Fourier Series:

Are series of cosine and sine terms and arise in the important practical task of representing general **periodic functions.**

Periodic functions:

A function $f(x)$ is called *periodic* if it is defined for all real x and if there is some positive No. $2l$ such that

$$f(x+2l) = f(x)$$

The No. $2l$ is called a period of $f(x)$.

Fourier said If $f(x+2l) = f(x)$, $2l$: periodic No. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{np}{l}x + b_n \sin\frac{np}{l}x \right)$$

Where a_0 , a_n & b_n are Fourier coefficients and

$$a_0 = \frac{1}{l}\int_A^B f(x)\; dx$$

$$a_n = \frac{1}{l}\int_A^B f(x)\; \cos\frac{npx}{l}\, dx$$

$$b_n = \frac{1}{l}\int_A^B f(x)\; \sin\frac{npx}{l}\, dx$$

$A \mathbf{p}\ x\ \mathbf{p}\ B$

Notes:

$\sin np = 0$, $(n = 0, \pm 1, \pm 2, ...)$, n integer No.

$$\cos np = \begin{cases} -1 & n = 1, 3, 5, ... \\ 1 & n = 0, 2, 4, ... \end{cases}$$

$\cos 2np = 1$ for all n , $(n = 0, \pm 1, \pm 2, \pm 3, ...)$

$\cos(-x) = \cos x$ *even*

$\sin(-x) = \sin x$ *odd*

EX.:

Write Fourier series for $f(x) = x \quad , \quad 0 \mathbf{p} x \mathbf{p} 2p$

Sol:

$\Rightarrow \quad 2l = 2p - 0 = 2p \Rightarrow l = p$

First we find a_0 , a_n & b_n

$$a_0 = \frac{1}{l} \int_A^B f(x) \; dx$$

$$= \frac{1}{p} \int_0^{2p} x \; dx = \frac{1}{p} \cdot \frac{x^2}{2} \Big|_0^{2p} = \frac{1}{2p} \left[4p^2 - 0 \right] = 2p$$

$$a_n = \frac{1}{l} \int_A^B f(x) \; \cos \frac{np}{l} x \, dx$$

$$= \frac{1}{p} \int_0^{2p} x \; \cos \frac{np}{p} x \, dx$$

$$= \frac{1}{p} \int_0^{2p} x \; \cos nx \, dx \qquad , \quad \text{by } u\mathrm{d}v$$

$$= \frac{1}{p} \left[x \cdot \frac{1}{n} \sin nx \Big|_0^{2p} - \int_0^{2p} \frac{1}{n} \sin nx \, dx \right]$$

$$= \frac{1}{p} \cdot \frac{1}{n^2} \cos nx \Big|_0^{2p} = \frac{1}{n^2 p} \left[\cos 2np - \cos 0 \right] = \frac{1}{n^2 p} (1-1) = 0$$

$\therefore \; \mathrm{a}_n = 0$

$$b_n = \frac{1}{p} \int_A^B x \; \sin \frac{np}{p} x \, dx$$

$$= \frac{1}{p} \int_A^B x \; \sin nx \, dx \qquad , \quad \mathrm{u} = x \quad , \quad \mathrm{dv} = \sin nx$$

$$= \frac{1}{p} \left[x \cdot \left(\frac{-1}{n} \cos nx \right) \Big|_0^{2p} - \int_0^{2p} \frac{-1}{n} \cos nx \, \mathrm{d}x \right]$$

$$= \frac{1}{p} \left[\frac{-1}{n} (2p \cdot 1 - 0) + \frac{1}{n^2} \sin nx \Big|_0^{2p} \right] = \frac{-2}{n}$$

$$b_1=\frac{-2}{1}\ ,\quad b_2=\frac{-2}{2}\quad ,\quad b_3=\frac{-2}{3}$$

$$\Rightarrow\ f(x)=\frac{2\pi}{2}+\sum_{n=1}^{\infty} b_n \sin nx \qquad ,\quad a_n=0$$

