Vector:

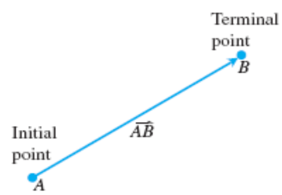
A vector is a matrix that has only one row – then we call the matrix a ***row vector*** – or only one column – then we call it a ***column vector***.

A row vector is of the form: $a = [a_1 \ \ a_2 \ \ \ldots \ \ a_n]$

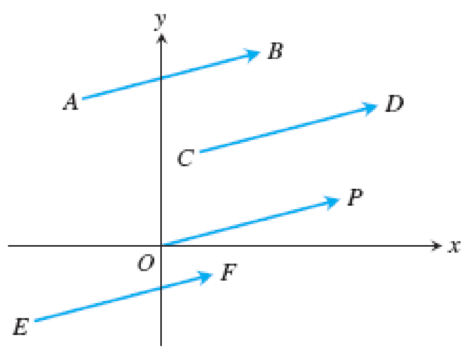
A column vector is of the form:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A quantity such as force, displacement, or velocity is called a vector and is represented by a directed line segment



A **vector** in the plane is directed line segment. The directed line segment $\overrightarrow{AB}$ has ***initial point*** A and ***terminal point*** B; its ***length*** is denoted by $|\overrightarrow{AB}|$. Two vectors are ***equal*** if they have the same length and direction.



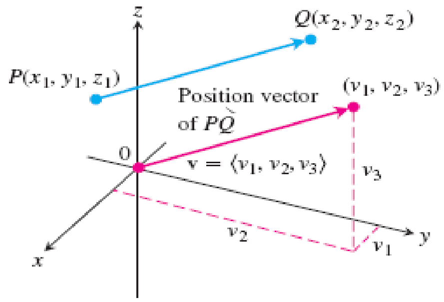
Component form

If v is a ***two dimensional*** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) **,**then the ***Component form*** of v is:

$$v = (v_1, v_2)$$

If v is a ***three dimensional*** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3)**,** then the ***Component form*** of v is:

$$v = (v_1, v_2, v_3)$$



The numbers v_1, v_2 and v_3 are called the components of v.

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$**,** the standard position vector $v = (v_1, v_2, v_3)$ equal to $\overrightarrow{PQ}$ is

$v = (x_2 - x_1 ,\ y_2 - y_1 \ ,\ z_2 - z_1)$

The ***magnitude*** or ***length*** of the vector $v = \overrightarrow{PQ}$ is the nonnegative number

$$|v| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length **0** is the zero vector $0 = (0,0)$ or $0 = (0,0,0)$. This vector is also the only vector with no specific direction.

Ex.: Find **a)** component form and **b)** length of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$

Solution:

a) $v = (-5+3,\ 2-4\ ,\ 2-1)$

The component form of $\overrightarrow{PQ}$ is $v = (-2, -2, 1)$

b) The length or magnitude of $v = \overrightarrow{PQ}$ is $|v| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3$

Vector Addition and Multiplication of a vector by a scalar

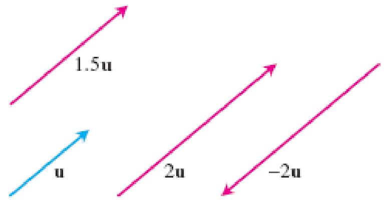
Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors with ***k*** a scalar.

Addition:

$u + v = (u_1 + v_1 ,\ u_2 + v_2 ,\ u_3 + v_3)$

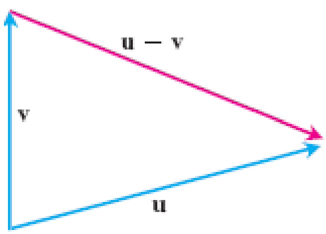
Scalar multiplication: $ku = (ku_1, ku_2, ku_3)$

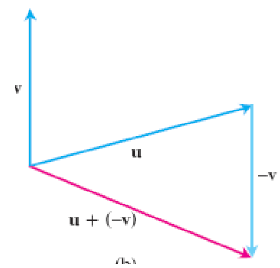
If the length of ku is the absolute value of the scalar k times the length of u. The vector $(-1)u = -u$ has the same length as u but points in the opposite direction.



