Chapter one

Functions

R: The set of all real numbers.

(a,b) or $a< x< b$

[a,b] or $a\leq x \leq b$

<u>Cartesian coordinates or xy-plane</u>

(x,y) is appoint in xy-plane

<u>Function</u>:

Afunction f from asset $D=\{x \backslash x\in R\}$ to asset Ris arule that a ssigns a single element of Rto each element in D, denoted by y=f(x).The set D is called domain of f and the set $R=\{y \backslash y\in R,\ y=f(x)\}$ is called the range of f.

<u>Axiom for domain:</u>

1- we does not take the values of x which introduce division by zero.
2- We does not take the values of x which introduce aroot anegative number.

Example:

Find the domain and the range of y=f(x)= $\sqrt{x-2}$

Solution:

$x-2\geq 0$

$x\geq 2$

$D=\{x\backslash x\in R\ ,\ x\geq 2\}$ and $R=\{y\backslash y\in R\ ,\ y\geq 0\}$ or $R=R^{+}$

Definition:

If f(x) and g(x) are two functions then f(x)+ g(x)

F(x).g(x) and $\frac{f(x)}{g(x)}$,$g(x)\neq 0$ are also functions.

Definition: The composition of f(x) and g(x) is denoted by f0g or g0f and is defined by : (g0f)(x)=g[f(x)]

<u>Example</u>

Write f0g if f(x)=$\sqrt{x}$ and g(x)=$\frac{x}{2}$

<u>Solution</u>

(f0g)(x)=f[g(x)]=f($\frac{x}{2}$)=$\sqrt{\frac{x}{2}}$

Some functions

Trigonometric functions:

$$\sin\theta = \frac{y}{r}$$

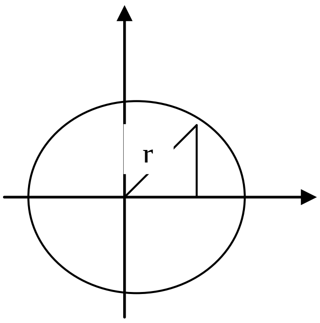
$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$$



1. $y = f(x) = \sin x, D = \Re, R = [y / y \in \Re, -1 \le y \le 1\}$
2. $y = f(x) = \cos x, D = \Re, R = [y / y \in \Re, -1 \le y \le 1\}$
3. $y = f(x) = \tan x, D = \{x / x \in \Re, x \ne \mp \frac{n\pi}{2}$, n, is odd number}

Absolute value function:

Let a be any real number,we define

$$|a| = \begin{cases} a & a \geq 0 \\ -a & a \prec 0 \end{cases}$$

Properties:

1- $|a+b| \leq \quad |a| + \quad |b|$

2- $|a| \leq r \qquad\qquad -r \leq a \leq r$

Example:

find the value of x satisfy $\left|\frac{3x+1}{2}\right| \prec 1$

Solution:

$$-1 \prec \frac{3x+1}{2} \prec 1$$

$$-2 \prec 3x+1 \prec 2$$

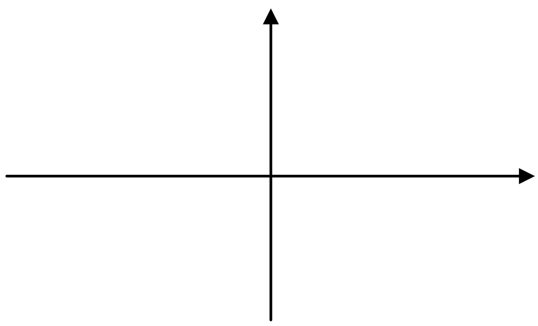
$$-3 \prec 3x \prec 1$$

$$-1 \prec x \prec \frac{1}{3}$$

Step function

Y=f(x)=[x] , $D = \Re$, $R = \Re$,[x]=the greatest integer less than or equal to x

Example



[1.9]=1,[2]=2,[0.5]=0,[-0.5]=-1,[-2.7]=-3

Limits

1-The limit of f(x) as x approaches a from the right is the number L if $\lim_{x\to a^+} f(x) = L.$

2-The limit of f(x) as x approaches a from the Left is the number L if $\lim_{x\to a^-} f(x) = L.$

3-The number L is the limit of f(x) as x approaches a , denoted by $\lim_{x\to a} f(x) = L.$ if and only if

$$\lim_{x\to a^+} f(x) = L. = \lim_{x\to a^-} f(x) = L.$$

Theorem 1:

If $\lim_{x\to a} f(x) = L1.$ and $\lim_{x\to a} g(x) = L2.$, then

1- $\lim_{x\to a}(f(x)\mp g(x)) = \lim_{x\to a} f(x) \mp \lim_{x\to a} g(x) = l_1 \mp l_2$

2- $\lim_{x\to a}(f(x).g(x)) = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = l_1.l_2$

3- $\lim_{x\to a} kf(x) = k \lim_{x\to a} = k.l_1.$

4- $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} f(x)} = \frac{l_1}{l_2}$ if $L_2 \neq 0$

Theorem 2

1- $\lim_{x\to a}(b_0 x^n + b_1 x^{n-1} + \ldots + b_n) = b_0 a^n + b_1 a^{n-1} + \cdots + b_n$

2- $\lim_{x\to 0} \frac{\sin x}{x} = 1$

3- If $g(x) \le f(x) \le h(x)$, are three functions such that $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$ then Lis the limit of f(x) as x apprpaches a.

Example:find

$$\lim_{x\to 3} \frac{x^2 + 2x + 4}{x + 2} = \frac{19}{5}$$

Infinity as a limit:

1- $\lim_{x\to 0^+} \frac{1}{x} = \infty$

2- $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

3- $\lim_{x\to \infty} \frac{1}{x} = 0$

4- $\lim_{x\to -\infty} \frac{1}{x} = 0$

Example: evaluate

$$\lim_{x\to\infty}\frac{x}{2x+4}=0=\frac{1}{2}$$
$$x=\cos\theta$$
$$y=\sin\theta$$

1. $\cos^2\theta+\sin^2\theta=1$
2. $1+\tan^2\theta=\sec^2\theta \;,1+\cot^2\theta=\csc^2\theta$
3. $\sin(\theta_1+\theta_2)\sin\theta_1\cos\theta_2+\cos\theta_1+\sin\theta_2$
4. $\cos(\theta_1+\theta_2)\cos\theta_1\cos\theta_2-\sin\theta_1+\sin\theta_2$
5. $\sin(2\theta)2\sin\theta\cos\theta$ $\qquad$ $,\cos(2\theta)\cos^2\theta-\sin^2\theta$
6. $\cos^2\theta=\dfrac{1+\cos(2\theta)}{2}$ $\qquad$ $,\ \sin^2\theta=\dfrac{1-\cos(2\theta)}{2}$

Example: evaluate

$$\lim_{x\to 0}\frac{1-\cos(2x)}{2x}$$

$$\lim_{x\to 0}\frac{1-\cos(2x)}{2x}=\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\sin x=1.0=0$$

Continuity

A function y=f(x) is said to be continuous at a point a if

1- f(a) exists

2- $\lim_{x\to a} f(x)$ Exists

3- $\lim_{x\to a} f(x)=f(a)$

Example: test the continuity of

$f(x)=\dfrac{1}{x-2}$ $\qquad$ a=2

Solution

F(x) does not exist of x=2 then f is not cont.

