Chapter 1

Functions

Definition of function

A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y.

Note

$\mathbb{R}$: the Set of all real numbers.

Domain of a function :

For a function $f : X \longrightarrow Y$ the domain of f is the set X, which is denoted by D_f.

Range of function :

For a function $f : X \longrightarrow Y$ the range of f is the set of y-values, which is denoted by R_f.

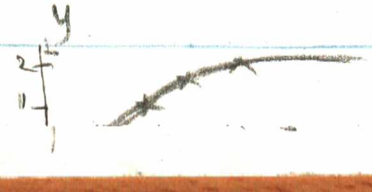
Example : Find the domain and the range of the following functions and then graph

① $y = f(x) = \sqrt{x-2}$

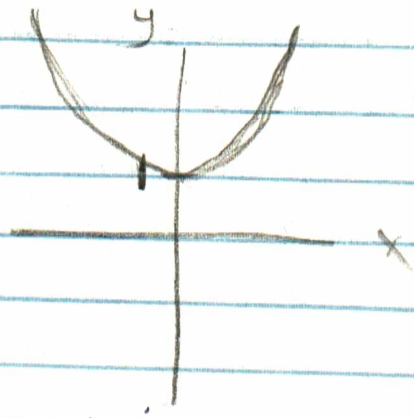
solution $x - 2 \geq 0 \Longrightarrow x \geq 2 \Longrightarrow D_f = \{x \mid x \in \mathbb{R}, x \geq 2\}$

or $D_f = [2, \infty)$

x	y
2	0
3	1



$R_f = \{y \mid y \in \mathbb{R} , y \geq 0\}$ or $R_f = [0, \infty)$



② $y = f(x) = x^2 + 1$

solution

$D_f = \mathbb{R}$

$R_f = y \geq 1$

③ $y = f(x) = -x^3 + 4x^2 - 4x$

$D_f = \mathbb{R}$

$R_f = \mathbb{R}$

Example : if $f(x) = 3x^2 - x + 1$, Solve the equation $f(x) = f(2)$?

sol/

$f(2) = 3 \cdot (2)^2 - 2 + 1 = 11$

$\therefore 3x^2 - x + 1 = 11$

$\Rightarrow 3x^2 - x - 10 = 0$

$\boxed{a = 3}$ $\boxed{b = -1}$ $\boxed{c = -10}$

$$\boxed{\begin{array}{l} ax^2 + bx + c = 0 \\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array}}$$

كل معادلة تربيعية لدستور

Inverse function

(١-١ متباينة)

① A function $f(x)$ is said to be one-to-one if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

② A function $f(x)$ is said to be onto if for every b on y-axis there exists a point a on x-axis such that $f(a) = b$.

③ If $f(x)$ is one-to-one and onto then there exists $f^{-1}(x)$ called the inverse of $f(x)$ such that

$$(f \circ f^{-1})(x) = x \quad \text{and} \quad (f^{-1} \circ f)(x) = x$$

Ex:// $y = f(x) = \frac{1}{x}$

sol// $yx = 1 \Longrightarrow x = \frac{1}{y}$

$\therefore x = f^{-1}(y) = \frac{1}{y}$

Ex// $y = f(x) = \frac{1}{4}x + 3$

$y = \frac{1}{4}x + 3 \Longrightarrow \frac{1}{4}x = y - 3$

$\Longrightarrow x = 4y - 12 = f^{-1}(y)$

or $x = 4y - 12 \Longrightarrow x + 12 = 4y \Rightarrow y = \frac{1}{4}(x+12) = f^{-1}(x)$

H.W/ (1) Find inverse of function $f(x) = 8x^3$?

(2) Find the domain of function $f(x) = \dfrac{x+5}{x^2-3x-4}$?

(3) Find D_f and R_f of function $y = \sqrt{x-1}$?

(4) Find inverse of function $y = f(x) = x^2$?
Determine the Domain and Range of $f(x)$?