If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, $u - v = (u_1 - v_1\ ,\ u_2 - v_2\ ,\ u_3 - v_3)$

Note that $(u - v) + v = u$ and the difference $u - v$ as the sum $u + (-v)$





Ex.:

Let $u = (-1,3,1)$ and $v = (4,7,0)$, find

a) $2u + 3v$ **b)** $u - v$ **c)** $\left|\frac{1}{2}u\right|$

Solution:

a) $2u + 3v = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$

b) $u - v = (-5, -4, 1)$

c) $\left|\frac{1}{2}u\right| = \left|\left(\frac{-1}{2}, \frac{3}{2}, \frac{1}{2}\right)\right| = \frac{1}{2}\sqrt{11}$

Properties of vector operations:

Let u , v and w be vectors and a and b be scalars.

1) $u + v = v + u$

2) $(u + v) + w = u + (v + w)$

3) $u + 0 = u$

4) $u + (-u) = 0$

5) $0u = 0$

6) $1u = u$

7) $a(bu) = (ab)u$ **8)** $a(u + v) = au + av$

9) $(a + b)u = au + bu$

Unit vectors

A vector v of length 1 is called ***unit vector***. The standard unit vectors are:

$i = (1,0,0) \quad , \quad j = (0,1,0) \quad , \quad k = (0,0,1)$

$$v = (v_1, v_2, v_3) = (v_1,0,0) + (0, v_2,0) + (0,0, v_3)$$
$$= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1)$$
$$= v_1 i + v_2 j + v_3 k$$

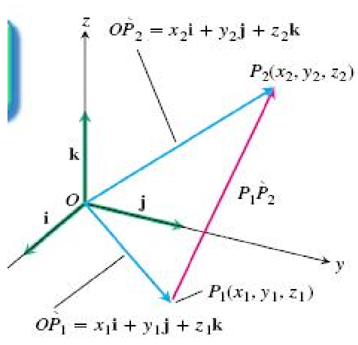
We call the scalar (or number) v_1 the ***i-component*** of the vector v , v_2 the ***j-component*** of the vector v, and v_3 the ***k-component***. In component form, $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathrm{i} + (y_2 - y_1)\mathrm{j} + (z_2 - z_1)k$

If $v \neq 0$, then

1) $u = \dfrac{v}{|v|}$ is a unit vector in the direction of v, called ***the direction*** of the nonzero vector v.

2) The equation $v = \dfrac{v}{|v|}|v|$ expresses v in terms of its ***length*** and ***direction***.



Ex.:

Find a unit vector u in the direction of the vector $P_1(1,0,1)$ and $P_2(3,2,0)$.

Solution

$$\overrightarrow{P_1P_2} = (3-1)\mathrm{i} + (2-0)\mathrm{j} + (0-1)k = 2\mathrm{i} + 2\mathrm{j} - \mathrm{k}$$
$$\overrightarrow{P_1P_2} = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$u = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2i + 2j - k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

The unit vector u is the ***direction*** of $\overrightarrow{P_1P_2}$.

Midpoint of a line segment

The Midpoint M of a line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

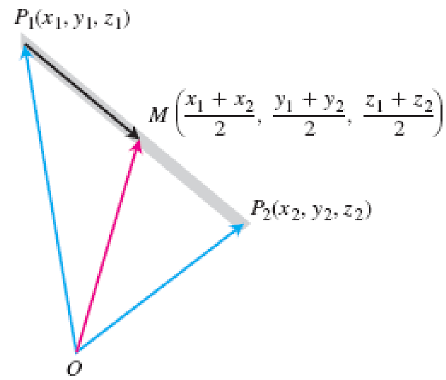
$$\left(\frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2}, \frac{(z_1 + z_2)}{2}\right)$$

$$\overrightarrow{\mathbf{OM}} = \overrightarrow{\mathbf{OP_1}} + \frac{1}{2}(\overrightarrow{\mathbf{P_1P_2}})$$

$$= \overrightarrow{\mathbf{OP_1}} + \frac{1}{2}(\overrightarrow{\mathbf{OP_2}} - \overrightarrow{\mathbf{OP_1}})$$

$$= \frac{1}{2}(\overrightarrow{\mathbf{OP_1}} + \overrightarrow{\mathbf{OP_2}})$$

$$= \frac{(\mathbf{x_1} + \mathbf{x_2})}{2}\mathbf{i} + \frac{(\mathbf{y_1} + \mathbf{y_2})}{2}\mathrm{j} + \frac{(\mathbf{z_1} + \mathbf{z_2})}{2}\mathbf{k}$$



Ex.:

The midpoint of the segment joining $P_1(3,-2,0)$ and $P_2(7,4,4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5,1,2)$$

Product of vectors

u & v are vectors,

There are two kinds of multiplication of two vectors:

1- The scalar product (dot product) u.v. The result is a **scalar**.

