Solving a system of linear equations

Let A be a matrix, X a column vector, B a column vector then the system of linear equations is denoted by AX=B.

The augmented matrix

The solution to a system of linear equations such as

$$x - 2y = -5$$
$$3x + y = 6$$

Depends on the coefficients of x and y and the constants on the right-hand side of the equation. The matrix of coefficients for this system is the 2×2 matrix

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

If we insert the constants from the right-hand side of the system into the matrix of coefficients, we get the 2×3 matrix

$$\left[\begin{array}{cc|c} 1 & -2 & -5 \\ 3 & 1 & 6 \end{array}\right]$$

We use a vertical line between the coefficients and the constants to represent the equal signs. This matrix is the **augmented matrix** of the system also it can be written as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

Note:

Two systems of linear equations are **equivalent** if they have the same solution set. Two augmented matrices are **equivalent** if the systems they represent are equivalent.

Ex.1:

Write the augmented matrix for each system of equations.

a)
$$x + y - z = 5$$
$$2x + z = 3$$
$$2x - y + 4z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 2 & 0 & 1 & 3 \\ 2 & -1 & 4 & 0 \end{array}\right]$$

b)
$$x + y = 1$$
$$y + z = 6$$
$$z = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

We'll take two methods to solve the system AX=B

1) Cramer's rule

The solution to the system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Is given by $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$ where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Provided that $D \neq 0$

Notes:

1. Cramer's rule works on systems that have **exactly one** solution.
2. Cramer's rule gives us a precise formula for finding the solution to an **independent** system.
3. Note that D is the determinant made up of the original coefficients of x and y. D is used in the denominator for both x and y. D_x is obtained by replacing the first (or x) column of D by the constants c_1 and c_2. D_y is found by replacing the second (or y) column of D by the constants c_1 and c_2.

Ex.1: Use Cramer's rule to solve the system:

$$3x - 2y = 4$$
$$2x + y = -3$$

Sol.:

First find the determinants D, D_x, and $\mathrm{D_y}$:

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7$$

$$D_x = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = 4 - 6 = -2, \qquad D_y = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -9 - 8 = -17$$

By Cramer's rule, we have

$$x = \frac{D_x}{D} = -\frac{2}{7} \quad \text{and} \quad \mathrm{y} = \frac{D_y}{D} = -\frac{17}{7}$$

Check in the original equations. The solution set is $\left\{(-\frac{2}{7}, -\frac{17}{7})\right\}$.

Ex.2: Solve the system:

$$2x+3y=9$$
$$-2x-3y=5$$

Sol.:

Cramer's rule does not work because

$$D=\begin{vmatrix}2 & 3\\ -2 & -3\end{vmatrix}=-6-(-6)=0$$

Because Cramer's rule fails to solve the system, we apply the addition method:

$$\begin{array}{r} 2x+3y=9\\ -2x-3y=5\\ \hline 0=14\end{array}$$

Because this last statement is false, the solution set is **empty**. The original equations are **inconsistent.**

Ex.3: Solve the system:

$$3x-5y=7$$
$$6x-10y=14$$

Sol.: Cramer's rule does not apply because

$$D=\begin{vmatrix}3 & -5\\ 6 & -10\end{vmatrix}=-30-(-30)=0$$

Multiply Eq.(1) by -2 and add it to Eq.(2)

$$\begin{array}{r} -6x+10y=-14\\ 6x-10y=14\\ \hline 0=0\end{array}$$

Because the last statement is an identity, the equations are **dependent**. The solution set is $\{(x,y)|3x-5y=7\}$.

Ex.4: Use Cramer's rule to solve the system:

$$2x-3(y+1)=-3$$
$$2y=3x-5$$

Sol.: First write the equations in standard form, $Ax+By=C$

$$2x-3y=0$$
$$-3x+2y=-5$$

Find D, D_x, and $\mathrm{D_y}$:

$$D=\begin{vmatrix}2 & -3\\ -3 & 2\end{vmatrix}=4-9=-5$$

$$D_x=\begin{vmatrix}0 & -3\\ -5 & 2\end{vmatrix}=0-15=-15, \qquad D_y=\begin{vmatrix}2 & 0\\ -3 & -5\end{vmatrix}=-10-0=-10$$

Using Cramer's rule, we get

$$x=\frac{D_x}{D}=\frac{-15}{-5}=3 \qquad \text{and} \qquad y=\frac{D_y}{D}=\frac{-10}{-5}=2$$

Because (3,2) satisfies both of the original equations, the solution se is $\{(3,2)\}$.