Derivatives:

The derivative of afunctiion y=f(x) is denoted by $f'(x)$ and defined by:

$f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ $\qquad$ if the limit exists.

Also $f'(x)$ is called the slope of the curve y=f(x)

Laws of derivatives:

the derivative of a constant is zero.

1. $(x^n)' = nx^{n-1}$
2. $(cf(x))' = cf(x)'$
3. $(f(x) \mp g(x))' = f(x)' + g(x)'$
4. $(f(x).g(x))' = f(x).g(x)' + f(x)' g(x)$
5. $[(f(x))^n]' = n(f(x)^{n-1} f(x)'$
6. $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f(x)' - f(x)g(x)'}{g(x)^2}$

Example: Find y'

$y = f(x) = x\sqrt{x^2 - 2}$

Solution

$y' = x[1/2(x^2 - 2)^{-1/2}(2x)] + \sqrt{x^2 - 2}$

Higher Derivatives

Implicit differentiation

Chain rule

<u>**Derivative of trigonometric functions**</u>:

1. $(\sin u)' = \cos u.du/dx$
2. $(\cos u)' = -\sin u.du/dx$
3. $(\tan u)' = \sec u^2.du/dx$
4. $(\cot u)' = -\csc u^2.du/dx$
5. $(\sec u)' = \sec u.\tan u.du/dx$
6. $(\csc u)' = -\csc u.\cot u.du/dx$

Example: find y'

$y = f(x) = \sin 3x$

Solution

$y' = 3\cos(3x)$

Inverse function

1. A function f(x) is said to be one to one if $f(x_1)=f(x_2)$ implies that $x_1=x_2$. Geometrically f(x) is one to one if we draw aline parallel to x-axis it will cute or intersect curve of f(x) in one point only.
2. A function f(x) is said to be onto if for every b on y-axis there exists a point a on x-axis such that f(a)=b
3. if f(x) is one to one and onto then there exists $f(x)^{-1}$ called the inverse of f(x) such that $fof^{-1} = f^{-1}of = x$

Example: find the inverse of y=f(x)=1/4x+3
Solution:
f^{-1}=4x-12

Application of Derivatives:
Increasing and Decreasing function:

Suppose that f(x) has derivative at every point x in an interval I then:

1- f increase on I if $f'(x) \succ 0$ for all x in I
2- f decrease on I if $f'(x) \prec 0$ for all x in I

Maximum and Minimum:

Let y=f(x) be a function such that y' and y'' exist first find critical points by solving y'=0. let a be one of these critical points
Test1
1- $f(a)$ is maximum if $f'(a-1) \succ 0 and f'(a+1) \prec 0$
2- f(a) is minimum if $f'(a-1) \prec 0 and f'(a+1) \succ 0$
Test2
1- $f(a)$ is maximum if $f''(a) \prec 0$
2- $f(a)$ is maximum $f''(a) \succ 0$

Concavity and points of inflection
F(a) is appoint of inflection to y=f(x) if f''(a)=0

1- the graph of f(x) is concave down on any interval where $y'' \prec 0$
2- the graph of f(x) is concave up on any interval where $y'' \succ 0$

example:
sketch the curve $y = 1/6(x^3 - 6x^2 + 9x + 6)$
solution
$y' = 1/6(3x^2 - 12x + 9) = 1/2(x^2 - 4x + 3) = 1/2(x-1)(x-3)$

y'=0 x=1 and x=3 are critical points
f(1)=5/3 is maximum
f(3)=1 is minimum
$y'' = 1/2(2x-4) = x-2$
y''=0, x=2 f(2)=4/3 is point of inflection

L'Hopital Rule:

$$if \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} or \frac{\infty}{\infty} \quad ,\text{then}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{'}(x)}{g^{'}(x)}$$

Example: find

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

Even and odd function:

1-Afunction f(x) is said to be an even function if f(-x)=f(x), geometrically the graph of f(x) must be symmetric about y-axis.
2-Afunction f(x) is said to be an odd function if f(-x)=-f(x),geometrically the graph of f(x) must be symmetric about origin.
Example
1-x^2 , cosx even function
2-x,x^3,sinx odd function

Chapter two

Integration

Laws of integration

1. $\int dx = x + c$
2. $\int kf(x)dx = k\int f(x)dx$
3. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
4. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ if n≠-1
5. $\int (f(x))^n f(x)'dx = \frac{(f(x)^{n+1})}{n+1} + c$ if n≠-1

Example: Evaluate

$\int \sqrt{x}dx = \int x^{1/2}dx = \frac{x^{3/2}}{3/2} + c$

<u>Integration of Trigonometric functions</u>

1- $\int \cos u du = \sin u + c$

2- $\int \sin u du = -\cos u + c$

3- $\int \sec^2 u du = \tan u + c$

4- $\int \csc^2 u du = -\cot u + c$

5- $\int \sec u \tan u du = \sec u + c$

6- $\int \csc u \cot u du = -\csc u + c$

Example: Evaluate

$\int \tan x \sec^2 x dx$

Solution

$\frac{\tan^2 x}{2} + c$

<u>Define Integral</u>:

The area under the curve y=f(x) from x=a to x=b is defined by:

A=$\int_a^b f(x)dx$

Properties::

1-$\int_a^b kf(x)dx = k\int_a^b f(x)dx$ k is constant

2-$\int_a^b f(x)dx \geq 0$

3- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

4- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ c is appoint in [a,b]

5- if f is continuous on [a,b] and f is any antderivatives of f on [a,b] then

$$\int_a^b f(x)dx = f(x)]_a^b = f(a) - f(b)$$

Remark

A= $\int_c^d f(y)dy$

Example

Find the area y=f(x)=sinx from x=0 to x=π

Solution

A= $\int_a^b f(x)dx = \int_0^\pi \sin xdx = -\cos x]_0^\pi = -\cos\pi + \cos 0 = 2$

Application of Define integral:

<u>Area between two curves</u>:

A= $\int_a^b [f_1(x) - f_2(x)]dx$

A= $\int_c^d [f_1(y) - f_2(y)]dy$

Example

Find the area bounded by y=2-x^2 and y=-x

Solution

2-x^2=-x

x^2-x-2=0=(x-2)(x+1)=0

x=-1 ,y=1or x=2,y=-2.

The iintersetion points are (-1,1) and (2,-2)

A= $\int_{-1}^2 (2 - x^2 + x)dx = 9/2$

Volume of Rotation

1- Disk Method

V= $\int_a^b \pi[f(x)]^2 dx$

F(x) is rotated about x-axis

V= $d\int_c^b \pi[f(y)]^2 dy$

F(y) is rotated about y-axis

Example find the volume of the solid generated by rotating the function $y=\sqrt{x}$ about x-axis from x=0 to x=4

Solution

V=$\int_a^b \pi[f(x)]^2\,dx=\int_0^4 \pi(\sqrt{x})^2\,dx=8\pi$

2-- V=$\int \pi[f_2(x)^2-f_1(x)^2]dx$

The area is rotated about x-axis

V=$\int \pi[f_2(y)^2-f_1(y)^2]dx$

The area is rotated about y-axis

Example

The region bounded by x=y is rotated about y-axis to generate asolid .find its volume

Solution

$y^2=y$

$y^2-y=0\rightarrow y(y-1)=0\rightarrow y=0 or y=1$

V=$\int_0^1 \pi(y^2-y^4)dy=\pi\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1=\pi\left[\frac{1}{3}-\frac{1}{5}\right]=\frac{2\pi}{15}$

