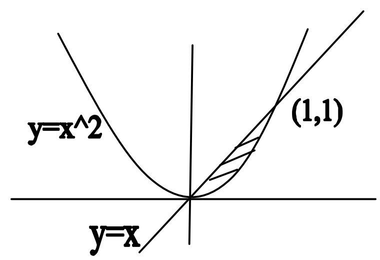
$$(\frac{\mathbf{y}^2}{2}+4\mathbf{y}+\frac{\mathbf{y}^2}{6})\Big|_{-3}^{-2}+\frac{-\mathbf{y}^2}{2}+\frac{\mathbf{y}^2}{6}\Big|_{-2}^{0}=2$$

Ex.3: Find the area of the region R bounded by y=x and y=x^2 in the first quadrant.

Solution:

$$\mathbf{A}=\int_0^1\int_{x^2}^{x}\text{dydx}$$

$$=\int_0^1(\mathbf{x}-\mathbf{x}^2)\mathbf{dx}=\frac{\mathbf{x}^2}{2}-\frac{\mathbf{x}^3}{3}\Big]_0^1=\frac{1}{6}$$

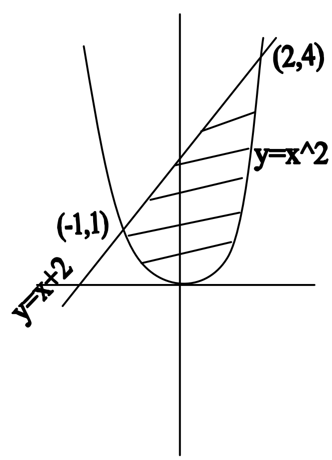


Ex.4: Find the area of the region R enclosed by the parabola y=x^2 and the line y=x+2.

Solution:

$$\mathbf{A}=\int_{-1}^{2}\int_{x^2}^{x+2}\text{dydx}=$$

$$\int_{-1}^{2}\mathbf{y}\Big]_{x^2}^{x+2}\mathbf{dx}=\int_{-1}^{2}(\mathbf{x}+2-\mathbf{x}^2)\mathbf{dx}=\frac{9}{2}$$



Another solution

$$\mathbf{A}=\int_0^1\int_{-\sqrt{y}}^{\sqrt{y}}\text{dxdy}+\int_1^4\int_{y-2}^{\sqrt{y}}\text{dxdy}$$