2) The Gaussian Elimination method

When we solve a single equation, we write simpler and simpler equivalent equations to get an equation whose solution is obvious. In the Gaussian elimination method we write simpler and simpler equivalent augmented matrices until we get an augmented matrix in which the solution to the corresponding system is obvious.

Because each row of an augmented matrix represents an equation, we can perform the **row operations** on the augmented matrix.

Elementary Row Operation:

1. Construct the augmented matrix (A:B).
2. Interchange two rows ($R_i \leftrightarrow R_j$).
3. Multiply any row by a constant different from zero ($R_i \leftrightarrow kR_i$)
4. Add a constant multiply of any row to another row ($R_i \leftrightarrow R_i + kR_j$)

Ex.1:

Use Gaussian elimination method to solve the system (two equations in two variables):

$$x - 3y = 11$$
$$2x + y = 1$$

Sol.:

Start with the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 0 & 7 & -21 \end{array}\right] \qquad \mathbf{R}'_2 = -2\mathrm{R}_1 + \mathbf{R}_2$$

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 0 & 1 & -3 \end{array}\right] \qquad \mathbf{R}'_2 = \frac{1}{7}\mathbf{R}_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \qquad \mathbf{R}'_1 = 3\mathbf{R}_2 + \mathbf{R}_1$$

This augmented matrix represents the system $x = 2$ and $y = -3$. So the solution set to the system is $\{(2,-3)\}$.

Ex.2: Use Gaussian elimination method to solve the system (three equations in three variables):

$$2x - y + z = -3$$
$$x + y - z = 6$$
$$3x - y - z = 4$$

Sol.:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & -3 \\ 1 & 1 & -1 & 6 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

$$R_1 \leftrightarrow \mathbf{R}_2 \begin{bmatrix} 1 & 1 & -1 & | & 6 \\ 2 & -1 & 1 & | & -3 \\ 3 & -1 & -1 & | & 4 \end{bmatrix} \quad \begin{matrix} \mathbf{R}'_2 = -2R_1 + \mathbf{R}_2 \\ \mathbf{R}'_3 = -3R_1 + \mathbf{R}_3 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & | & 6 \\ 0 & -3 & 3 & | & -15 \\ 0 & -4 & 2 & | & -14 \end{bmatrix}$$

$$R'_2 = -\frac{1}{3}R_2 \begin{bmatrix} 1 & 1 & -1 & | & 6 \\ 0 & 1 & -1 & | & 5 \\ 0 & -4 & 2 & | & -14 \end{bmatrix} \quad \begin{matrix} \mathbf{R}'_1 = -R_2 + \mathbf{R}_1 \\ \mathbf{R}'_3 = 4R_2 + \mathbf{R}_3 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & -1 & | & 5 \\ 0 & 0 & -2 & | & 6 \end{bmatrix}$$

$$R'_3 = -\frac{1}{2}R_3 \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & -1 & | & 5 \\ 0 & 0 & 1 & | & -3 \end{bmatrix} \quad R'_2 = R_3 + \mathbf{R}_2 \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -3 \end{bmatrix}$$

This augmented matrix represents the system $x = 1,\ y = 2$ and $z = -3$. So the solution set to the system is $\{(1,2,-3)\}$.

Ex.3: Solve the system

$$x - y = 1$$
$$-3x + 3y = 4$$

Sol.:

$$\begin{bmatrix} 1 & -1 & | & 1 \\ -3 & 3 & | & 4 \end{bmatrix} \to \mathbf{R}'_2 = 3R_1 + \mathbf{R}_2 \begin{bmatrix} 1 & -1 & | & 1 \\ 0 & 0 & | & 7 \end{bmatrix}$$

R_2 corresponds to the equation 0 = 7. So the equations are **inconsistent**, and there is no solution to the system.

Matrix Inverse

The matrix A has an inverse denoted by A^{-1} if $|A| \neq 0$ where $A.A^{-1} = I$. We'll take two methods to find A^{-1} where A is an n×n matrix.