Chapter three

Transcendental Functions

<u>Inverse of Trigonometric Functions</u>

$y=f(x)=\sin^{-1}x, D=\{x/-1\le x\le 1\}, R=\{y/-\pi/2\le y\le \pi\}$

$y=f(x)=\cos^{-1}x, D=\{x/-1\le x\le 1\}, R=\{y/0/2\le y\le \pi\}$

$y=f(x)=\tan^{-1}x, D=\Re, r=\{y/-\pi/2\le y\le \pi/2\}$

$y=f(x)=\cot^{-1}x, D=\Re, R=\{y/0\le y\le \pi\}$

$y=f(x)=\sec^{-1}x, D=\{x/x\ge 1, x\le -1\}, R=\{y/y\ne n(\pi/2)\}$

$y=f(x)=\csc^{-1}x, D=\{x/x\ge 1, x\le -1\}, R=\{y/0\le /y/\le \pi/2\}$

Properties

$\sin^{-1}(-x) = -\sin^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

$\sin^{-1} x + \cos^{-1} x = \pi / 2$

$\cot^{-1} x = \pi / 2 - \tan^{-1} x$

$\sec^{-1} x = \cos^{-1}(1/x)$

$\csc^{-1} x = \sin^{-1}(1/x)$

$\csc^{-1}(-x) = \pi - \sec^{-1} x$

Derivatives

$(\sin^{-1} u)' = \dfrac{du/dx}{\sqrt{1-u^2}}$ 1-

$(\cos^{-1} u)' = \dfrac{-du/dx}{\sqrt{1-u^2}}$ 2-

$(\tan^{-1} u)' = \dfrac{du/dx}{1+u^2}$ 3-

$(\cot^{-1} u)' = \dfrac{-du/dx}{1+u^2}$ 4-

$(\sec^{-1} u)' = \dfrac{du/dx}{|u|\sqrt{u^2-1}}$ 5-

$(\csc^{-1} u)' = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}$ 6-

Example: find y'

$y = \cos^{-1}(2x)$

Solution

$y' = \dfrac{-2}{\sqrt{1-(2x)^2}}$

Integration

$\int \dfrac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$ 1-

$\int \dfrac{du}{1+u^2} = \tan^{-1} u + c$ 2-

$\int \dfrac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + c$ 3-

Example

$$\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\tan^{-1} x^3 + c$$

Natural Logarithm:

$$y = \ln x = \int_1^x dt/t$$

Properties:

1. ln1=0
2. ln(ax)=lna+lnx
3. ln(x/a)=lnx-lna
4. $\ln x^a = a\ln x$
5. ln1/x=-lnx

Derivative: $(\ln u)' = \dfrac{du/dx}{u}$

Integration: $\displaystyle\int \frac{du}{u} = \ln|u| + c$

Examples:

1- $y = \ln(6x^2 + 2x + 1)$ $\qquad y' = \dfrac{12x+2}{6x2+2x+1}$

2- $\displaystyle\int \frac{xdx}{1+x^2} = 1/2\int \frac{2xdx}{1+x^2} = 1/2\ln\left|1+x^2\right| + c$

The Exponential function

$y = f(x) = e^x, D = \Re, R = \Re^+$

It is the inverse of y=lnx

Properties

1 $\lim\limits_{x\to\infty} e^x = \infty$

2- $\lim\limits_{x\to-\infty} e^x = 0$

3- $e^a \cdot e^x = e^{a+x}$

4- $e^{-x} = \dfrac{1}{e^x}$

5- $\ln e^x = x$

6- $e^{\ln x} = x$

Derivative:- $\left(e^u\right)' = e^u \dfrac{du}{dx}$

Integration:- $\int e^u du = e^u + c$

The Function: a^u

$y = f(x) = a^x, D = \Re, R = \Re^+, a > 0$

Derivative: $(a^u)' = a^u \ln a.\frac{du}{dx}$

Integration: $\int a^u du = \frac{a^u}{\ln a} + c$

The function $y = \log_a u$

$y = f(x) = \log_a(x)$ is the inverse of $y = a^x$

Properties:

1-$\log_a x = \frac{\ln x}{\ln a}$

2-$\log_a a^x = x$

3-$a^{\log_a x} = x$

Derivative: $\left(\log_a u\right)' = \frac{du/dx}{u \ln a}$

Chapter four

Integration Methods

1-Integration by parts

$\int udv = udv - \int vdu$

Example

$\int \ln xdx$

Solution

$u = \ln x, dv = dx$

$du = dx/x, v = x$

$\int \ln dx = x \ln x - \int x.dx/x = x \ln x - x + c$

2-partial fraction

$\sqrt{a^2 + u^2}$,$\sqrt{u^2 - a^2}$ $\sqrt{a^2 - u^2}$, $a^2 - u^2$, $a^2 + u^2$ 3-Integral involves

put u=a sinθ then use 1-sin^{2}θ=cos^{2}θ $a^2 - u^2$,.

put u=a tanθ then use 1+tan^{2}θ=sec^{2}θ $a^2 + u^2$

put u=a secθ then use 1+tan^{2}θ=sec^{2}θ $u^2 - a^2$

4-powers of Trigonometric functions

Sinx or cosx of odd power

$$\int \cos x^2 dx = \int \cos x^2 \cos x dx = \int (1-\sin x^2)\cos x dx$$

$$= \int \cos x dx - \int \sin x^2 \cos x dx$$

$$= \sin x - \sin x^3/3 + c$$

Sinx or cosx of even power

$$\int \cos x^2 dx = \int [1+\cos 2x/2]dx = 1/2x + 1/4\sin 2x + c$$

$$\sin x^a or \cos x^b$$

5- $\sin ax \cos bx, \sin ax \sin bx, \cos ax \cos bx$

Sinax sinbx=1/2[cos(a-b)x-cos(a+b)x]

Sinax cosbx=1/2[sin(a-b)x+sin(a+b)x]

Cosax cosbx=1/2[cos(a-b)x+cos(a+b)x]

Example

$$\int \sin 5x \cos 3x dx = 1/2\int \sin 2x + \sin 8x dx = -1/4\cos 2x - 1/16\cos 8x + c$$ 5-

6- a- tanx,cotx ,secx and cscx of even power

Use $\tan^2 x=\sec^2 x-1$

b- tanx or cotx of odd power

use $\cot^2 x=\csc^2 x-1$

c-secx or cscx of odd power

use integration by part (udv)

Chapter five

Matrices:

When a system of equations has more than two equations, it is difficult to discuss them without using matrices and vectors.