(5) Find an equation for the inverse for each of the following function :-

(a) $f(x) = 3x+2$

(b) $f(x) = \dfrac{x+4}{3x-5}$

(c) $f(x) = e^{5x-1}$

(d) $f(x) = \ln(2x-1)$

(6) Determine the domain and range of the following function :-

(a) $f(x) = x-2$

(b) $f(x) = x^3$

(c) $f(x) = -2\sin x$

(d) $f(x) = e^x - 2$

Definition: If $f(x)$ and $g(x)$ are two functions then $f(x) \mp g(x)$ and $\frac{f(x)}{g(x)}$ (where $g(x) \neq 0$) are also functions.

Definition: The Composition of $f(x)$ and $g(x)$ is denoted by fog or gof and is defined by: $(gof)(x) = g[f(x)]$

Example: Write fog and gof if $f(x) = \sqrt{x}$ and $g(x) = \frac{x}{2}$

Solution: $(fog)(x) = f[g(x)]$

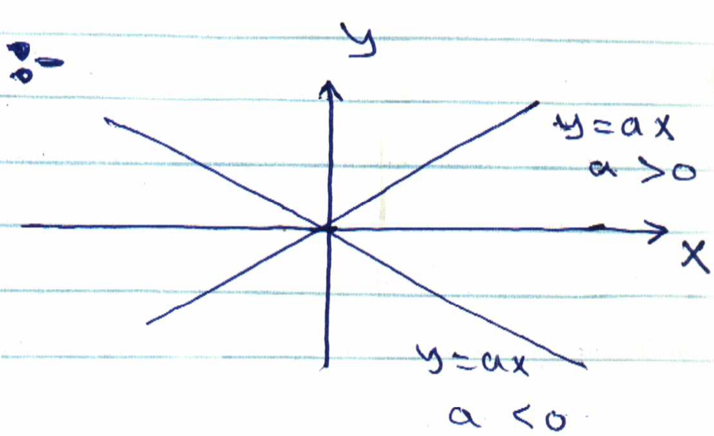
$$= f\left(\frac{x}{2}\right) = \sqrt{\frac{x}{2}}$$

and $(gof)(x) = g[f(x)] = g(\sqrt{x}) = \frac{\sqrt{x}}{2}$

notice that $(gof)(x) \neq (fog)(x)$.

