Depertement of Applied science
Iraq - Baghdad
Third class

رياضيات متقدمة

chapter one : Differential equation with variable coefficients.

1. Finding the second Solution using another
2. The cauchy - Euler Differential equation.

chapter two: The Gamma and Beta functions.

1. The Gamma function.
2. The Beta function.

Chapter three : Power series Solution of Differential equat

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1. wave equation.
2. Heat equation
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Chapter 5. Complex analysis.

1. Algebraic of Complex number
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3. Analytic function.
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1) Finding the second solution using another by using the formula.

$$y_2 = y_1 \int \frac{e^{-\int \frac{a_1(x)}{a_0(x)} dx}}{y_1^2} dx$$

where y_1, y_2 are the solutions of

$$a_0(x) y'' + a_1(x) y' + a_2(x) y = 0 \quad \text{---} \quad (1.1)$$

Exercise.

① For each the following D.E. use the given y_1 to find y_2.

$$y'' - 4y' + 13y = 0 \quad , \quad y_1 = e^{2x} \cos 3x.$$

② $y'' - xy' + y = 0 \quad , \quad y_1 = x$

3) $y'' - 2(1 + \tan^2 x) y = 0 \quad , \quad y_1 = \tan x.$

2) The Cauchy-Euler Differential equation

A differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, $f(x)$ is a function of x called Cauchy-Euler equation

EX $\quad x^3 \frac{d^3 y}{dx^3} - 3x^2 \frac{d^2 y}{dx^2} - 5y = x$ is Cauchy-Euler.

② The Cauchy-Euler Differential equation.

A differential equation of the form

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where $a_0, a_1, a_2, \cdots, a_n$ are constants, $f(x)$ is a function of x called Cauchy-Euler equation

<u>Ex</u> $x^3 \frac{d^3 y}{dx^3} - 3x^2 \frac{d^2 y}{dx^2} - 5y = x$ is Cauchy-Euler

<u>Ex</u> Solve $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$.

Sol. This is Cauchy-Euler equation

Let $x = e^t \rightarrow t = \ln x$

$$x \frac{dy}{dx} = Dy \quad , \quad D = \frac{d}{dt}.$$

$$x^2 \frac{d^2 y}{dx^2} = (D^2 - D)y$$
$$= D(D-1)y$$

Similiarly

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

$$x^4 \frac{d^4 y}{dx^4} = D(D-1)(D-2)(D-3)y$$

and so on.

∴ The solution of example is

$$(D^2 - D - 2D - 4)y = 0$$

$$(D^2 - 3D - 4)\,y = 0 \qquad \text{homogenous.}$$

$$\therefore \quad m^2 - 3m - 4 = 0.$$

$$(m-4)(m+1) = 0.$$

$$m_1 = 4,\ m_2 = -1 \longrightarrow c_1 e^{m_1 t} + c_2 e^{m_2 t} \text{ is a solution.}$$

$$\therefore\ y = c_1 e^{4t} + c_2 e^{-t}$$

$$\therefore\ y = c_1 x^4 + c_2 x^{-1}$$

H.W Solve.

① $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} - \dfrac{2}{x^2}y = 0.$

② $x^2y'' - 2xy' - 4y = x^2 + 2\ln x$

③ $x^2y'' - 2xy' + 2y = x^3\cos x$

④ $x^2y'' + xy' - y = x^2e^x$

5) $x^2y'' - 2xy' + 4y = 2x^2.$

Special functions

Gamma and Beta functions

① The Gamma function

For any $n > 0$, the Gamma function denoted by $\Gamma(n)$ and defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

Ex $\Gamma(1) = 1$

Example: Show that $\Gamma(n+1) = n\Gamma(n)$

Proof:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$$

$$u = x^n \quad , \quad dv = e^{-x}$$

$$du = nx^{n-1}dx \quad\quad v = -e^{-x}$$

$$= -x^n e^{-x}\Big|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\,dx$$

$$= 0 - 0 + n\Gamma(n)$$

$$= n\Gamma(n)$$

$$\therefore \quad \Gamma(n+1) = n\Gamma(n)$$

a) If n is positive integer

$$\Gamma(n) = (n-1)!$$

b) If n is positive but not integer then using $\Gamma(\frac{1}{2})$

c) If n is negative and not integer. then

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1)$$

d) If n is negative and integer then $\Gamma(n)$ is not defined, also $\Gamma(0)$ is not defined.