The size of the matrix is described by the number of its row and columns. A matrix of *n* rows and *m* columns is said to be $n \times m$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n\times m} = [a_{ij}], \quad i=1,2,\ldots,n \quad , \quad j=1,2,\ldots,m.$$

Types of matrices:

Square matrix: It is a matrix whose number of rows are equal to the number of columns ($n=m$). For example:

$$A=\begin{bmatrix}1 & 5\\ 2 & 4\end{bmatrix}_{2\times 2} \quad , \quad B=\begin{bmatrix}1 & 3 & 0\\ 3 & 2 & 1\\ 1 & 8 & 0\end{bmatrix}_{3\times 3}$$

Diagonal matrix: It is a square matrix which all its elements are zero except the elements on the main diagonal. For example:

$$A=\begin{bmatrix}4 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Identity matrix: It is a diagonal matrix whose elements on the main diagonal are equal to 1, and it is denoted by I_n. For example:

$$I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad , \quad I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Transpose matrix: Transpose of A is denoted by (A^T), means that write the rows of A as columns in A^t. For example:

$$A=\begin{bmatrix}3 & 7 & 1\\ -2 & 1 & -3\end{bmatrix}_{2\times 3} \quad , \quad A^T=\begin{bmatrix}3 & -2\\ 7 & 1\\ 1 & -3\end{bmatrix}_{3\times 2}$$

Matrix addition and multiplication

If $A=\lfloor a_{ij}\rfloor$ and $B=\lfloor b_{ij}\rfloor$ and both A & B are $n\times m$ matrices, then

$$A+B=\lfloor a_{ij}\rfloor+\lfloor b_{ij}\rfloor=\lfloor a_{ij}+b_{ij}\rfloor$$

Ex.1:

$$\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}+\begin{bmatrix}2 & 3\\ 4 & 5\end{bmatrix}=\begin{bmatrix}3 & 2\\ 4 & 7\end{bmatrix}$$

For any scalar (number) c , we can multiply A by c as follows:

$$cA=c\lfloor a_{ij}\rfloor=\lfloor ca_{ij}\rfloor$$

Ex.2:

$$3\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}3 & -3\\ 0 & 6\end{bmatrix}$$

A matrix with only one column, $n\times 1$ in size, is called **a column vector**, and one of only one row, $1\times m$ in size, is called **a row vector**.

Matrices multiplication

Let A be an n× k matrix and B be a k×m matrix then C=AB is an n×m matrix, where the element in the i^{th} row and j^{th} column of AB is the sum

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad i = 1,2,\ldots,m \ and \ j = 1,2,\ldots,p.$$

Ex.3

Suppose $\mathbf{A} = \begin{bmatrix} 3 & 7 & 1 \\ -2 & 1 & -3 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 5 & -2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix}_{3\times 2}$ then

$$AB = \begin{bmatrix} 16 & 14 \\ -13 & 10 \end{bmatrix}_{2\times 2}$$

$$BA = \begin{bmatrix} 19 & 33 & 11 \\ -6 & 3 & -9 \\ 5 & 6 & 4 \end{bmatrix}_{3\times 3}$$

Determinants

With each square matrix A we associate a number det(A) or $|a_{ij}|$ called the determinant of A, calculated from the entries of A as follows:

For n=1, det(a)=a,

For n =2, $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Minors

To each element of a 3×3 matrix there corresponds a 2×2 matrix that is obtained by deleting the *row and column* of that element. The determinant of the 2×2 matrix is called the **minor** of that element.

For a matrix of dimension 3×3, we define

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\cdot\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\cdot\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

where $\begin{vmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} \\ \mathbf{a}_{32} & \mathbf{a}_{33} \end{vmatrix}$ is the minor of a_{11}, $\begin{vmatrix} \mathbf{a}_{21} & \mathbf{a}_{23} \\ \mathbf{a}_{31} & \mathbf{a}_{33} \end{vmatrix}$ is the minor of a_{12},

and $\begin{vmatrix} \mathbf{a}_{21} & \mathbf{a}_{22} \\ \mathbf{a}_{31} & \mathbf{a}_{32} \end{vmatrix}$ is the minor of a_{13}.

Ex.4:

Find the determinant of each matrix

a) $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$

$$\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} = 1(5) - 3(-2) = 5 + 6 = 11$$

b) $\begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$

$$\begin{vmatrix} 2 & 4 \\ 6 & 12 \end{vmatrix} = 2(12) - 4(6) = 0$$

Ex.5: Find the determinant of A where:

$$A = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

Sol.: By choosing the first column we get

$$\det(A) = \begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 6 \\ -7 & 9 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & -5 \\ 4 & 6 \end{vmatrix}$$

$$= 1 \cdot [36 - (-42)] + 2 \cdot (27 - 35)$$

$$= 78 - 16 = 62$$

Ex.6: Evaluate the determinant of A if:

$$A = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{bmatrix}$$

Solution:

By choosing the second row we get

$$\det(A) = \begin{vmatrix} 1 & 3 & -5 \\ -2 & 4 & 6 \\ 0 & -7 & 9 \end{vmatrix} = -(-2)\begin{vmatrix} 3 & -5 \\ -7 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & -5 \\ 0 & 9 \end{vmatrix} - 6 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -7 \end{vmatrix}$$

$$= 2(27 - 35) + 4(9 - 0) - 6(-7 - 0)$$

$$= -16 + 36 + 42 = 62$$

Note that 62 is the same value that was obtained for this determinant in Example above.

Note:

If a matrix A is triangular (either upper or lower), its determinant is just the product of the diagonal elements:

Solving a system of linear equations

Let A be a matrix, X a column vector, B a column vector then the system of linear equations is denoted by AX=B.

The augmented matrix

The solution to a system of linear equations such as

$$\begin{aligned} x-2y&=-5\\ 3x+y&=6 \end{aligned}$$

Depends on the coefficients of x and y and the constants on the right-hand side of the equation. The matrix of coefficients for this system is the 2×2 matrix

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

If we insert the constants from the right-hand side of the system into the matrix of coefficients, we get the 2×3 matrix

$$\left[\begin{array}{cc|c} 1 & -2 & -5 \\ 3 & 1 & 6 \end{array}\right]$$

We use a vertical line between the coefficients and the constants to represent the equal signs. This matrix is the **augmented matrix** of the system also it can be written as:

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}=\begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

Note:

Two systems of linear equations are **equivalent** if they have the same solution set. Two augmented matrices are **equivalent** if the systems they represent are equivalent.

Ex.1:

Write the augmented matrix for each system of equations.

a) $$\begin{aligned} x+y-z&=5\\ 2x+z&=3\\ 2x-y+4z&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 2 & 0 & 1 & 3 \\ 2 & -1 & 4 & 0 \end{array}\right]$$

b) $$\begin{aligned} x+y&=1\\ y+z&=6\\ z&=0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

We'll take two methods to solve the system AX=B

1) Cramer's rule

The solution to the system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Is given by $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$ where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad \text{and} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Provided that $D \neq 0$

Notes:

1. Cramer's rule works on systems that have **exactly one** solution.
2. Cramer's rule gives us a precise formula for finding the solution to an **independent** system.
3. Note that D is the determinant made up of the original coefficients of x and y. D is used in the denominator for both x and y. D_x is obtained by replacing the first (or x) column of D by the constants c_1 and c_2. D_y is found by replacing the second (or y) column of D by the constants c_1 and c_2.

Ex.1: Use Cramer's rule to solve the system:

$$3x - 2y = 4$$
$$2x + y = -3$$

Sol.:

First find the determinants $D, D_x,$ and D_y:

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7$$

$$D_x = \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} = 4 - 6 = -2, \qquad D_y = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -9 - 8 = -17$$

By Cramer's rule, we have

$$x = \frac{D_x}{D} = -\frac{2}{7} \quad \text{and} \quad y = \frac{D_y}{D} = -\frac{17}{7}$$

Check in the original equations. The solution set is $\left\{ (-\frac{2}{7}, -\frac{17}{7}) \right\}$.