2- The vector product (cross product) u×v. The result is a **vector**.

1) The dot product

In this section, we show how to calculate easily the angle between two vectors directly from their components. The dot product is also called ***inner*** or ***scalar*** products because the product results in scalar, not a vector.

Def.: The dot product $u \cdot v$ (u dot v) of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is:

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Note:

$$\left.\begin{array}{c} i \cdot i \\ j \cdot j \\ k \cdot k \end{array}\right] = 1.1 = 1 \quad , \quad \left.\begin{array}{c} i \cdot j \\ j \cdot k \\ k \cdot j \end{array}\right] = 0$$

Ex.:

a)

$(3,5) \cdot (-1,2) = 3(-1) + 5(2) = 7$ scalar

$(3i + 5j) \cdot (-i + 2j) = 7$

b)

$(1,-3,4) \cdot (1,5,2) = 1 - 15 + 8 = -6$ scalar

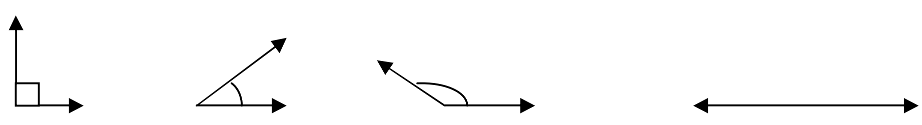
$(i - 3j + 4k) \cdot (i + 5j + 2k) = -6$

Angle between two vectors

The angle θ between two nonzero vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is given by

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}\right) \qquad \text{where } \theta \quad (0 \le \theta \le \pi)$$



u.v=0 u.v>0 u.v<0 Counter clockwise

Ex.: Find the angle between two vectors in space

$\vec{u} = 2\vec{i} - \vec{j} + 2\vec{k} \quad , \quad \vec{v} = \vec{i} - 2\vec{j} + 2\vec{k}$

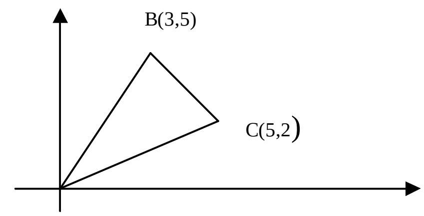
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{2 + 2 + 4}{\sqrt{4 + 1 + 4} \cdot \sqrt{1 + 4 + 4}}$$

$$\cos\theta = \frac{8}{9} \quad \Rightarrow \quad \theta = \cos^{-1}\frac{8}{9}$$

Ex.:

Find the angle q in the triangle ABC determined by the vertices

$A=(0,0)$, B(3,5) and C(5,2)



$$\overrightarrow{CA}=(-5,-2) \quad \text{and} \quad \overrightarrow{CB}=(-2,3)$$
$$\overrightarrow{CA}\cdot\overrightarrow{CB}=(-5)(-2)+(-2)(3)=4$$
$$\left|\overrightarrow{CA}\right|=\sqrt{(-5)^2+(-2)^2}=\sqrt{29}$$
$$\left|\overrightarrow{CB}\right|=\sqrt{(-3)^2+(3)^2}=\sqrt{13}$$
$$q=\cos^{-1}\left(\frac{4}{\sqrt{29}\cdot\sqrt{13}}\right)$$

Orthogonal vectors

Vectors $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ are ***orthogonal*** (or ***perpendicular***) if and only if $u\cdot v=0$

Ex.:

a) $u=(3,-2)$ and $v=(4,6)$ are orthogonal because $u\cdot v=0$

b) $u=3i-2j+k$ and $v=2j+4k$ are orthogonal because $u\cdot v=0$

c) **0** is orthogonal to every vector **u** since

$$0\cdot u=(0,0,0)\cdot(u_1,u_2,u_3)$$
$$=0$$

Properties of the Dot product

If u , v and w are any vectors and c is a scalar, then

1) $u\cdot v=v\cdot u$
2) $(cu)\cdot v=u\cdot(cv)=c(u\cdot v)$
3) $u\cdot(v+w)=u\cdot v+u\cdot w$
4) $u\cdot u=|u|^2$

5) $0 \cdot u = 0$

Vector projection

Vector projection of *u* onto v

$$proj_v\, u = \left(\frac{u \cdot v}{|v|^2}\right) v \qquad \ldots\ldots \quad (1)$$

$proj_v\, u$ ***("The vector projection of u onto v"***

)

Scalar component of *u* in the direction of v

$$|u| \cos q = \frac{u \cdot v}{|v|} = \mathrm{u} \cdot \frac{v}{|v|} \qquad \ldots\ldots. \quad (2)$$

Ex.:

Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v.