Many integrals can be evaluated by using the Gamma functions.

Ex $\int_0^\infty x^5 e^{-x} dx = \Gamma(6) = 5! = 120.$

② The Beta Function

The Beta function is denoted by $B(m,n)$ and defined as

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad , m, n > 0.$$

Note, $B(m,n) = B(n,m)$.

Relation between B-function and Γ-function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Example: Find $\int_0^1 x^3 (1-x)^2 \, dx = B(4,3)$

$$B(4,3) = \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = \frac{3!\,2!}{6!} = \frac{1}{5*4*3} = \frac{1}{60}$$

another form of Beta function.

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta .$$

Example: Evaluate $\int_0^{\pi/2} \sin^6\theta \, d\theta$.

Sol. $2m-1 = 6 \Rightarrow m = \frac{7}{2}$

$2n-1 = 0 \Rightarrow n = \frac{1}{2}$.

$$\therefore \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{1}{2} B(m,n) = \frac{1}{2} \Gamma\left(\frac{7}{2},\frac{1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma(\frac{7}{2}) \cdot \Gamma(\frac{1}{2})}{\Gamma(4)} = \frac{1}{2}\left(\frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{3!}\right)$$

$$= \frac{5}{32}\pi .$$

Power series Solution of Differential equation

Theorem:

Suppose $x=0$ is an ordinary points of the D.Eq.

$$b_0(x)y''+b_1(x)y'+b_2(x)y=0.$$

then, there is a solution

$$y=\sum_{n=0}^{\infty} a_n x^n$$

that contains two arbitrary constants, namely a_0 & a_1 and converges inside a circle with center at $x=0$ and extending out to the nearest singular points.

Example: Use power series to solve

$$y''+xy'+(x^2+2)y=0.$$

Sol.

$x=0$, is an ordinary point of the D.Eq.

Let $y=\sum_{n=0}^{\infty} a_n x^n$ be a solution.

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

substituting in the equation.

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2+2)\sum_{n=0}^{\infty} a_n x^n = 0$$

rewrite the summation, so that x has exponent (n)

$$\therefore \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}X^n+\sum_{n=1}^{\infty} n a_n X^n+\sum_{n=2}^{\infty} a_{n-2}X^n+\sum_{n=0}^{\infty} 2a_nX^n=$$

write the few first terms, so that n start from the common range for each summation,

then,

$$(2a_2+2a_0)+(6a_3+3a_1)x+\sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2}+na_n$$

$$+a_{n-2}+2a_n\Big]X^n=0.$$

$$2a_2+2a_0=0 \longrightarrow a_2=-a_0.$$

$$6a_3+3a_1=0 \longrightarrow a_3=-\frac{1}{2}a_1$$

$$(n+2)(n+1)a_{n+2}+(n+2)a_n+a_{n-2}=0 \quad \forall n\geq 2.$$

this is called Recurrence relation.

$$a_{n+2}=\frac{-(n+2)a_n-a_{n-2}}{(n+2)(n+1)}, \quad n\geq 2.$$

after some computation, we get.

$$a_4=\frac{1}{4}a_0, \quad a_5=\frac{3a_1}{40}, \quad a_6=\frac{-a_0}{60}, \quad a_7=\frac{-a_1}{1680}.$$

by substuation, we get.

$$y=a_0(1-x^2+\frac{1}{4}x^4-\frac{1}{60}x^6+\cdots)+a_1(x-\frac{1}{2}x^3+\frac{3}{40}x^5-\cdots)$$

H-w

Use Power series method to solve.

① $(x^2+1)\,y'' + 2x\,y' = 0$.

② $2y'' + x y' - 4y = 0$.

