مادة الرياضيات ع. ١

المرحلة الثانية صباحي / مسائي

فرع الفيزياء التطبيقية

محاضرات

- Differential Equations.

I- First Order First Degree Differential Equations.

1.1- Variable Separable

1.2- Homogenous

1.3- Linear

1.4- Exact

1.5- Some Applications

1.6- Exercises

Differential Equations

A differential equation is an equation that involves one or more derivatives, or diferentials, there are two types of diff. equ. ordinary and partial.

Order: The order of the differential equation is the highest order derivative that occurs in the equation.

Degree: The degree of the diff. equ. is the power of the exponent of the highest order derivative.

Examples:

write order and degree of the following:

① $\frac{dy}{dx} = 5x + 3$ $\quad O = 1,\ D = 1$

② $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$ $\quad O = 3,\ D = 2$

③ $4\frac{d^3y}{dx^3} + \sin x \frac{d^2y}{dx^2} + 5xy = 0$ $\quad O = 3,\ D = 1$

Exercise: Find the order and degree of these diff. equ.

① $\frac{dy}{dx} + \cos x = 0$

② $3\,dx + 4y^2\,dy = 0$

③ $\frac{d^2y}{dx^2} + x = y^2$

④ $(y'')^2 + 2y' = x^2$

⑤ $y''' + 2(y'')^2 = xy$

Solution :- The solution of the differential eq. is a function $y=f(x)$ that satisfies the eq.

Example : Show that $y=3e^{2x}-e^{-2x}$ is a solution to $y''-4y=0$

Sol.

$y=3e^{2x}-e^{-2x}$

$y'=6e^{2x}+2e^{-2x}$

$y''=12e^{2x}-4e^{-2x}$

$\Rightarrow y''-4y=12e^{2x}-4e^{-2x}-4[3e^{2x}-e^{-2x}]=12e^{2x}-4e^{-2x}+4e^{-2x}=0$

$\therefore$ $y=3e^{2x}-e^{-2x}$ is a solution to $y''-4y=0$

Types of solutions:-

① **General solution :-** Any solution contains an arbitrary constant is called a general sol.

② **Particular Solution :-** Any solution comes from general solution by giving values to the arbitrary constant is called particular sol.

Example : Show that ① $y=\sin(x+c)$ ② $y=\sin(x+1)$ are solutions to $(y')^2+y^2=1$ and give its name

Sol.

① $y=\sin(x+c) \Rightarrow y'=\cos(x+c)$

$(y')^2+y^2=\cos^2(x+c)+\sin^2(x+c)=1$

$\therefore y=\sin(x+c)$ is general solution.

② $y = \sin(x+1)$ is particular solution $c = 1$

Exercise

① show that $y = c_1 \sin 2x + c_2 \cos 2x$ is the solution of $y'' + 4y =$

② Determine wether $y(x) = 2e^{-x} + xe^{-x}$ is a solution of $y'' + 2y' + y =$

③ Determine wether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$

1- First Order First Degree Diff. eq.

Types of first order first degree Diff. eq.

i) Variable separable.
ii) Homogeneous.
iii) Linear.
iv) Exact.

i) Variable Separable.

The general form of the first order first degree diff. eq. is

$$\boxed{M\,dx + N\,dy = 0} \quad \text{--- ①}$$

where M and N are functions of x and y.

If M is a function of only x or constant and
N is a function " " y or " ;

Then the eq (1) is variable separable an,

$$\boxed{\int M\,dx + \int N\,dy = c}$$ is the general solution

Example: solve $\frac{dy}{dx} = \frac{x(2\ln x+1)}{(\sin y + y\cos y)}$

Sol:

$x(2\ln x+1)\,dx = (\sin y + y\cos y)\,dy$

is variable separable.