$$=\pi+(b_1\sin x+b_2\sin 2x+b_3\sin 3x+\ldots)$$

$$=\pi+\left(\frac{-2}{1}\sin x+\frac{-2}{2}\sin 2x+\frac{-2}{3}\sin 3x+\ldots\right)$$

$$=\pi-2\left(\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots\right)$$

Fourier even & odd functions

Let f(x) is periodic function of period 2*l* and f(x) defined on (-*l*,*l*) then

1) If $f(x)$ is even, then

i) $b_n=0$

ii) $a_0=\frac{1}{l}\cdot 2\int_0^{l} f(x)\,dx$

iii) $a_n=\frac{1}{l}\cdot 2\int_0^{l} f(x)\cos\frac{n\pi}{l}x\,dx$

2) If $f(x)$ is odd, then

i) $a_0=a_n=0$

ii) $b_n=\frac{1}{l}\cdot 2\int_0^{l} f(x)\sin\frac{n\pi}{l}x\,dx$

Def.:

A function $f(x)$ is even if $f(-x)=f(x)$ for all x. For example,
$f(x)=x^2$, $f(x)=\cos x$, $f(x)=|x|$, $f(x)=constant$, $f(x)=x^2+5$.

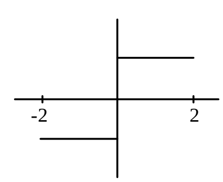
A function $f(x)$ is odd if $f(-x)=-f(x)$ for all x. For example, $f(x)=x^3$
$f(x)=\sin x$, $f(x)=x^5+x$

Notes:

- If $f(x)$ symmetric about y-axis $\Rightarrow$ f(x) is even.
- If $f(x)$ symmetric about origin $\Rightarrow$ f(x) is odd.

EX.:

Write Fourier series for $f(x)=\begin{cases}-1 & -2<x<0\\ 1 & 0<x<2\end{cases}$



Sol:

$\Rightarrow\ 2l=2-(-2)=4\Rightarrow l=2$

i) From sketch $\Rightarrow$ f(x) symmetric about origin $\Rightarrow$ f(x) odd.

$\Rightarrow\quad a_0=a_n=0$

$$b_n = \frac{1}{2}\cdot 2\int_0^B 1\cdot \sin\frac{np}{2}x\,dx$$

$$= \int_0^B 1\cdot \sin\frac{np}{2}x\,dx$$

$$= \frac{-2}{np}\cos\frac{np}{2}x\Big|_0^2 \quad = \frac{-2}{np}(\cos np - 1)$$

$$\therefore\ b_n = \frac{2}{np}(1-\cos np) \qquad \ldots.\ (1)$$

To find b_1 , put $n = 1$ in eq. (1)

$$\therefore\ b_1 = \frac{2}{p}(1-(-1)) = \frac{4}{p} \qquad , \qquad b_2 = \frac{2}{2p}(1-1)) = 0$$

$$b_3 = \frac{2}{3p}(1+1)) = \frac{4}{3p} \qquad , \qquad b_4 = \frac{2}{5p}(1-1)) = 0$$

$$b_5 = \frac{2}{5p}(1+1)) = \frac{4}{5p}$$

$$\Rightarrow\ f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{np}{2}x$$

$$= \frac{4}{p}\sin\frac{p}{2}x + 0 + \frac{4}{3p}\sin\frac{3p}{2}x + 0 + \frac{4}{5p}\sin\frac{5p}{2}x + \ldots..$$

$$= \frac{4}{p}(\sin\frac{p}{2}x + \frac{1}{3}\sin\frac{3p}{2}x + \frac{1}{5}\sin\frac{5p}{2}x + \ldots..)$$

Notes:

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\underline{\cos(x-y) = \cos x\cos y + \sin x\sin y} \qquad \text{add}$$
$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$
$$\cos x\cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

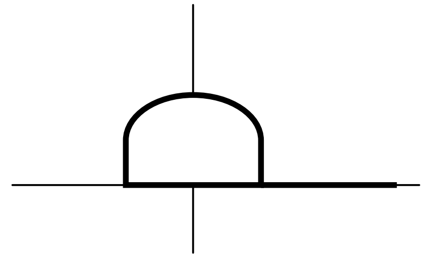
We can obtain $\sin x\sin y$ by subtraction.