Legender Polynomial

The Di. Eq. of the form $(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0$ with λ constant, is called Legender D. Eq.

Example $(1-x^2)\,y'' - 2x y' + 6y = 0$.

It's solution is called Legender function

$x=0$ is an ordinary point.

by use power series method, we get.

$$(2a_2 + (\lambda^2+\lambda)\,a_0) + (6a_3 - 2a_1 + (\lambda^2+\lambda)\,a_1)\,x +$$
$$\sum_{n=2}^{\infty} \{ (n+2)(n+1)\,a_{n+2} - n(n-1)\,a_n - 2n\,a_n + (\lambda^2+\lambda)\,a_n \} x^n = 0.$$

after some computation, we get.

$$a_2 = \frac{-(\lambda^2+\lambda)}{2}\,a_0$$

$$a_3 = \frac{2-(\lambda^2+\lambda)}{6}\,a_1$$

$$a_{n+2} = \left\{ \frac{n(n+1) - (\lambda^2+\lambda)}{(n+1)(n+2)} \right\} a_n \qquad n \geq 2.$$

$\therefore$ the solution is

$$y = a_0\left[1 - \left(\frac{\lambda^2+\lambda}{2}\right)x^2 + \left(\frac{6-(\lambda^2+\lambda)}{12}\right)\left(\frac{-(\lambda^2+\lambda)}{2}\right)x^4 + \left(\frac{20-(\lambda^2+\lambda)}{30}\right)\left(\frac{6-(\lambda^2+\lambda)}{12}\right)\left(\frac{-(\lambda^2+\lambda)}{2}\right)x^6 + \cdots\right].$$

$$+ a_1\left[x + \frac{2-(\lambda^2+\lambda)}{6}x^3 + \frac{12-(\lambda^2+\lambda)}{20}\cdot\frac{2-(\lambda^2+\lambda)}{6}x^5 + \cdots\right]$$

$$= a_0 F(x) + a_1 G(x).$$

Clearly if λ is even then $a_0 F(x)$ is a polynomial of degree λ containing only even powers of x & if λ is odd then $a_1 G(x)$ is a polynomial of degree λ containing only odd powers of x.

Choose a_0 & a_1 such that each solution has value 1 at $x=0$.

The polynomial are called Legender polynomials

In general

$$P_n(x) = \frac{1}{2^n}\sum_{i=0}^{[n]} \frac{(-1)^i}{i!}\cdot\frac{(2n-2i)!}{(n-2i)!\,(n-i)!}\,x^{n-2i}$$

where

$$[n] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

For example

$$P_4 = \frac{1}{2^4}\sum_{i=0}^{2} \frac{(-1)^i\,(8-2i)!}{i!\,(4-2i)!\,(4-i)!}\,x^{4-2i}$$

Legender Polynomial can be written as

$$P_n(X) = \frac{1}{2^n n!} \frac{d^n}{dX^n}\left[(X^2-1)^n\right].$$

This is called the Rodrigue's formula.

<u>Ex</u> $$P_5(X) = \frac{1}{2^5\, 5!} \frac{d^5}{dX^5}\left[(X^2-1)^5\right]$$

Partial differential equation

A Partial differential equation is an equation containg an unknown function of two or more variable and its Partial derivatives with respect to those variable

Example:

① $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

② $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$

Order of Parial differential equation:

is the order of the highest order derivative appearing in the eq.

A Solution of P.D.E. is any function which satisfies the eq.

Ex Show that $u = x^2 - y^2$ is a solution of

$u_{xx} + u_{yy} = 0.$

Sol.

$u_x = 2x \quad , \quad u_y = -2y$

$u_{xx} = 2 \quad , \quad u_{yy} = -2$

$\therefore u = x^2 - y^2$ is a Solution.

Solution of P.D.E. using method of separation of variables.

Ex Solve $\frac{\partial u}{\partial x} = y \frac{\partial u}{\partial y}$.

