C. Higher order Differential Equations:

How to find roots of an equation:

Let $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$ be an eq. of degree n, we denote this eq. by $f(x) = 0$ then:

1) r is a root of the eq. $f(x) = 0$ if $f(r) = 0$.
2) r is repeated root of the eq. $f(x) = 0$ if $f'(r) = 0$.
3) If r is a root of the eq. $f(x) = 0$, then r must be a factor of a_n.
4) If r is a root of the eq. $f(x) = 0$, then $(x - r)$ divides $f(x)$.

<u>Ex.:</u> Find all roots of $x^3 + 4x^2 - 3x - 18 = 0$

Solution:

$a_n = 18 : \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$f(2) = 8 + 16 - 6 - 18 = 0$, 2 is a root of the eq.

There are two methods to factorize f(x): *long division* & *fast division*.

First method: Fast division

	1	4	-3	-18
2	↓	2	12	18
	1	6	9	0

$\Rightarrow \quad x^2 + 6x + 9 = 0$

$\Rightarrow (x-2)(x^2 + 6x + 9) = 0$

$(x-2)(x+3)(x+3) = 0$

$x_1 = 2, \quad x_2 = -3, \quad x_3 = -3$

The roots are 2, -3, -3

Second method: long division

$x^3 + 4x^2 - 3x - 18 = 0$

$\Rightarrow (x-2)(x^2 + 6x + 9) = 0$

$(x-2)(x+3)(x+3) = 0$

$$x^2 + 6x + 9$$
$$(x-2)\overline{)\, x^3 + 4x^2 - 3x - 18}$$
$$\underline{\mp x^3 \pm 2x^2}$$
$$6x^2 - 3x$$
$$\underline{\mp 6x^2 \pm 12x}$$
$$9x - 18$$
$$\underline{\mp 9x \pm 18}$$
$$\mathbf{0}$$

Higher order linear Differential Equations:

The general form of with constant coefficient

is: $y^{(n)} + a_1 y^{(n-1)} + \mathbf{K} + a_{n-1} y + a_n y = F(x)$ ......(1)

If $F(x) = 0$ then (1) is called *homogenous, otherwise* (1) is called *nonhomogenous.*

<u>The general solution</u>

The methods of solving second order homogenous D.Eqs. with constant coefficients can be extended to solve higher order homogenous and nonhomogenous D.Eq. with constant coefficients.

a) Homogenous: the characteristic equation of nth order homogenous D. Eq.:

$y^{(n)} + a_1 y^{(n-1)} + \mathbf{K} + a_{n-1} y + a_n y = 0$ is:

$r^n + a_1 r^{n-1} + \mathbf{K} + a_{n-1} r + a_n = 0$

Let r_1, r_2, …, rn be the roots of characteristic equation then:

1) If r_1 , r_2 , ... , r_n are all distinct then the solution is:

$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \mathbf{K} + c_n e^{r_n x}$

2) If r_1 repeated m times, then y_h will contain the terms:

$c_1 e^{r_1 x} + c_2 x e^{r_1 x} + \ldots + c_m x^{m-1} e^{r_1 x}$

3) If some of roots are complex ($r = a \mathbf{m} ib$) then y_h will contain

$(c_1 \cos bx + c_2 \sin bx) e^{ax}$

Ex.1: solve y''' -3y''+2y'=0

Solution:

$$r^3 - 3r^2 + 2r = 0 \quad \Rightarrow \quad r(r^2 - 3r + 2) = 0$$

$$r(r-2)(r-1) = 0$$

$$\Rightarrow \quad r_1 = 0 \ , \ r_2 = 2 \ , \ r_3 = 1$$

are all distinct

$$\Rightarrow \quad y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x}$$

$$y_h = c_1 + c_2 e^{2x} + c_3 e^x$$

Ex.2:

$y^{(4)}-3y'''+3y''-y'=0$

$$r^4-3r^3+3r^2-r=0 \quad \Rightarrow \quad r(r^3-3r^2+3r-1)=0$$

$$r(r-1)^3=0$$

$$\Rightarrow \quad r_1=0 \ , \ r_2=r_3=r_4=1 \qquad \Rightarrow \ m=3$$

$$\Rightarrow y_h=c_1e^{r_1x}+(c_2x^{m-1}+c_3x^{m-2}+c_4)e^{rx}$$

$$y_h=c_1+c_2e^x+c_3xe^x+c_4x^2e^x$$

b) Nonhomogenous: the general form of nth order nonhomogenous differential equation is:

$$y^{(n)}+a_1y^{(n-1)}+\mathbf{K}+a_{n-1}y+a_ny=F(x) \qquad ......(1)$$

The general solution is $y_g=y_h+y_p$

<u>Methods of finding y_p:</u>

1) Undetermined coefficients

We can extend the methods of solving second order non homogenous D.Eqs. with constant coefficients to solve higher order nonhomogenous D.Eq. with constant coefficients.