$\Rightarrow \int x(2\ln x+1)\,dx + \int(-\sin y - y\cos y)\,dy = 0$

$\int 2x\ln x\,dx + \int x\,dx - \int \sin y\,dy - \int y\cos y\,dy = 0$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$, Let $u = y \Rightarrow du = dy$

$dv = 2x\,dx \Rightarrow v = x^2$ $\quad dv = \cos y\,dy \Rightarrow v = \sin y$

$\int u\,dv = uv - \int v\,du$

$\Rightarrow x^2\ln x - \int x^2\frac{dx}{x} + \int x\,dx - \int \sin y\,dy - (y\sin y - \int \sin y\,dy) = c$

$x^2\ln x - \int x\,dx + \int x\,dx - \int \sin y\,dy - y\sin y + \int \sin y\,dy = c$

$\boxed{x^2\ln x - y\sin y = c}$ is general solution

Example: solve $e^{x+y}\,dx = \frac{dy}{x}$

Sol.

$e^{x+y}\,dx - \frac{dy}{x} = 0$ $\quad [e^x . e^y = e^{x+y}]$

$e^x . e^y\,dx - \frac{dy}{dx} = 0$

$e^x\,dx - \frac{dy}{xe^y} = 0 \Rightarrow xe^x\,dx - \frac{dy}{e^y} = 0$

$\therefore$ is variable separable

$\int xe^x\,dx - \int \frac{dy}{e^y} = c \Rightarrow \int xe^x\,dx - \int e^{-y}\,dy = c$

let $u = x$; $dv = e^x\,dx$

$du = dx$ $\quad v = e^x$

$xe^x - \int e^x\,dx - \int e^{-y}\,dy = c$

$\boxed{xe^x - e^x + e^{-y} = c}$ is general Sol.

ii) Homogeneous :-

A function $f(x,y)$ is said to be homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$.

A diff. eq. $M dx + N dy = 0$ is called hom. if M and N are both homo. functions of same degree.

Example 1: The function $f(x,y) = x^2 + xy + y^2$ are homogeneous function of degree 2 since:

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^2 + \lambda x \lambda y + (\lambda y)^2 \\ &= \lambda^2 x^2 + \lambda^2 xy + \lambda^2 y^2 \\ &= \lambda^2 (x^2 + xy + y^2) \\ &= \lambda^2 f(x,y) \end{aligned}$$

Example 2: The function $f(x,y) = x^3 + xy$ is not hom. Since:

$$f(\lambda x, \lambda y) = \lambda^3 x^3 + \lambda^2 x^2 = \lambda^2(\lambda x^3 + x^2) \neq \lambda^2 f(x,y)$$

The General Solution of hom. Diff. eq.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) = F(v) \quad (1)$$ this eq. called homo.

Let $v = \frac{y}{x}$ then $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ---(2)

from (1) and (2) we get

$$v + x\frac{dv}{dx} = F(v) \quad \text{or} \quad \boxed{\frac{dx}{x} + \frac{dv}{v - F(v)} = 0}$$

Example 1: Solve $x\,dy - 2y\,dx = 0$

Sol.:

$$[x\,dy = 2y\,dx] * \frac{1}{x\,dx} \Rightarrow \frac{dy}{dx} = \frac{2y}{x} = 2v = F(v)$$

$$\frac{dx}{x} + \frac{dv}{v-2v} = 0$$

$$\frac{dx}{x} + \frac{dv}{-v} = 0 \Rightarrow \ln|x| - \ln|v| = c \Rightarrow \frac{x}{v} = c$$

$$\because v = \frac{y}{x} \Rightarrow \frac{x^2}{y} = c \Rightarrow y = \frac{x^2}{c} \quad \text{or} \quad y = kx^2$$

Example 2: Solve $(x^2-y^2)dx + 2xy\,dy = 0$

Sol.:

$$M = f(x,y) = x^2 - y^2$$

$$\rightarrow f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 y^2 = \lambda^2 (x^2-y^2) = \lambda^2 f(x,y)$$

it is hom. of degree 2.

$$N = g(x,y) = 2xy$$

$$g(\lambda x, \lambda y) = 2\lambda x \lambda y = 2\lambda^2 xy = \lambda^2 g(x,y)$$

it is hom. of degree 2.