Graphs of functions:-

① $y = f(x) = ax$



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Graphs of functions:-

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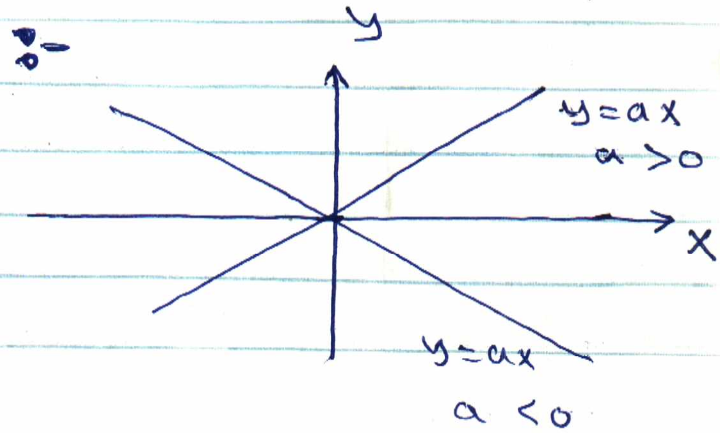
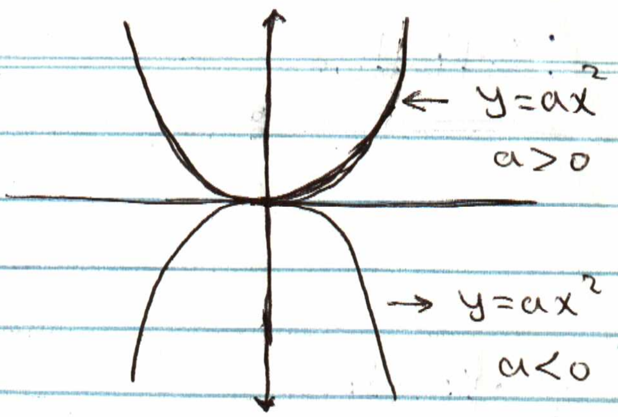
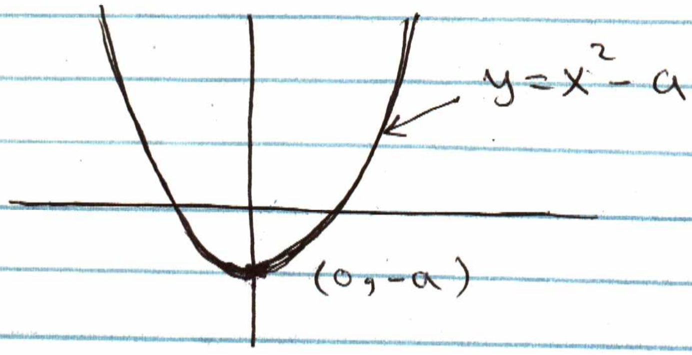
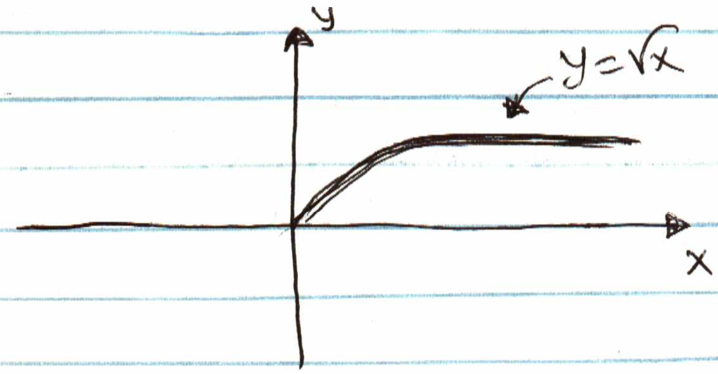