Ex.1: $y^{(4)}-8y''+16y=-18\sin x$

Solution:

$y_g=y_h+y_p$

$y^{(4)}-8y''+16y=0$

$r^4-8r^2+16=0 \Rightarrow (r^2-4)^2=0 \Rightarrow r^2=4 \Rightarrow r=\pm 2$

$y_h=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+c_4xe^{-2x}$

let $y_p=A\cos x+B\sin x$, $y'_p=-A\sin x+B\cos x$, $y''_p=-A\cos x-B\sin x$

$y'''_p=A\sin x-B\cos x$, $y^{(4)}_p=A\cos x+B\sin x$

$A\cos x+B\sin x+8A\cos x+8B\sin x+16A\cos x+16B\sin x=-18\sin x$

$25A\cos x+25B\sin x=-18\sin x$

$25A=0 \Rightarrow A=0$

$25B=-18 \Rightarrow B=-18/25$

$$y_g=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+c_4xe^{-2x}-\frac{18}{25}\sin x$$

2) Variation of parameters

In this method, the particular solution yp has the form $y_p=v_1u_1+v_2u_2+\ldots+v_nu_n$

Where $u_1, u_2, \ldots, u_n$ are taken from $y_h=c_1u_1+c_2u_2+\ldots+c_nu_n$.

To find $v_1, v_2, \ldots, v_n$, we must solve the following linear eqs. For $v'_1, v'_2, \ldots, v'_n$:

$$\begin{aligned}
v'_1u_1+v'_2u_2+\ldots+v'_nu_n&=0\\
v'_1u'_1+v'_2u'_2+\ldots+v'_nu'_n&=0\\
&\vdots\\
v'_1u_1^{(n-2)}+v'_2u_2^{(n-2)}+\ldots+v'_nu_n^{(n-2)}&=0\\
v'_1u_1^{(n-1)}+v'_2u_2^{(n-1)}+\ldots+v'_nu_n^{(n-1)}&=f(x)
\end{aligned}$$

Ex2: solve y'''+y'=secx

Solution:

Let y'''+y'=0

$r^3+r=0 \Rightarrow r(r^2+1)=0 \Rightarrow r=0,\ r^2=-1 \Rightarrow r1=0,\ r=\pm i$

$y_h=c_1+c_2\cos x+c_3\sin x$

$u_1=1,\ u_2=\cos x,\ u_3=\sin x,\ f(x)=\sec x$

$$\begin{aligned}
v'_1+v'_2\cos x+v'_3\sin x&=0\\
v'_1*0+v'_2(-\sin x)+v_3(\cos x)&=0\\
v'_1*0+v'_2(-\cos x)-v'_3(\sin x)&=\sec x
\end{aligned}$$

$$D=\begin{vmatrix}1 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ 0 & -\cos x & -\sin x\end{vmatrix}=\sin^2 x+\cos^2 x=1$$

$$D_1=\begin{vmatrix}0 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ \sec x & -\cos x & -\sin x\end{vmatrix}=\sec x(\sin^2 x+\cos^2 x)=\sec x$$

$$D_2=\begin{vmatrix}1 & 0 & \sin x\\ 0 & 0 & \cos x\\ 0 & \sec x & -\sin x\end{vmatrix}=\begin{vmatrix}0 & \cos x\\ \sec x & -\sin x\end{vmatrix}=-\cos x\sec x=-1$$

$$D_3=\begin{vmatrix}1 & \cos x & 0\\ 0 & -\sin x & 0\\ 0 & -\cos x & \sec x\end{vmatrix}=\begin{vmatrix}-\sin x & 0\\ -\cos x & \sec x\end{vmatrix}=-\sin x\sec x=-\tan x$$

$$v_1'=\frac{D_1}{D}=\sec x \Rightarrow v_1=\int \sec x\,dx=\ln(\sec x+\tan x)$$

$$v_2'=\frac{D_2}{D}=-1 \Rightarrow v_2=\int -1\,dx=-x$$

$$v_3' = \frac{D_3}{D} = -\tan x \Rightarrow v_3 = -\int \tan x \, dx = -\ln \cos x$$

y_p= ln (secx+tanx)-x cosx-ln cosx sinx

y_g= c_1+$c_2$cosx+$c_3$sinx+ ln (secx+tanx)-x cosx - ln cosx sinx