M and N are hom. functions of same degree = 2

$\therefore (x^2-y^2)dx + 2xy\,dy = 0$ is hom. Diff. eq.

$$[2xy\,dy = -(x^2-y^2)dx] * \frac{1}{(2xy)*dx}$$

$$\frac{dy}{dx} = -\frac{x^2-y^2}{2xy} \qquad \therefore \frac{dy}{dx} = \left[\frac{y^2-x^2}{2xy}\right] \div x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y^2}{x^2} - 1}{2\frac{y}{x}} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)} = \frac{v^2-1}{2v} = F(v)$$

using:

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0 \Rightarrow v - F(v) = v - \frac{v^2-1}{2v}$$

$$\Rightarrow v - F(v) = \frac{v(2v)-(v^2-1)}{2v} = \frac{2v^2-v^2+1}{2v} = \frac{v^2+1}{2v}$$

$$\therefore \frac{dx}{x} + \frac{dv}{\frac{v^2+1}{2v}} = 0$$

$$\int \frac{dx}{x} + \int \frac{2v\,dv}{v^2+1} = c \Rightarrow \ln x + \ln(v^2+1) = c$$

$$\ln[x(v^2+1)] = c \Rightarrow x(v^2+1) = c_1 \qquad [c_1 = e^c]$$

but $v = \frac{y}{x} \Rightarrow \boxed{x\left[\left(\frac{y}{x}\right)^2 + 1\right] = c_1}$ The general SoL.

Exercise: Show that the following equations are hom. and solve.

1- $(x^2+y^2)\,dx + 2xy\,dy = 0$ ans: $x^3\left(1+3\frac{y^2}{x^2}\right) = c$

2- $x^2dy + (y^2 - xy)\,dx = 0$ ans: $y = \frac{x}{\ln x - c}$

3- $(x^2+y^2)\,dx + xy\,dy = 0$ ans: $x^2(x^2+2y^2) = c$

iii) Linear

The equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of only x or constant is called Linear in y and $\frac{dy}{dx}$.

Let $\boxed{\rho = e^{\int P\,dx}}$ then $\boxed{\rho . y = \int \rho Q\,dx + c}$ is the general solution.

Example: Solve $(1+x^2)\,dy + (y - \tan^{-1}x)\,dx = 0$

Sol.:

$$(1+x^2)\frac{dy}{dx} + y - \tan^{-1}x = 0$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} - \frac{\tan^{-1}x}{1+x^2} = 0$$

$$\Rightarrow y' + \frac{1}{1+x^2}\, y - \frac{\tan^{-1}}{1+x^2} = 0 \qquad \text{linear.}$$

where $P = \dfrac{1}{1+x^2}$, $Q = \dfrac{\tan^{-1}}{1+x^2}$

$$\rho = e^{\int P\,dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

$$\rho . y = \int \rho Q\,dx + c$$

$$e^{\tan^{-1}x} \cdot y = \int e^{\tan^{-1}x} \cdot \frac{\tan^{-1}x}{1+x^2}\,dx + c$$

Let $z = \tan^{-1}x \Rightarrow dz = \dfrac{dx}{1+x^2}$

$$\Rightarrow y e^{z} = \int z e^{z}\,dz + c$$

let $u = z$ $\quad du = dz$

$dv = e^{z}dz \Rightarrow v = e^{z}$

$$\therefore y e^{z} = z e^{z} - \int e^{z}\,dz + c$$

$$y e^{z} = z e^{z} - e^{z} + c$$

$$\therefore \boxed{y e^{\tan^{-1}x} = (\tan^{-1}x)\, e^{\tan^{-1}x} - e^{\tan^{-1}x} + c}$$ is the general sol.