Ex.2: Solve the system:

$$2x + 3y = 9$$
$$-2x - 3y = 5$$

Sol.:

Cramer's rule does not work because

$$D = \begin{vmatrix} 2 & 3 \\ -2 & -3 \end{vmatrix} = -6 - (-6) = 0$$

Because Cramer's rule fails to solve the system, we apply the addition method:

$$\begin{array}{r} 2x + 3y = 9 \\ \underline{-2x - 3y = 5} \\ 0 = 14 \end{array}$$

Because this last statement is false, the solution set is **empty**. The original equations are **inconsistent.**

Ex.3: Solve the system:

$$3x - 5y = 7$$
$$6x - 10y = 14$$

Sol.: Cramer's rule does not apply because

$$D = \begin{vmatrix} 3 & -5 \\ 6 & -10 \end{vmatrix} = -30 - (-30) = 0$$

Multiply Eq.(1) by -2 and add it to Eq.(2)

$$\begin{array}{r} -6x + 10y = -14 \\ \underline{6x - 10y = 14} \\ 0 = 0 \end{array}$$

Because the last statement is an identity, the equations are **dependent**. The solution set is $\{(x, y) | 3x - 5y = 7\}$.

Ex.4: Use Cramer's rule to solve the system:

$$2x - 3(y + 1) = -3$$
$$2y = 3x - 5$$

Sol.: First write the equations in standard form, $Ax + By = C$

$$2x - 3y = 0$$
$$-3x + 2y = -5$$

Find D, D_x, and D_y:

$$D = \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} = 4 - 9 = -5$$

$$D_x = \begin{vmatrix} 0 & -3 \\ -5 & 2 \end{vmatrix} = 0 - 15 = -15, \qquad D_y = \begin{vmatrix} 2 & 0 \\ -3 & -5 \end{vmatrix} = -10 - 0 = -10$$

Using Cramer's rule, we get

$$x = \frac{D_x}{D} = \frac{-15}{-5} = 3 \qquad \text{and} \qquad y = \frac{D_y}{D} = \frac{-10}{-5} = 2$$

Because (3,2) satisfies both of the original equations, the solution se is $\{(3,2)\}$.

2) The Gaussian Elimination method

When we solve a single equation, we write simpler and simpler equivalent equations to get an equation whose solution is obvious. In the Gaussian elimination method we write simpler and simpler equivalent augmented matrices until we get an augmented matrix in which the solution to the corresponding system is obvious.

Because each row of an augmented matrix represents an equation, we can perform the **row operations** on the augmented matrix.

Elementary Row Operation:

1. Construct the augmented matrix (A:B).
2. Interchange two rows ($R_i \leftrightarrow R_j$).
3. Multiply any row by a constant different from zero ($R_i \leftrightarrow kR_i$)
4. Add a constant multiply of any row to another row ($R_i \leftrightarrow R_i + kR_j$)

Ex.1:

Use Gaussian elimination method to solve the system (two equations in two variables):

$$x-3y=11$$
$$2x+y=1$$

Sol.:

Start with the augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 0 & 7 & -21 \end{array}\right] \qquad \mathbf{R}'_2 = -2R_1 + \mathbf{R}_2$$

$$\left[\begin{array}{cc|c} 1 & -3 & 11 \\ 0 & 1 & -3 \end{array}\right] \qquad \mathbf{R}'_2 = \frac{1}{7}\mathbf{R}_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right] \qquad \mathbf{R}'_1 = 3\mathbf{R}_2 + \mathbf{R}_1$$

This augmented matrix represents the system $x=2$ and $y=-3$. So the solution set to the system is $\{(2,-3)\}$.

Ex.2: Use Gaussian elimination method to solve the system (three equations in three variables):

$$2x-y+z=-3$$
$$x+y-z=6$$
$$3x-y-z=4$$

Sol.:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & -3 \\ 1 & 1 & -1 & 6 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

$$R_1 \leftrightarrow \mathbf{R}_2 \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -1 & 1 & -3 \\ 3 & -1 & -1 & 4 \end{array}\right] \quad \begin{array}{c} \mathbf{R}'_2 = -2R_1 + \mathbf{R}_2 \\ \mathbf{R}'_3 = -3R_1 + \mathbf{R}_3 \end{array} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & -1 & 6 \\ 0 & -3 & 3 & -15 \\ 0 & -4 & 2 & -14 \end{array}\right]$$

$$R'_2 = -\frac{1}{3}R_2 \left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 5 \\ 0 & -4 & 2 & -14 \end{array}\right] \quad \begin{array}{c} \mathbf{R}'_1 = -R_2 + \mathbf{R}_1 \\ \mathbf{R}'_3 = 4R_2 + \mathbf{R}_3 \end{array} \left[\begin{array}{ccc|c} \mathbf{1} & 1 & 0 & 1 \\ 0 & \mathbf{1} & -1 & 5 \\ 0 & 0 & -2 & 6 \end{array}\right]$$

$$R'_3 = -\frac{1}{2}R_3 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad R'_2 = R_3 + \mathbf{R}_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

This augmented matrix represents the system $x = 1$, $y = 2$ and $z = -3$. So the solution set to the system is $\{(1,2,-3)\}$.

Ex.3: Solve the system

$$x - y = 1$$
$$-3x + 3y = 4$$

Sol.:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -3 & 3 & 4 \end{array}\right] \to \mathbf{R}'_2 = 3R_1 + \mathbf{R}_2 \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 7 \end{array}\right]$$

R_2 corresponds to the equation 0 = 7. So the equations are **inconsistent**, and there is no solution to the system.

Ex.4: Solve the system

$$3x + y = 1$$
$$6x + 2y = 2$$

Sol.:

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 6 & 2 & 2 \end{array}\right] \to \mathbf{R}'_2 = -2R_1 + R_2 \left[\begin{array}{cc|c} 3 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

In the R_2 of the augmented matrix we have the equation 0 = 0. So the equations are **dependent**. For ordered pair that satisfies the first equation satisfies both equations. The solution set is $\{(x, y) | 3x + y = 1\}$

Exercises: Solve the following systems:

1) $$x + y = 3$$
$$-3x + y = -1$$

2) $$x + 2y = 1$$
$$3x + 6y = 3$$

3) $$2x + y + z = 4$$
$$x + y - z = 1$$
$$x - y + 2z = 2$$

Matrix Inverse

The matrix A has an inverse denoted by A^{-1} if $|A| \neq 0$ where $A.A^{-1}=I$. We'll take two methods to find A^{-1} where A is an n×n matrix.

2) **By Cofactor Method** (Using determinant of the matrix)

The cofactor of the element a_{ij} of the matrix $A = (a_{ij})$ is defined by $c_{ij} = (-1)^{i+j} A_{ij}$ where A_{ij} is the determinant of the matrix that remains when the row i and the column j are deleted.

