Matrices:

When a system of equations has more than two equations, it is difficult to discuss them without using matrices and vectors.

The size of the matrix is described by the number of its row and columns. A matrix of n rows and m columns is said to be $n\times m$ matrix.

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n\times m} = \left[a_{ij}\right], \quad \mathrm{i}=1,2,...,\mathrm{n} \quad , \quad \mathrm{j}=1,2,...,\mathrm{m}.$$

Types of matrices:

Square matrix: It is a matrix whose number of rows are equal to the number of columns ($n=m$). For example:

$$A=\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}_{2\times 2} \quad , \quad B=\begin{bmatrix} 1 & 3 & 0 \\ 3 & 2 & 1 \\ 1 & 8 & 0 \end{bmatrix}_{3\times 3}$$

Diagonal matrix: It is a square matrix which all its elements are zero except the elements on the main diagonal. For example:

$$A=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrix: It is a diagonal matrix whose elements on the main diagonal are equal to 1, and it is denoted by I_n. For example:

$$I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Transpose matrix: Transpose of A is denoted by (A^T), means that write the rows of A as columns in A^t. For example:

$$A=\begin{bmatrix} 3 & 7 & 1 \\ -2 & 1 & -3 \end{bmatrix}_{2\times 3} \quad , \quad \mathrm{A}^{\mathrm{T}}=\begin{bmatrix} 3 & -2 \\ 7 & 1 \\ 1 & -3 \end{bmatrix}_{3\times 2}$$

Matrix addition and multiplication

If $A = \lfloor a_{ij} \rfloor$ and $B = \lfloor b_{ij} \rfloor$ and both A & B are $n \times m$ matrices, then
$A + B = \lfloor a_{ij} \rfloor + \lfloor b_{ij} \rfloor = \lfloor a_{ij} + b_{ij} \rfloor$

Ex.1:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$$

For any scalar (number) c, we can multiply A by c as follows:

$$cA = c\lfloor a_{ij} \rfloor = \lfloor ca_{ij} \rfloor$$

Ex.2:

$$3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}$$

A matrix with only one column, $n \times 1$ in size, is called **a column vector**, and one of only one row, $1 \times m$ in size, is called **a row vector**.

Matrices multiplication

Let A be an n× k matrix and B be a k×m matrix then C=AB is an n×m matrix, where the element in the ith row and jth column of AB is the sum

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,p.$$

Ex.3

Suppose $\mathbf{A} = \begin{bmatrix} 3 & 7 & 1 \\ -2 & 1 & -3 \end{bmatrix}_{2\times3}$, $\mathsf{B} = \begin{bmatrix} 5 & -2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix}_{3\times2}$ then

$$AB = \begin{bmatrix} 16 & 14 \\ -13 & 10 \end{bmatrix}_{2\times2}$$

$$BA = \begin{bmatrix} 19 & 33 & 11 \\ -6 & 3 & -9 \\ 5 & 6 & 4 \end{bmatrix}_{3\times3}$$

Determinants

With each square matrix A we associate a number det(A) or $|a_{ij}|$ called the determinant of A, calculated from the entries of A as follows:

For n=1, det(a)=a,

For n =2, $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Minors

To each element of a 3×3 matrix there corresponds a 2×2 matrix that is obtained by deleting the *row and column* of that element. The determinant of the 2×2 matrix is called the **minor** of that element.

For a matrix of dimension 3×3, we define

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

where $\begin{vmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{32} & \mathbf{a}_{33} \end{vmatrix}$ is the minor of a_{11}, $\begin{vmatrix} \mathbf{a}_{21} & \mathbf{a}_{23} \\ \mathbf{a}_{31} & \mathbf{a}_{33} \end{vmatrix}$ is the minor of a_{12},

and $\begin{vmatrix} \mathbf{a}_{21} & \mathbf{a}_{22} \\ \mathbf{a}_{31} & \mathbf{a}_{32} \end{vmatrix}$ is the minor of a_{13}.

Ex.4:

Find the determinant of each matrix

a) $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

$$\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = 1(5) - 3(-2) = 5 + 6 = 11$$

b) $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

$$\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 2(12) - 4(6) = 0$$

Ex.5: Find the determinant of A where:

$$A = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

Sol.: By choosing the first column we get

$$\det(A) = \begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} = 1\cdot\begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - (-2)\cdot\begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 0\cdot\begin{vmatrix} 3 & -5 \\ 4 & 6 \end{vmatrix}$$

$$= 1\cdot[36 - (-42)] + 2\cdot(27 - 35)$$

$$= 78 - 16 = 62$$

Ex.6: Evaluate the determinant of A if:

$$A = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

Solution:

By choosing the second row we get

$$\det(A) = \begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} = -(-2)\begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & -5 \\ 0 & 9 \end{vmatrix} - 6 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -7 \end{vmatrix}$$

$$= 2(27 - 35) + 4(9 - 0) - 6(-7 - 0)$$

$$= -16 + 36 + 42 = 62$$

Note that 62 is the same value that was obtained for this determinant in Example above.

Note:

If a matrix A is triangular (either upper or lower), its determinant is just the product of the diagonal elements:

Linearly Dependent and Linearly Independent

Definition: the vectors $v_1, v_2, \ldots, v_m$ are linearly dependent if $|\, v_1 \;\; v_2 \;\; \ldots \;\; v_m| = 0$, and if $|\, v_1 \;\; v_2 \;\; \ldots \;\; v_m| \neq 0$ then $v_1, v_2, \ldots, v_m$ are linearly independent.

Ex1: Let $v_1=(3, 6, -1)$; $v_2=(8, 2, -4)$; $v_3=(1, -1, 1)$, determine whether v_1, v_2, v_3 are linearly dependent or not.

Sol: Since

$$\begin{vmatrix} 3 & 8 & 1 \\ 6 & 2 & -1 \\ -1 & -4 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} - 8\begin{vmatrix} 6 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ -1 & -4 \end{vmatrix} = 3(2-4) - 8(6-1) + (-24+2) = -68 \neq 0$$

then v_1, v_2, v_3 are linearly independent

Ex2: Let $v_1=(2, 4, 6)$; $v_2=(1, 3, 3)$; $v_3=(1, 2, 3)$, determine whether v_1, v_2, v_3 are linearly dependent or not.

Sol: Since

$$\begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 3 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 3 & 3 \end{vmatrix} - \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 6 & 3 \end{vmatrix} = 2(9-6) - (12-12) + (12-18) = 6 - 0 - 6 = 0$$

then v_1, v_2, v_3 are linearly dependent

Exercise:

1) Determine whether the given vectors are linearly dependent or linearly **independent**.

   a) (3,2);(1,-1)

   b) (4,-3,1);(10,-3,0);(2,-6,3)

2) Find determinant of the following matrices

a) $\begin{pmatrix} -4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 3 & -2 & 1 \\ 1 & 10 & -1 \\ -3 & -2 & 1 \end{pmatrix}$

References:

1- Calculus & Analytic Geometry (Thomas).

2- Calculus (Haward Anton).

3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**