Exercise :

1. $\frac{dy}{dx} + 2y = e^{-x}$ ans: $y = e^{-x} + c\, e^{-2x}$

2- $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$ ans: $x^3 y = c - \cos x$

3- $x\, dy + y\, dx = y\, dy$ ans: $x = \frac{y}{2} + \frac{c}{y}$

Remark: The equation $\frac{dx}{dy} + P Q = Q$; where P and Q are functio of only y or constant, is said to be linear in x and $\frac{dx}{dy}$, and here $\rho = e^{\int P dy}$ and the general sol. is :

$$\rho . x = \int \rho . Q \, dy + c$$

Example :-

Solve $e^{2y} dx + 2(x e^{2y} - y) dy = 0$

Sol. :

$$\left[e^{2y} \frac{dx}{dy} + 2x e^{2y} - 2y = 0 \right] * e^{-2y} \quad (\text{or} \div e^{2y})$$

$$\frac{dx}{dy} + 2x - 2y e^{-2y} = 0$$

$$\frac{dx}{dy} + 2x = 2y e^{-2y}$$ is linear in x and $\frac{dx}{dy}$

$P = 2$, $Q = 2y e^{-2y}$, $\rho = e^{\int P(x) dy} \Rightarrow \rho = e^{2y}$

$$\rho . x = \int \rho \, Q(x) \, dy + c$$

$$e^{2y} . x = \int \cancel{e^{2y}} \, 2y \cancel{e^{-2y}} \, dy + c$$

$$\Rightarrow e^{2y} . x = \cancel{2} \frac{y^2}{\cancel{2}} + c \Rightarrow \boxed{e^{2y} x = y^2 + c}$$ is the general Sol.

Bernoulli's equation

It has the form:

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^{n} \quad \cdots (1)$$

where n is any real number

Note

1- If $n=0$ then equ. (1) becomes linear

2- If $n=1$ then equ. (1) becomes Variable separable

To solve equ. (1) transfer it to linear by:-

1) Divide both sides by y^n

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \quad --- (2)$$

2) put $z = y^{1-n}$, $\frac{dz}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$ where $n \neq 1$

$$\frac{y^n}{(1-n)} \cdot \frac{dz}{dx} = \frac{dy}{dx}$$; substitute in (2)

$$\left[\frac{y^n}{1-n} \cdot \frac{dz}{dx} + P(x) \cdot z = Q(x)\right] * (1-n)$$

$$\frac{dz}{dx} + (1-n)P(x) \cdot z = (1-n)\,Q(x)$$, which is linear w.r.t z and $\frac{dz}{dx}$ then solve the resulting linear for z interms of x

Example: solve $\frac{dy}{dx} + xy = xy^2$

Sol.:

$$y^{-2}\frac{dy}{dx} + xy^{-1} = x$$

$$z = y^{-1}, \quad \frac{dz}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^{2}\frac{dz}{dx}$$

$$\left[-y^{-2}y^{2}\frac{dz}{dx} + xz = x\right] * -1$$

$$\frac{dz}{dx} + (-xz) = -x \quad \text{linear in } z \text{ and } \frac{dz}{dx}$$

$$P = -x, \quad Q = -x, \quad f = e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$$

$$e^{-\frac{x^2}{2}} \cdot z = \int -e^{-x^2/2}\, x\,dx + c$$

$$e^{-x^2/2} \cdot z = e^{-x^2/2} + c$$

$$\Rightarrow z = 1 + ce^{x^2/2}, \quad \frac{1}{y} = 1 + ce^{x^2/2}$$

$$\therefore \boxed{y = \frac{1}{1 + ce^{x^2/2}}} \quad \text{is general sol.}$$

Exercise: solve $\frac{dy}{dx} - \frac{3}{x}y = x^4 y^{1/3}$ ans: $y^{2/3} = cx^2 + \frac{2}{9}x^5$

v) Exact

The equation $Mdx+Ndy=0$ is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

تعريف (الاشتقاق الجزئي) :-

تشتق الدالة جزئياً بالنسبة لـ x عندما تكون دالة بمتغيرين أي $f(x,y)$، تثبت y ونعتبر x متغير. ويرمز للاشتقاق الجزئي $\frac{\partial}{\partial x}$، وتشتق الدالة $f(x,y)$ جزئياً بالنسبة لـ y بثبوت x ويرمز للاشتقاق الجزئي $\frac{\partial}{\partial y}$.