To find the inverse of a matrix whose determinant is not zero

1- construct the matrix of cofactors of A, cof (A) = c_{ij}
2- Construct the transposed matrix of cofactors called the adjoin of A = adj (A) = cof $(A)^T$
3- then $A^{-1} = \dfrac{1}{\det(\mathbf{A})}$ adj A
4- to check your answer $A.A^{-1} = I$ or $A^{-1}.A = I$

<u>Ex.</u>: Use determinant to find A^{-1} where $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{adj}(\mathbf{A})$$

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7$$

$$\text{Cof}(A) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

$C_{11} = (-1)^{1+1}\ |4| = 4$
$C_{12} = (-1)^{1+2}\ |1| = -1$
$C_{21} = (-1)^{2+1}\ |1| = -1$
$C_{22} = (-1)^{2+2}\ |2| = 2$

$$\text{Adj}(A) = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}^{\mathbf{T}} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & \frac{-1}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix}$$

Ex2: Find A^{-1} if $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{bmatrix}$

Solution

$$|\mathbf{A}| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 0 & 2 \end{vmatrix} = 2+8=10$$

$$\mathbf{cof}(\mathbf{A}) = \begin{pmatrix} 0 & -10 & 0 \\ 2 & -8 & -4 \\ 2 & 7 & 1 \end{pmatrix}$$

$\mathbf{c}_{11} = 0, \quad \mathbf{c}_{12} = (-1)10 = -10, \quad \mathbf{c}_{13} = 0,$

$\mathbf{c}_{21} = -(-2) = 2, \quad \mathbf{c}_{22} = -8, \quad \mathbf{c}_{23} = -4,$

$\mathbf{c}_{31} = 2, \quad \mathbf{c}_{32} = (-1)(-7) = 7, \quad \mathbf{c}_{33} = 1$

$$\text{Adj(A)} = \text{cof}^{t} = \begin{pmatrix} 0 & 2 & 2 \\ -10 & -8 & 7 \\ 0 & -4 & 1 \end{pmatrix}$$

$$\therefore \mathbf{A}^{-1} = \begin{bmatrix} 0 & \frac{1}{5} & \frac{1}{5} \\ -1 & \frac{-4}{5} & \frac{7}{10} \\ 0 & \frac{-2}{5} & \frac{1}{10} \end{bmatrix}$$

Problems:

1) write the augment matrix to the following systems then find the solution:

a) $$\begin{aligned} x - y + z &= 1 \\ 2x - 2y + 2z &= 2 \\ -3x + 3y - 3z &= -3 \end{aligned}$$

b) $$\begin{aligned} x + y - z &= 2 \\ 2x - y + z &= 1 \\ 3x + 3y - 3z &= 8 \end{aligned}$$

c) $$\begin{aligned} 2\mathbf{x}_2 &= 1 \\ \mathbf{x}_2 - 2\mathbf{x}_3 &= 2 \\ \mathbf{x}_1 - 2\mathbf{x}_4 &= -10 \\ \mathbf{x}_1 + \mathbf{x}_4 &= 5 \end{aligned}$$

2) Find the inverse of each following matrix

a) $$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 5 & 2 \\ 5 & -2 & 4 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 1 & -3 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

Chapter six

Hyerbolic Function

Coshu=$\dfrac{e^{u}+e^{-u}}{2}$ (hyperbolic cosine of u)

Sinhu= $\dfrac{e^{u}-e^{-u}}{2}$ (hyperbolic sine of u)

tanhu=$\dfrac{cohu}{\sinh u}=\dfrac{e^{u}+e^{-u}}{e^{u}-e^{-u}}$

sechu=$\dfrac{1}{\cosh u}=\dfrac{2}{e^{u}+e^{-u}}$

cothu=$\dfrac{\cosh u}{\sinh u}=\dfrac{e^{u}+e^{-u}}{e^{u}}$

$\csc hu=\dfrac{1}{\sinh u}=\dfrac{2}{e^{u}-e^{-u}}$

properties

1- $\cosh^{2}\theta-\sinh^{2}\theta=1$

2- $1-\tanh^{2}\theta=\sec h^{2}\theta$, $\coth^{2}\theta-1=\csc h^{2}\theta$

3- $\sinh(\theta_1+\theta_2)=\sinh\theta_1\cosh\theta_2+\cosh\theta_1+\sinh\theta_2$

4- $\cosh(\theta_1+\theta_2)\cosh\theta_1 coh\theta_2+\sinh\theta_1+\sinh\theta_2$

5- $\sinh(2\theta)=2\sinh\theta\cosh\theta$, $\cosh(2\theta)=\cosh^{2}\theta+\sinh^{2}\theta$

6- $\cosh^{2}\theta=\dfrac{1+\cosh(2\theta)}{2}$, $\sinh^{2}\theta=\dfrac{\cos(2\theta)-1}{2}$

7- $\cosh\theta-\sinh\theta=e^{-\theta}$, $\cosh\theta+\sinh\theta=e^{\theta}$

Derivatives

1- $(\sinh u)'=\cos uh.du/dx$

2- $(\cosh u)'=\sinh u.du/dx$

3- $(\tanh)'=\sec hu^{2}du/dx$

4- $(\coth u)'=-\csc u^{2}h.du/dx$

5- $(\sec hu)'=-\sec hu.\tanh u.du/dx$

6- $(\csc hu)'=-\csc hu.\coth u.du/dx$

Example

y=sinh(3x)

y'=3cosh(3x

Integrations:

1- $\int \sinh u \, du = \cosh u + c$

2- $\int \cosh u \, du = \sinh u + c$

3.- $\int \operatorname{sech}^2 u \, du = \tanh u + c$

3- $\int c\mathrm{sch}^2 u \, du = -\coth u + c$

4- $\int \operatorname{sech} u \tanh u \, du = -\sec h\, u + c$

5- $\int c\mathrm{sch}\, u \coth u \, du = -\csc h\, u + c$

Example

$$\int \sin(3x)dx = 1/3\cosh(3x) + c$$

The inverse hyberbolic Functions

1- $y = f(x) = \sinh x^{-1}, D = \Re, R = \Re$

2- $y = f(x) = \cosh x^{-1}, D = \{x / x \in \Re, x \geq 1\}, R = \Re$

3- $y = f(x) = \tanh x^{-1}, D = [-1,1], R = \Re$

4- $y = f(x) = \coth x^{-1}, D = \{x / x \in \Re, x \succ 1, x \prec 1\}, R = \Re \,|\, \{0\}$

5- $y = f(x) = \sec hx^{-1}, D = [0,1], R = \Re$

6- $y = f(x) = \csc hx^{-1}, D = \Re \,|\, \{0\}, R = \Re \,|\, \{0\}$

Derivatives

$(\sinh^{-1} u)' = \dfrac{du/dx}{\sqrt{1+u^2}}$ 1-

$(\cosh^{-1} u)' = \dfrac{du/dx}{\sqrt{u^2-1}}$ 2-

$(\tanh^{-1} u)' = \dfrac{du/dx}{1-u^2}$ 3-

$(\coth^{-1} u)' = \dfrac{du/dx}{1-u^2}$ 4-

$(\sec h^{-1} u)' = \dfrac{-du/dx}{|u|\sqrt{1-u^2}}$ 5-

$(\csc h^{-1} u)' = \dfrac{-du/dx}{|u|\sqrt{1+u^2}}$ 6-

Example

$$y = \sinh(2x)^{-1} = y' = \frac{2}{\sqrt{1+(2x)^2}}$$

Integrations:

1- $\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$

2- $\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$

3- $\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c, if\,|u| \prec 1 \\ \coth^{-1} u + c, if\,|u| \succ 1 \end{cases}$

4- $\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sec}h^{-1}|u| + c$

5- $\int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csc}h^{-1}|u| + c$

Properties

1- $\operatorname{sec}h^{-1} x = \cosh^{-1}(\frac{1}{x})$

2- $\operatorname{csc}h^{-1} x = \sinh^{-1}(\frac{1}{x})$

3- $\coth^{-1} x = \tanh^{-1}(\frac{1}{x})$

Chapter seven

Complex Numbers

The general form of a complex number is $z = a + bi$ Where a and b are real numbers , $i\sqrt{-1}$. The number a is called real part denoted by Re(z)=a, and Im(z)=b is called the imaginary part.