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \text{subtraction}$$
$$\underline{\sin(x-y) = \sin x\cos y - \cos x\sin y}$$
$$\sin(x+y) + \sin(x-y) = 2\cos x\sin y$$
$$\cos x\sin y = \frac{1}{2}\sin(x+y) + \sin(x-y)$$

EX.:

Write Fourier series for $f(x) = \begin{cases} \cos x & \frac{-p}{2}\ \mathbf{p}\ x\ \mathbf{p}\ \frac{p}{2} \\ 0 & \frac{p}{2}\ \mathbf{p}\ x\ \mathbf{p}\ \frac{3p}{2} \end{cases}$

Sol:



$\Rightarrow \; 2l = \frac{3p}{2} + \frac{p}{2} = 2p \Rightarrow l = p$

$f(x)$ is even in $\frac{-p}{2} \,\mathbf{p}\, x \,\mathbf{p}\, \frac{p}{2} \quad \Rightarrow \; b_n = 0$

This is true if and only if the other interval = 0

$$a_0 = \frac{1}{l}\cdot 2\int_0^{\frac{p}{2}} f(x)dx = \frac{1}{p}\cdot 2\int_0^{B} \cos x\, dx = \frac{2}{p}\ \sin x\Big|_0^{\frac{p}{2}} = \frac{2}{p}[1-0] = \frac{2}{p}$$

$$a_n = \frac{1}{l}\cdot 2\int_0^{\frac{p}{2}} f(x)\cos\frac{np}{p}x\, dx$$

$$= \frac{1}{p}\cdot 2\int_0^{\frac{p}{2}} \cos x \cos nx\, dx \qquad ....(1)$$

$$= \frac{2}{p}\int_0^{\frac{p}{2}} \frac{1}{2}[\cos(x+nx) + \cos(x-nx)]\, dx$$

$$= \frac{1}{p}\left[\frac{1}{1+n}\sin(x+nx) + \frac{1}{1-n}\sin(x-nx)\right]_0^{\frac{p}{2}}$$

$$= \frac{1}{p}\left[\frac{1}{1+n}\left\{\sin(\frac{p}{2} + n\frac{p}{2}) - \sin 0\right\} + \frac{1}{1-n}\left\{\sin(\frac{p}{2} - n\frac{p}{2}) - 0\right\}\right] \qquad ....(2)$$

To find a_1 , put n = 1 in eq. (1)

$$\therefore\ a_1 = \frac{2}{p}\cdot\int_0^{\frac{p}{2}} \cos x^2\ dx = \frac{2}{p}\cdot\int_0^{\frac{p}{2}} \frac{1}{2}(1+\cos 2x)\, dx = \frac{1}{p}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{p}{2}}$$

$$= \frac{1}{p}(\frac{p}{2} + 0 - (0+0)) = \frac{1}{2}$$

in eq.(2)

$$a_2 = \frac{1}{p}\left[\frac{1}{3}\ \{\sin\frac{3p}{2}\} - \{\sin\frac{-p}{2}\}\right] = \frac{1}{p}(\frac{-1}{3} + 1) = \frac{2}{3p}$$

$$a_3 = \frac{1}{p}\left[\frac{1}{4}\ \{0 - \frac{1}{2}\{0\}\}\right] = 0$$

$$a_4 = \frac{1}{p}\left[\frac{1}{5}\ \{\sin\frac{5p}{2}\} - \frac{1}{3}\{\sin\frac{-3p}{2}\}\right] = \frac{1}{p}(\frac{1}{5} - \frac{1}{3}) = \frac{-2}{15p}$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx$$

$$= \frac{1}{p} + \left(\frac{1}{2}\cos x + \frac{2}{3p}\cos 2x - \frac{2}{15p}\cos 4x + ...\right)$$