Solution:

We find $proj_v\, u$ from eq.(1):

$$proj_v\, u = \left(\frac{u \cdot v}{|v|^2}\right) v = \frac{u \cdot v}{v \cdot v} v = \frac{6 - 6 - 4}{1 + 4 + 4}(i - 2j - 2k) = \frac{-4}{9}(i - 2j - 2k) = \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k$$

We find the scalar component of *u* in the direction of *v* from eq.(2):

$$|u| \cos q = \mathrm{u} \cdot \frac{v}{|v|} = 6i + 3j + 2k \cdot \left(\frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k\right) = 2 - 2 - \frac{4}{3} = -\frac{4}{3}$$

Problems:

1) Let $u = (3,-2)$ and $\mathrm{v} = (-2,5)$. Find the **a)** component form and **b)** magnitude (length) of the vector.

1. $-2u + 5v$
2. $\frac{3}{5} u + \frac{4}{5} v$

2) Find the component form of the vector:

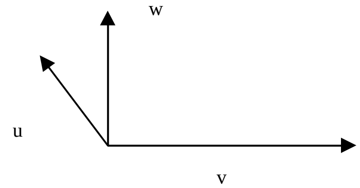
a. The vector $\overrightarrow{PQ}$ where $P = (1,3)$ and $\mathrm{Q}(2,\text{-}1)$.

b. The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS, where $R = (2,\text{-}1)$ and $\mathrm{S} = (-4,3)$.

c. The vector from the point $A = (2,3)$ to the origin.

d. The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where
$A = (1,-1)$, $B = (2,0)$, $C = (-1,3)$ and $D = (-2,2)$

3) Let v , u and w as in the figure: find a**)** $u+v$, **b)** $u+v+w$, **c)** $u-v$ and **d)** $u-w$



4) Express each vector as a product of its length and direction

a. $2i + j - 2\ k$

b. $5\ k$

c. $\frac{3}{5}i + \frac{4}{5}\ k$

d. $\frac{1}{\sqrt{6}}i - \frac{1}{\sqrt{6}}\ j - \frac{1}{\sqrt{6}}\ k$

5) Find the vectors whose lengths and directions are given. Try to do the calculation without writing:

	Length	Direction
a.	2	i
b.	$\sqrt{3}$	- k
c.	$\frac{1}{2}$	$\frac{3}{5}j + \frac{4}{5}k$
d.	7	$\frac{6}{7}i - \frac{2}{7}j + \frac{3}{7}k$

6) Find **a)** the direction of $\overrightarrow{P_1P_2}$ and **b)** the midpoint of line segment P_1P_2.

a. $P_1(-1,1,5)$ and $P_2(2,5,0)$

b. $P_1(0,0,0)$ and $P_2(2,-2,-2)$

7) Find $v \cdot u$, $|v|$, $|u|$, the cosine of the angle between v *and* u , the scalar component of ***u*** in the direction of **v** and the vector $proj_v\ u$.

a) $v = 2i - 4j + \sqrt{5}\ k$ **,** $u = -2i + 4j - \sqrt{5}\ k$

b) $v = (\frac{3}{5})i + (\frac{4}{5})k$ **,** $u = 5i + 12j$

c) $v = -i + j$ **,** $u = \sqrt{2}i + \sqrt{3}j + 2\ k$

d) $v = 5i + j$ **,** $u = 2i + \sqrt{17}j$

e) $v = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$ **,** $u = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{3}}\right)$

8) Find the angles between the vectors:

a) $u = 2i - 2j + k$, $v = 3i + 4k$

b) $u = \sqrt{3}i - 7j$, $v = \sqrt{3}i + j - 2k$

c) $u = i + \sqrt{2}j - \sqrt{2}k$, $v = -i + j + k$

9) Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1,0)$, B(0,3) , C(3,4) and D(4,1)

References:

1- Advanced Engineering Mathematics (Erwin Kreyszic)- 8th Edition.
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**