<u>Algebra of complex numbers:</u>

Let z_1=a+bi and z_2=c+di be two complex numbers

1-Equality: $z_1=z_2 \rightarrow$a=c and b=d

2-Addition: z_1+z_2=(a+c)+(b+d)i

3-Multiplication:$z_1.z_2$=(a+bi).(c+di)=(ac-bd)+(ad+bc)i

4- complex conjugate: $\bar{z}_1 = \overline{(a+bi)} = a - bi$

5-Division: $\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2}$

6-lengh of $z_1$is $|z_1| = \sqrt{a^2 + b^2}$

Polar form

$a = r\cos\theta, b = r\sin\theta$ then

$z = a + bi = r\cos\theta + ir\sin\theta$ **or** $z = r(\cos\theta + i\sin\theta)$ **is called the polar form.**

The angle θ is called argument of z and written arg(z)= θ

Exponential form:

We know that $e^{i\theta}=\cos\theta+i\sin\theta$, $z=re^{i\theta}$ is called exponential form.

Theorem:

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

1- $z_1.z_2 = r_1.r_2[(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

2- $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

<u>Demoivers Theorem:</u>

$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$

Powers: let $z = r(\cos\theta + i\sin\theta)$, then

$Z^n=r^n[\cos(n\theta)+i\sin(n\theta)]$

Roots: let $z = r(\cos\theta + i\sin\theta)$,then

$W_k=\sqrt[n]{z} = \sqrt[n]{r}[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})]$

K=0,1,2,….,n-1

Chapter eight

Vector:

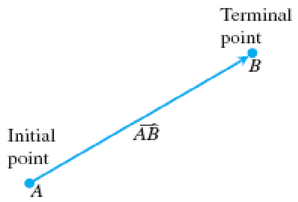
A vector is a matrix that has only one row – then we call the matrix a ***row vector*** – or only one column – then we call it a ***column vector***.

A row vector is of the form: $a = [a_1 \ \ a_2 \ \ \ldots \ \ a_n]$

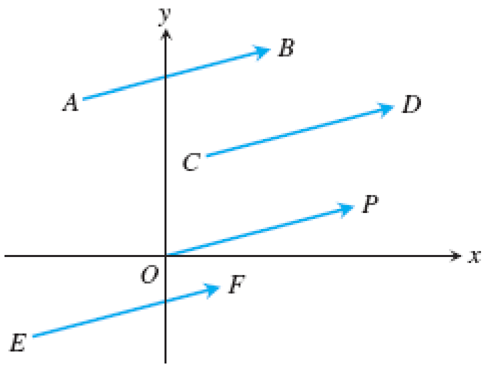
A column vector is of the form:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A quantity such as force, displacement, or velocity is called a vector and is represented by a directed line segment



A **vector** in the plane is directed line segment. The directed line segment $\overrightarrow{AB}$ has ***initial point*** A and ***terminal point*** B; its ***length*** is denoted by $\left|\overrightarrow{AB}\right|$. Two vectors are ***equal*** if they have the same length and direction.



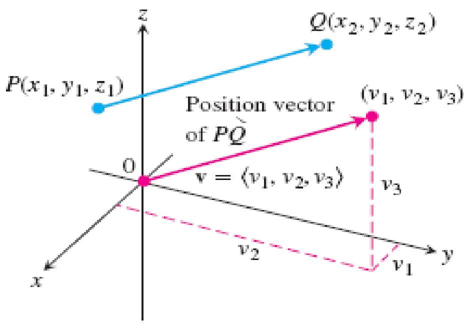
Component form

If v is a ***two dimensional*** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) ,then the ***Component form*** of v is:

$$v = (v_1, v_2)$$

If v is a ***three dimensional*** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) **,** then the ***Component form*** of v is:

$$v = (v_1, v_2, v_3)$$



The numbers v_1, v_2 and v_3 are called the components of v.

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$**,** the standard position vector $v = (v_1, v_2, v_3)$ equal to $\overrightarrow{PQ}$ is

$v = (x_2 - x_1\,,\ y_2 - y_1\ ,\ z_2 - z_1)$

The ***magnitude*** or ***length*** of the vector $v = \overrightarrow{PQ}$ is the nonnegative number

$$|v| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length **0** is the zero vector $0 = (0,0)$ or $0 = (0,0,0)$. This vector is also the only vector with no specific direction.

Ex.: Find **a)** component form and **b)** length of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$

Solution:

a) $v = (-5+3,\ 2-4\ ,\ 2-1)$

The component form of $\overrightarrow{PQ}$ is $v = (-2, -2, 1)$

b) The length or magnitude of $v = \overrightarrow{PQ}$ is $|v| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3$

Vector Addition and Multiplication of a vector by a scalar

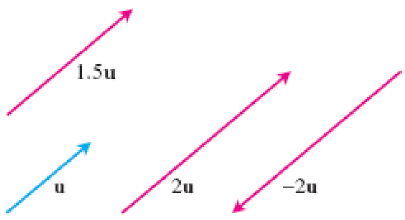
Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors with ***k*** a scalar.

Addition:

$u + v = (u_1 + v_1\,,\ u_2 + v_2\,,\ u_3 + v_3)$

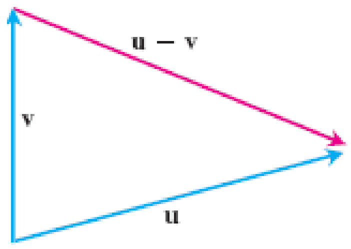
Scalar multiplication: $ku = (ku_1, ku_2, ku_3)$

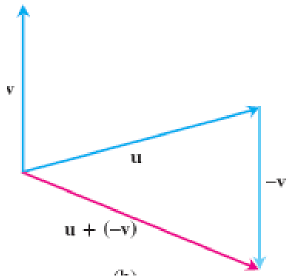
If the length of ku is the absolute value of the scalar k times the length of u. The vector $(-1)u = -u$ has the same length as u but points in the opposite direction.



If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, $u - v = (u_1 - v_1\,,\ u_2 - v_2\,,\ u_3 - v_3)$

Note that $(u - v) + v = u$ and the difference $u - v$ as the sum $u + (-v)$





Ex.:

Let $u = (-1,3,1)$ and $v = (4,7,0)$, find

a) $2u + 3v$ **b)** $u - v$ **c)** $\left|\frac{1}{2}u\right|$

Solution:

a) $2u + 3v = (-2, 6, 2) + (12, 21, 0) = (10, 27, 2)$

b) $u - v = (-5, -4, 1)$

c) $\left|\frac{1}{2}u\right| = \left|\left(\frac{-1}{2}, \frac{3}{2}, \frac{1}{2}\right)\right| = \frac{1}{2}\sqrt{11}$

Properties of vector operations:

Let u , v and w be vectors and a and b be scalars.