Sol. Let $u(x,y) = X(x) \cdot Y(y)$.

$$\frac{\partial u}{\partial x} = X' \cdot Y \quad , \quad \frac{\partial u}{\partial y} = X \cdot Y'$$

$$\left(X' \cdot Y = y\, X \cdot Y' \right) * \frac{1}{XY}$$

$$\frac{X'}{X} = y \frac{Y'}{Y} = K.$$

by Solving $\frac{X'}{X} = K$ and $y \frac{Y'}{Y} = K$, we get.

$$X(x) = A e^{Kx} \quad \text{and} \quad Y(y) = B y^{K}.$$

$$\therefore u(x,y) = X(x) \cdot Y(y)$$

$$= A e^{Kx} \cdot B y^{K}$$

$$= c\, y^{K} e^{Kx}$$

Kinds of P.D.E.

1- wave equation

2- Heat equation

3- Laplace equation.

1- Solutions of wave equation.

The form of this equation is.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with}$$

$u(0,t)=0$, $u(l,t)=0$ for all t and $u(x,0)=f(x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

Let $u(x,t)=F(x)\cdot G(t)$.

by some stepes, we get

$$u(x,t)=\sum_{n=1}^{\infty} u_n(x,t)=\sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi}{l}x$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$B_n^* = \frac{2}{cn\pi}\int_0^l g(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

2- Solutions of Heat equation.

The general form is.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{with}$$

$u(0,t)$, $u(l,t)=0$ for all t, $u(x,0)=f(x)$.

after some steps, we get

$$u(x,t)=\sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{l}x\right) e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{cn\pi}{l}$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$$

3- Laplace equation

The general form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \begin{matrix} 0 \le x \le a \\ 0 \le y \le b \end{matrix}$$

$$u(x,y) = \begin{cases} 0 & x = 0 & 0 \le y \le b \\ 0 & x = a & 0 \le y \le b \\ 0 & y = b & 0 \le x \le a \\ f(x) & y = 0. & 0 \le x \le a \end{cases}$$

The general solution is obtained using separation of variable, which is.

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

$$B_n = \frac{\frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right)}{\sinh\left(\frac{n\pi b}{a}\right)}.$$

Complex analysis

Complex number is a number of the form $a+bi$, where $i=\sqrt{-1}$ which is a solution of the algebric equation. $x^2+cx+d=0$.

$$x=\frac{-c\mp\sqrt{c^2-4d}}{2}, \text{ where } c^2<4d$$

i is called the imaginary unit.

a is called the real part of the complex number.

b is called the imaginary part " " "

If $z=a+ib$ is a complex number, then the complex conjugate of z denoted by $\bar{z}$ is $\bar{z}=a-ib$.

Algebra of complex number.

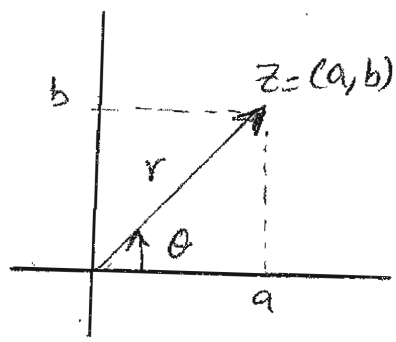
If $z=a_1+ib_1$, $z_2=a_2+ib_2$ are two complex numbers then.

1- $z_1\mp z_2=(a_1\mp a_2)+i(b_1\mp b_2)$

2- $z_1z_2=(a_1a_2-b_1b_2)+i(b_1a_2+a_1b_2)$

3- $\dfrac{z_1}{z_2}=\dfrac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+i\dfrac{b_1a_2-a_1b_2}{a_2^2+b_2^2}$.

Polar form of complex number.



$z=a+ib$

$r=\sqrt{a^2+b^2}=|z|$

$b=r\sin a$

$a=r\cos\theta$.

$\theta=\tan^{-1}\frac{b}{a}$.

$\therefore z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

A complex quantity $z = x+iy$ which x & y are real variables is called a complex variable.

- If $z = x+iy$ & $w = u+iv$ are two Complex variables. and suppose that a relation is given so that to each val in Some regions R. (R is a part of xy-plane), there is assig a single value of w, we say that w is a function of z and write $w = f(z)$.