Example 1:- find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x,y)=x^2+xy+y^2$

Sol.:

$$\frac{\partial f}{\partial x} = 2x+y+0 \qquad , \qquad \frac{\partial f}{\partial y} = 0+x+2y$$

Example 2:- find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of $f(x,y)=\sin(xy)+e^{xy}+x\sin y$

Sol.:-

$$\frac{\partial f}{\partial x} = y\cos(xy)+y e^{xy}+\sin y$$

$$\frac{\partial f}{\partial y} = x\cos(xy)+x e^{xy}+x(\cos y).$$

General solution: The exact diff. equ. $Mdx+Ndy=0$ has a general solution $F(x,y)=c$ where

$$\partial F = Mdx+Ndy \quad , \quad M=\frac{\partial F}{\partial x} \quad , \quad N=\frac{\partial F}{\partial y}$$

$$\therefore \frac{\partial F}{\partial x}=M \Longrightarrow F=\int M dx + f(y)$$

substitute in $\frac{\partial F}{\partial y}=N$ then the integration gives $f(y)$ and $F(x,y)=c$ is the general solution.

Example¹ :- find the sol. of the equation:

$$\underbrace{(x+y)}_{M}dx+\underbrace{(x+y^2)}_{N}dy=0$$

sol.:

$\frac{\partial M}{\partial y}=1$, $\frac{\partial N}{\partial x}=1$ the diff. equ. is exact

$$F=\int M\,dx+f(y)=\int (x+y)\,dx+f(y)$$

$F=\frac{x^2}{2}+xy+f(y)$ ← لإيجاد $f(y)$ نشتق

$\frac{\partial F}{\partial y}=x+f'(y)=N=x+y^2$

$\therefore \cancel{x}+f'(y)=\cancel{x}+y^2 \Rightarrow f'(y)=y^2$ نكامل الطرفين نخص $\Rightarrow f(y)=\frac{y^3}{3}$

$\therefore F=\frac{x^2}{2}+xy+\frac{y^3}{3}$

then the solution is $\boxed{\frac{x^2}{2}+xy+\frac{y^3}{3}=c}$

Example² :-

$$\underbrace{(2xe^{y}+e^{x})}_{M}dx+\underbrace{(x^2+1)e^{y}}_{N}dy=0$$

sol.

$\left.\begin{array}{l}\frac{\partial M}{\partial y}=2xe^{y}\\ \frac{\partial N}{\partial x}=2xe^{y}\end{array}\right\}$ the diff is exact

$$F=\int M\,dx+f(y)=\int(2xe^{y}+e^{x})\,dx+f(y)$$

$\Rightarrow F=e^{y}x^2+e^{x}+f(y)$, to find $f(y)$

$\frac{\partial F}{\partial y}=x^2e^{y}+f'(y)=N=(x^2+1)e^{y}$

$\Rightarrow \cancel{x^2e^{y}}+f'(y)=\cancel{x^2e^{y}}+e^{y} \Rightarrow f'(y)=e^{y} \xRightarrow{\text{نكامل}} f(y)=e^{y}$