1) $u + v = v + u$ **2)** $(u + v) + w = u + (v + w)$

3) $u + 0 = u$ **4)** $u + (-u) = 0$

5) $0u = 0$ **6)** $1u = u$

7) $a(bu) = (ab)u$ **8)** $a(u+v) = au + av$

9) $(a+b)u = au + bu$

Unit vectors

A vector v of length 1 is called ***unit vector***. The standard unit vectors are:

$i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$

$$\begin{aligned} v = (v_1, v_2, v_3) &= (v_1,0,0) + (0,v_2,0) + (0,0,v_3) \\ &= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) \\ &= v_1 i + v_2 j + v_3 k \end{aligned}$$

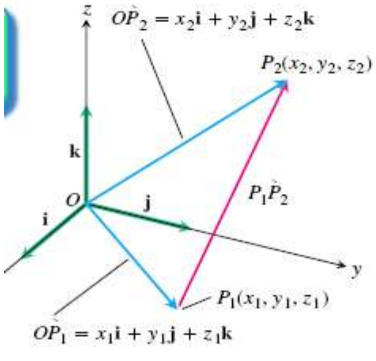
We call the scalar (or number) v_1 the ***i-component*** of the vector v , v_2 the ***j-component*** of the vector v, and v_3 the ***k-component***. In component form, $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathrm{i} + (y_2 - y_1)\mathrm{j} + (z_2 - z_1)k$$

If $v \neq 0$, then

1) $u = \dfrac{v}{|v|}$ is a unit vector in the direction of v, called ***the direction*** of the nonzero vector v.

2) The equation $v = \dfrac{v}{|v|}|v|$ expresses v in terms of its ***length*** and ***direction***.



Ex.:

Find a unit vector u in the direction of the vector $P_1(1,0,1)$ and $P_2(3,2,0)$.

Solution

$$\overrightarrow{P_1P_2} = (3-1)\mathrm{i} + (2-0)\mathrm{j} + (0-1)k = 2\mathrm{i} + 2\mathrm{j} - \mathrm{k}$$

$$\overrightarrow{P_1P_2} = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$u = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2i + 2j - k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

The unit vector u is the ***direction*** of $\overrightarrow{P_1P_2}$.

Product of vectors

u & v are vectors,

There are two kinds of multiplication of two vectors:

1- The scalar product (dot product) u.v. The result is a **scalar**.

2- The vector product (cross product) u×v. The result is a **vector**.

1) The dot product

In this section, we show how to calculate easily the angle between two vectors directly from their components. The dot product is also called ***inner*** or ***scalar*** products because the product results in scalar, not a vector.

Def.: The dot product $u \cdot v$ (u dot v) of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ is:

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$$

Note:

$$\left.\begin{matrix} i \cdot i \\ j \cdot j \\ k \cdot k \end{matrix}\right] = 1.1 = 1 \quad , \quad \left.\begin{matrix} i \cdot j \\ j \cdot k \\ k \cdot j \end{matrix}\right] = 0$$

Ex.:

a)

$(3,5) \cdot (-1,2) = 3(-1) + 5(2) = 7$ scalar

$(3i + 5j) \cdot (-i + 2j) = 7$

b)

$(1,-3,4) \cdot (1,5,2) = 1 - 15 + 8 = -6$ scalar

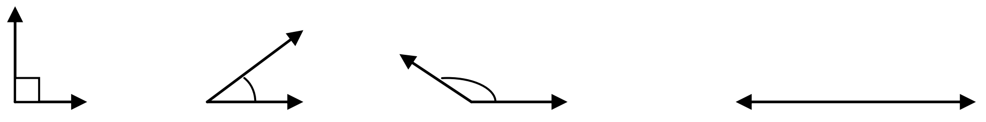
$(i - 3j + 4k) \cdot (i + 5j + 2k) = -6$

Angle between two vectors

The angle θ between two nonzero vectors $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ is given by

$$\vec{u}\cdot\vec{v}=\left|\vec{u}\right|\cdot\left|\vec{v}\right|\cdot\cos\theta$$

$$\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\cdot\left|\vec{v}\right|}\right) \qquad \text{where } \theta \quad (0\le\theta\le\pi)$$



u.v=0 u.v>0 u.v<0 Counter clockwise

Ex.: Find the angle between two vectors in space

$$\vec{u}=2\vec{i}-\vec{j}+2\vec{k} \qquad , \qquad \vec{v}=\vec{i}-2\vec{j}+2\vec{k}$$

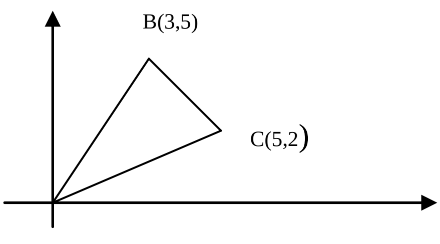
$$\cos\theta=\frac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\cdot\left|\vec{v}\right|}=\frac{2+2+4}{\sqrt{4+1+4}\cdot\sqrt{1+4+4}}$$

$$\cos\theta=\frac{8}{9} \quad \Rightarrow \quad \theta=\cos^{-1}\frac{8}{9}$$

Ex.:

Find the angle θ in the triangle ABC determined by the vertices

$A=(0,0)$, B(3,5) and C(5,2)



$\overrightarrow{CA} = (-5,-2)$ and $\overrightarrow{CB} = (-2,3)$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$

$\left|\overrightarrow{CA}\right| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$

$\left|\overrightarrow{CB}\right| = \sqrt{(-3)^2 + (3)^2} = \sqrt{13}$

$\theta = \cos^{-1}\left(\frac{4}{\sqrt{29} \cdot \sqrt{13}}\right)$

Orthogonal vectors

Vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are ***orthogonal*** (or ***perpendicular***) if and only if $u \cdot v = 0$

Ex.:

a) $u = (3,-2)$ and $\mathrm{v} = (4,6)$ are orthogonal because $u \cdot v = 0$

b) $\mathrm{u} = 3i - 2j + k$ and $\mathrm{v} = 2j + 4k$ are orthogonal because $u \cdot v = 0$

c) **0** is orthogonal to every vector **u** since

$$\begin{aligned} 0 \cdot u &= (0,0,0) \cdot (u_1, u_2, u_3) \\ &= 0 \end{aligned}$$

Properties of the Dot product

If u , v and w are any vectors and c is a scalar, then

1) $u \cdot v = v \cdot u$
2) $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
3) $u \cdot (v + w) = u \cdot v + u \cdot w$
4) $u \cdot u = |u|^2$
5) $0 \cdot u = 0$

Vector projection

Vector projection of u onto v

$$proj_v\, u = \left(\frac{u \cdot v}{|v|^2}\right) v \qquad \ldots\ldots \quad (1)$$

$proj_v\, u$ ***("The vector projection of u onto v"***

)

Scalar component of u in the direction of v

$$|u|\cos\theta = \frac{u \cdot v}{|v|} = \text{u} \cdot \frac{v}{|v|} \quad \ldots\ldots\ (2)$$

Ex.:

Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v.

Solution:

We find $proj_v\, u$ from eq.(1):

$$proj_v\, u = \left(\frac{u \cdot v}{|v|^2}\right) v = \frac{u \cdot v}{v \cdot v} v = \frac{6 - 6 - 4}{1 + 4 + 4}(i - 2j - 2k) = \frac{-4}{9}(i - 2j - 2k) = \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k$$

We find the scalar component of u in the direction of v from eq.(2):

$$|u|\cos\theta = \text{u} \cdot \frac{v}{|v|} = 6i + 3j + 2k \cdot \left(\frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k\right) = 2 - 2 - \frac{4}{3} = -\frac{4}{3}$$