- any complex function can be decomposed into two real-valu function of x & y. This is called the uv-plane.

Remark:

1- $\cos y = \dfrac{e^{iy} + e^{-yi}}{2}$, $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$

2- $\cosh y = \dfrac{e^{y} + e^{-y}}{2}$, $\sinh y = \dfrac{e^{y} - e^{-y}}{2}$

3- $\cos(iy) = \cosh y$

$\sin(iy) = i\sinh y$.

H.w write in uv-form.

1. $w = e^{z}$
2. $w = \sin z$
3. $w = \cosh z$
4. $w = \frac{1}{z}$

Theorem: Given $w = f(z) = u(x,y) + iv(x,y)$, suppose that

① the function u, v, u_x, v_x, u_y, v_y are continous at z_0.

② $u_x = v_y$ & $v_x = -u_y$ at z_0.

then

$$f'(z) \text{ exists and } f'(z) = u_x + iv_x$$

$$\text{or } f'(z) = v_y - iu_y.$$

Ex: Show that $f(z) = z^2$ is differentiable everwhere in the z-plane and find $f'(z)$?

$$f(z) = (x^2 - y^2) + i2xy$$

$$u_x = 2x, \quad u_y = -2y$$

$$v_x = 2y, \quad v_y = 2x$$

$\therefore$ $\left.\begin{aligned} u_x &= 2x = v_y \\ v_x &= 2y = -u_y \end{aligned}\right\}$ $\therefore f(z)$ is differentiable everwhere.

$$f'(z) = u_x + iv_x$$

$$= 2(x + iy) = 2z.$$

Theorem: Let $f(z) = u(r,\theta) + iv(r,\theta)$, suppose that

1. u, v, u_r, u_θ, v_r & v_θ are continous at $z_0 = (r_0, \theta_0)$

2. $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ at z_0, then

$f(z)$ is differentiable at z_0 and

$$f'(z) = e^{-i\theta_0}\left[u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)\right]$$

Analytic function

A function $f(z)$ is said to be analytic at a point z_0 of its derivative exists not only at z_0 but also at each point z in some circle with center z_0.

EX Some examples of analytic function:

① $f(z) = z^2$

② $f(z) = e^z$

Complex integration

Let z and a be two complex number, then

(i) $|z-a| = \rho$ is a circle c of radius ρ with center a.

(ii) $|z-a| < \rho$ interior of c.

iii) $|z-a| \leq$ (closed cicular disk).

(iv) $|z| = 1$ is unit circle.

v) $\rho_1 < |z-a| < \rho_2$ represent a region between two concentric circules of raduis ρ_1 and ρ_2 $(\rho_2 > \rho_1)$.

Line integral

we define $\int_c f(z)\,dz$ to be a line integral of $f(z)$ along the oriented curve c.

Given $f(z) = u + iv$, where $u = u(x,y)$, $v = v(x,y)$. then

$\int f(z)\,dz = \int_c (u+iv)(dx + i\,dy)$

Cauchy's integral theorem

theorem: Let C be a simple closed curve. If $f(z)$ is analytic within the region bounded by C as well as C. then we have

$$\oint_C f(z)\,dz = 0.$$

EX: Find $\oint_C e^z dz$, where C is any closed curve.

Sol. by above theorem $\oint_C e^z\,dz = 0$.

Definition: simply connected region.

A region R is called simply connected region if every simple closed curve in R encloses only point in R.

Cauchy's - Integral formula

Let $f(z)$ be analytic in a simply connected region R, then for any point z_0 in R and simple closed curve in R which encloses z_0 we have.

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi f(z_0).$$

Ex Evaluet $\oint_C \frac{\sin z}{z-2i}\,dz$, $c: |z|=3$

Sol. $\oint_C \frac{\sin z}{z-2i}\,\cancel{dz} = 2\pi i\,[\sin(z_0)]_{z_0=2i} = 2\pi i \sin(2i)$

Note:
For the n^{th} derivative of $f(z)$ at $z = z_0$ is given by

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$