$\therefore F=e^{y}x^2+e^{x}+e^{y}$

The sol. is $\boxed{e^{y}x^2+e^{x}+e^{y}=c}$

Exercise: Solve

1- $(2+ye^{xy})\,dx + (xe^{xy} - 2y)\,dy = 0$ ans: $c = 2x + e^{xy} - y^2$

2- $(\tan x + \tan y)\,dy + (y\sec^2 x + \sec x \tan x)\,dx = 0$

ans: $c = y\tan x - \ln\cos y + \sec x$

3- $(2xy + y^2)\,dx + (x^2 + 2xy - y)\,dy = 0$

ans: $x^2 y + y^2 x - \frac{y^2}{2} = c$

Integrating Factor:

The diff. equ. $M\,dx + N\,dy = 0$ which is not exact ie: $\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$ can be some times changed into exact diff. equ. by multiplying it by a function of x and y which is called an Integrating Factor (I.F).

Finding I.F by Calculation

a) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) * \frac{1}{N} = f(x)$ or constant then:

$$I.F = e^{\int f(x)\,dx}$$

b) If $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) * \frac{1}{M} = g(y)$ or constant then:

$$I.F = e^{-\int g(y)\,dy}$$

Example 1 :- solve $(4xy + 3y^2 - x)\,dx + x(x+2y)\,dy = 0$

$$\underbrace{(4xy + 3y^2 - x)}_{M}\,dx + \underbrace{x(x+2y)}_{N}\,dy = 0$$

Sol. :

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4x + 6y \\ \frac{\partial N}{\partial x} &= 2x + 2y \end{aligned}\right\} \quad \because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

the the diff. eq. is not exact

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) * \frac{1}{N} = (4x + 6y - 2x - 2y) * \frac{1}{x(x+2y)}$$

$$= \frac{(2x+4y)}{x(x+2y)} = \frac{2\cancel{(x+2y)}}{x\cancel{(x+2y)}} = \frac{2}{x} = f(x)$$

$$\therefore I.F = e^{\int f(x)\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = \cancel{e}^{\cancel{\ln} x^2} = x^2$$

$$\Rightarrow x^2(4xy + 3y^2 - x)\,dx + x^2(x(x+2y))\,dy = 0$$

$$\underbrace{(4x^3y + 3x^2y^2 - x^3)}_{M}\,dx + \underbrace{x^3(x+2y)}_{N}\,dy = 0 \quad \text{(the new diff equ.)}$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4x^3 + 6x^2y \\ \frac{\partial N}{\partial x} &= 4x^3 + 6x^2y \end{aligned}\right\} \text{ the diff equ. is exact}$$

Now we find the sol.

$$\frac{\partial f}{\partial x} = M \Rightarrow F = \int M\,dx + f(y)$$

$$= \int (4x^3y + 3x^2y^2 - x^3)\,dx + f(y)$$

$$\Rightarrow F = x^4y + x^3y^2 - \frac{x^4}{4} + \underline{\underline{f(y)}}$$

$$\frac{\partial F}{\partial y} = \cancel{x^4} + \cancel{2x^3y} + f'(y) = N = \cancel{x^4} + \cancel{2yx^3}$$

$$\therefore f'(y) = 0 \Rightarrow f(y) = c_1$$

$$\therefore F = x^4y + x^3y^2 - \frac{x^4}{4} + c_1$$

$\Rightarrow$ The general sol is $\boxed{x^4y + x^3y^2 - \frac{x^4}{4} + c_1 = c}$

Example 2:-

Solve $\underbrace{(y^4+2y)}_{M}\,dx + \underbrace{(xy^3+2y^4-4x)}_{N}\,dy = 0$

sol.:

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 4y^3+2 \\ \frac{\partial N}{\partial x} &= y^3-4\end{aligned}\right\} \text{ is not exact}$$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) * \frac{1}{M} = (4y^3+2-y^3+4) * \frac{1}{y^4+2y}$$

$$= \frac{3y^3+6}{y^4+2y} = \frac{3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}} = \frac{3}{y} = \underline{g(y)}$$ دالة بدلالة (y)

$$I.F = e^{-\int g(y)dy} = e^{-\int \frac{3}{y}dy} = e^{-3\ln y} = y^{-3} = \frac{1}{y^3}$$