1) By Gauss elimination method(Using row operations)**:**

1. Construct the augment matrix (A:I)
2. Use row operation until we have ($I:A^{-1}$)

Ex1: Use Row operation to find A^{-1} if A= $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 1 & 4 & | & 0 & 1 \end{bmatrix} \to \mathbf{R}_1 = \frac{1}{2}\mathbf{R}_1 \to \begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 1 & 4 & | & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}'_2 = \mathbf{R}_2 - \mathbf{R}_1 \to \begin{bmatrix} 1 & \frac{1}{2} & | & \frac{1}{2} & 0 \\ 0 & \frac{7}{2} & | & \frac{-1}{2} & 1 \end{bmatrix}$$

$$\mathbf{R}_2' = \frac{2}{7}\mathbf{R}_2 \quad \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{-1}{7} & \frac{2}{4} \end{array}\right]$$

$$\mathbf{R}_1' = \mathbf{R}_1 - \frac{1}{2}\mathbf{R}_2 = \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{7} & \frac{-1}{7} \\ 0 & 1 & \frac{-1}{7} & \frac{2}{7} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}$$

Ex2: Find A^{-1} if $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \mathbf{R}_1 = \frac{1}{2}\mathbf{R}_1 \left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow R_2 = R_2 - R_1$$

$$\rightarrow R_3 = R_3 - 4R_1 \left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{-7}{2} & \frac{-1}{2} & 1 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{array}\right] \rightarrow R_2 = 2R_2 \left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 2 & -4 & -2 & 0 & 1 \end{array}\right]$$

$$\rightarrow R_1 = \frac{1}{2}\mathbf{R}_2 + \mathbf{R}_1 \rightarrow R_3 = -2R_2 + R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{-4}{2} & 0 & 1 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 10 & 0 & -4 & 1 \end{array}\right] \rightarrow \mathbf{R}_3 = \frac{1}{10}\mathbf{R}_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -7 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 & \frac{-2}{5} & \frac{1}{10} \end{array}\right] \rightarrow R_1 = 2R_3 + R_1 \rightarrow R_2 = R_2 + 7R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & 0 & 1 & 0 & \frac{-2}{5} & \frac{1}{10} \end{array}\right]$$

$$\therefore \mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$$

2) **By Cofactor Method** (Using determinant of the matrix)

The cofactor of the element a_{ij} of the matrix $A = (a_{ij})$ is defined by $c_{ij} = (-1)^{i+j} A_{ij}$ where A_{ij} is the determinant of the matrix that remains when the row i and the column j are deleted.

To find the inverse of a matrix whose determinant is not zero

1- construct the matrix of cofactors of A, cof (A) = c_{ij}
2- Construct the transposed matrix of cofactors called the adjoin of A = adj (A) = cof $(A)^T$
3- then $A^{-1} = \dfrac{1}{\det(\mathbf{A})}$ adj A
4- to check your answer $A.A^{-1} = I$ or $A^{-1}.A = I$

<u>Ex.</u>: Use determinant to find A^{-1} where $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{adj}(\mathbf{A})$$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7$$

$$\text{Cof}(A) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

$C_{11} = (-1)^{1+1}\,|4| = 4$
$C_{12} = (-1)^{1+2}\,|1| = -1$
$C_{21} = (-1)^{2+1}\,|1| = -1$
$C_{22} = (-1)^{2+2}\,|2| = 2$

$$\text{Adj}(A) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix}$$

Ex2: Find A^{-1} if $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$

Solution:

$$|\mathbf{A}| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{vmatrix} = 2 + 8 = 10$$

$$\mathbf{cof}(\mathbf{A}) = \begin{pmatrix} 0 & -10 & 0 \\ 2 & -8 & -4 \\ 2 & 7 & 1 \end{pmatrix}$$

$$\mathbf{c}_{11} = 0, \quad \mathbf{c}_{12} = (-1)10 = -10, \quad \mathbf{c}_{13} = 0,$$
$$\mathbf{c}_{21} = -(-2) = 2, \quad \mathbf{c}_{22} = -8, \quad \mathbf{c}_{23} = -4,$$
$$\mathbf{c}_{31} = 2, \quad \mathbf{c}_{32} = (-1)(-7) = 7, \quad \mathbf{c}_{33} = 1$$

$$\text{Adj(A)} = \text{cof}^{t} = \begin{pmatrix} 0 & 2 & 2 \\ -10 & -8 & 7 \\ 0 & -4 & 1 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$$