الان نضرب طرفي المعادلة التفاضلية بـ I.F

$$\frac{1}{y^3}\left[(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0\right]$$

$$\Rightarrow \left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 1-4y^{-3} \\ \frac{\partial N}{\partial x} &= 1-4y^{-3}\end{aligned}\right\} \text{ is exact}$$

$$F = \int M\,dx + g(y) = \int \left(y+\frac{2}{y^2}\right)dx + g(y)$$

$$\Rightarrow F = yx + \frac{2x}{y^2} + g(y)$$

$$\frac{\partial F}{\partial y} = \cancel{x} + \cancel{4xy^{-3}} + g'(y) = \cancel{x} + 2y - \cancel{4xy^{-3}}$$

$$\Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2$$

$$\Rightarrow F = yx + \frac{2x}{y^2} + y^2$$

$\Rightarrow$ The general sol. is $\boxed{yx + \frac{2x}{y^2} + y^2 = c}$

Exercise: solve

① $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

ans: $I.F = \frac{1}{x^4}$, $c = \frac{-y^2}{2x^2} + \frac{1}{3}e^{1/x^3}$

② $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

ans: $I.F = \frac{1}{y^3}$, $c = yx + \frac{2x}{y^2} + y^2$

بعض الملاحظات المهمة عن معرفة نوع المعادلة التفاضلية من الرتبة والدرجة الأولى

$Mdx + Ndy = 0$

① لكي تكون المعادلة التفاضلية منفصلة المتغيرات يجب ان تكون M دالة بدلالة x فقط او حاصل ضرب دالتين احداهما تكون بدلالة x والاخرى بدلالة y لكي يسهل فصلهما وهذا الشرط ينطبق على N ايضا.

② لكي تكون المعادلة التفاضلية متجانسة يجب ان تكون كل من M و N دالتين متجانستين وبنفس الدرجة وفي حالة ظهور دوال مثلثية او زوايا فيجب ان تكون الزاوية حاصل قسمة المتغيرين x و y وبنفس الدرجة.

مثال: $\cos\frac{y}{x}$ ، $\sin\frac{x^2}{y^2}$ ، $e^{x/y}$ ، ...

وفي حالة متعدد الحدود تتم معرفة التجانس من حساب درجة كل حد ظاهر في هذا المتعدد

مثال:

x^2+xy+y^2 دالة متجانسة

اما x^2+xy+1 و x^2y+y^2+x دوال غير متجانسة

③ لمعرفة كون المعادلة التفاضلية خطية ام لا ونكون خطية بالنسبة لـ x اي $\frac{dx}{dy}$ او لـ y اي $\frac{dy}{dx}$ فاذا كانت بالشكل

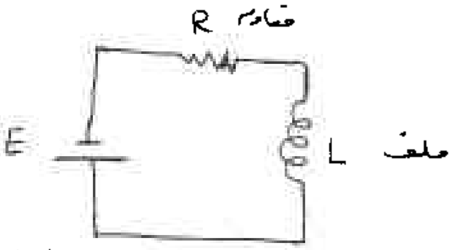
$y\,dx + (x+y)\,dy = 0$

نجيب القسمة على dy لان معامل dy يحتوي على حدين في حين معامل dx يحتوي على حد واحد فقط.

Some Applications

① Simple RL electrical circuits, the basic equation governing the amount of current I in a simple RL circuit consisting of a resistance R and inductor L, and an electromotive force E is:

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E}{L}$$



حسب قانون كيرشوف لحساب فولتية دائرة كهربائية فإن فولتية المصدر = مجموع فولتية المقاوم والملف

(مجموع الفولتيات في دائرة كهربائية = صفر)

مق = ف / ت

نلاحظ ان المعادلة الاخيرة هي خطية بالنسبة لـ $\frac{dI}{dt}$ و I وعليه يمكن ايجاد التيار بشكل دالة للزمن.

$$L\frac{dI}{dt} + RI = E(t)$$

$$\frac{dI}{dt} + \frac{RI}{L} = \frac{E}{L}$$

$$P = \frac{R}{L}, \quad Q = \frac{E}{L}$$

$$\rho = e^{\int P(t)dt} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

$$\rho . I = \int \rho . Q\, dt + c$$

$$e^{\frac{R}{L}t} . I = \frac{1}{R}\int e^{\frac{R}{L}t} . \frac{RE}{L} + c$$

$$I(t) = \frac{E}{R} + c\, e^{-\frac{R}{L}t}$$

when $t=0$, $I=0$ $[I(0)=0]$

$$0 = \frac{E}{R} + c\, e^{0}$$

$$0 = \frac{E}{R} + c \Rightarrow c = \frac{-E}{R}$$

$$I(t) = \frac{E}{R} - \frac{E}{R} e^{-R/L . t}$$

$$= \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

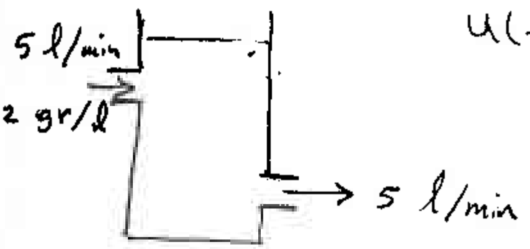
② Dilution problems   مسائل تخفيف التراكيز

Example:

مثال :- خزان يحتوي 100 لتر من محلول ملحي بتركيز 1 غم/لتر يضاف اليه من فتحه علويه محلول ملحي اخر بتركيز 2 غم/لتر وبمعدل 5 لتر/دقيقه ويخرج المحلول من فتحه سفليه في الخزان وبمعدل 5 لتر/دقيقه. ① اوجد كميه الملح في الخزان في كل دقيقه. ② احسب الوقت اللازم للوصول الى كميه 150 غم من الملح في الخزان.

الحل :-

لتكن مقدار الملح في الخزان عند الزمن t يساوي $u(t)$



$$\therefore \frac{du}{dt} = \underset{\text{داخل}}{In} - \underset{\text{خارج}}{out}$$ تغير مقدار كمية الملح بالنسبه للزمن

$$In = \frac{2gr}{l} * \frac{5l}{min} = 10 \ gr/min$$

$$out = 10 - \frac{u(t)}{20} \Rightarrow \frac{du}{dt} + \frac{u(t)}{20} = 10$$ which is linear in $\frac{du}{dt}$

$$P = \frac{1}{20}, \ Q = 10, \ \rho = e^{\int P(t)dt} = e^{\int 1/20 \, dt} = e^{1/20 \, t}$$

$$\rho . u = \int \rho . Q(t) \, dt + c \Rightarrow e^{1/20 t} . u = \int 10 e^{1/20 t} dt \Rightarrow e^{1/20 t} . u = 200 e^{1/20 t} + c$$

$$\therefore u(t) = 200 + c e^{-1/20 t}$$

① when $t = 0$, $u = 100 \Rightarrow u(0) = 100$

$$\therefore 100 = 200 + c e^{0} \Rightarrow 100 = 200 + c \Rightarrow c = -100$$

$$\therefore \boxed{u(t) = 200 - 100 e^{-1/20 t}}$$

② when $u(t) = 150$

$$150 = 200 - 100 e^{-1/20 t} \Rightarrow 100 e^{-1/20 t} = 50$$

$$\therefore \ln e^{-1/20 t} = \ln 1/2 \Rightarrow \frac{-1}{20} t = \ln 1/2 \Rightarrow -\frac{1}{20} t = -\ln 2$$

$$\therefore \frac{1}{20} t = \ln 2 \Rightarrow t = 20 \ln 2 \Rightarrow t = 13.9 \ min$$