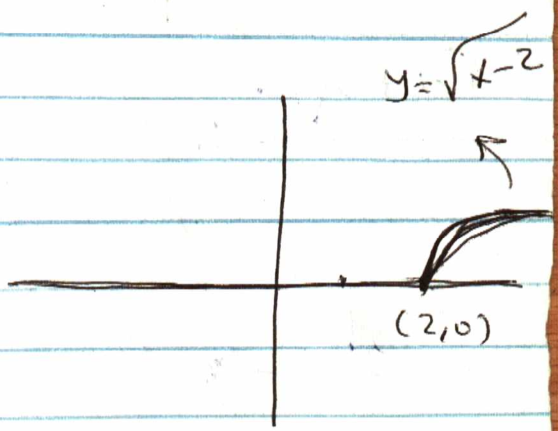
② $y = f(x) = ax^2$



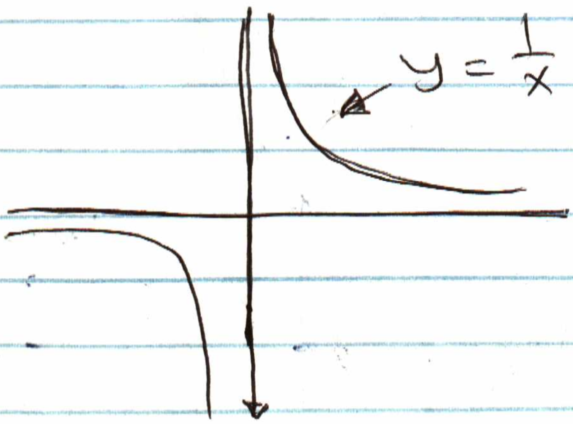


③ $y = f(x) = \sqrt{x}$





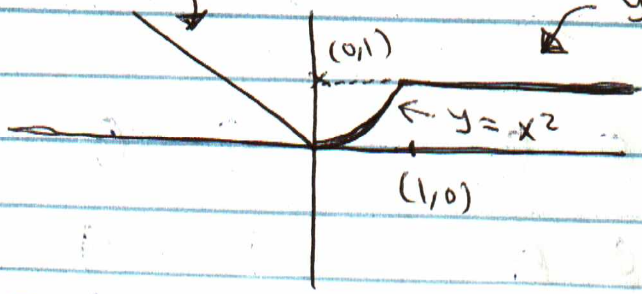
④ $y = f(x) = \frac{1}{x}$



Example : Sketch the graph of

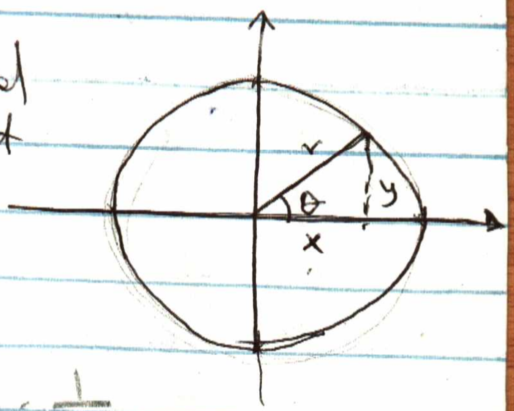
$$y = f(x) = \begin{cases} -x & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Solution :



Trigonometric Relations (Rules)

Let θ be any angle in standard position and (x, y) is any point We define



$\sin\theta = \dfrac{y}{r}$ $\qquad$ $\cos\theta = \dfrac{x}{r}$

$\sec\theta = \dfrac{r}{x} = \dfrac{1}{\cos\theta}$ $\qquad$ $\csc\theta = \dfrac{r}{y} = \dfrac{1}{\sin\theta}$

$\tan\theta = \dfrac{y}{x}$ $\qquad$ $\cot\theta = \dfrac{x}{y} = \dfrac{\cos\theta}{\sin\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Angle Measurment

The angle degree $\longleftrightarrow$ radian

$$360^\circ = 2\pi$$

Ly

$1^\circ = \frac{\pi}{180}$ rad   **Ex** $30^\circ = \frac{\pi}{180} \cdot 30 = \frac{\pi}{6}$

$1(\text{rad}) = \frac{180}{\pi}$ **Ex** $\frac{\pi}{2} = \frac{\pi}{2} \cdot \frac{180}{\pi} = 90^\circ$

$x = \cos\theta$, $y = \sin\theta$

$x^2 + y^2 = 1$

(Some Trigonometric Relations)

① $\cos^2\theta + \sin^2\theta = 1$

② $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$

③ $\sin(A \mp B) = \sin A \cos B \mp \cos A \sin B$

④ $\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$

⑤ $\sin(2\theta) = 2\sin\theta \cos\theta$

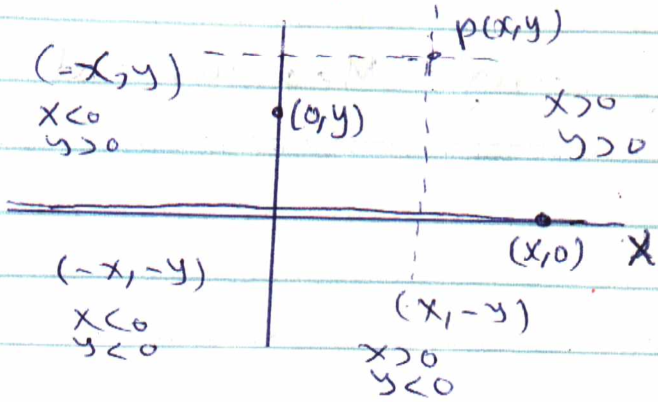
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

⑥ $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$, $\sin^2 x = \frac{1-\cos(2\theta)}{2}$

Analytic Geometry

Cartesian plane , Coordinates plane, x-y-plane

Every point in the plane corresponds to an order pair of real number x and y called Coordinates (x, y)

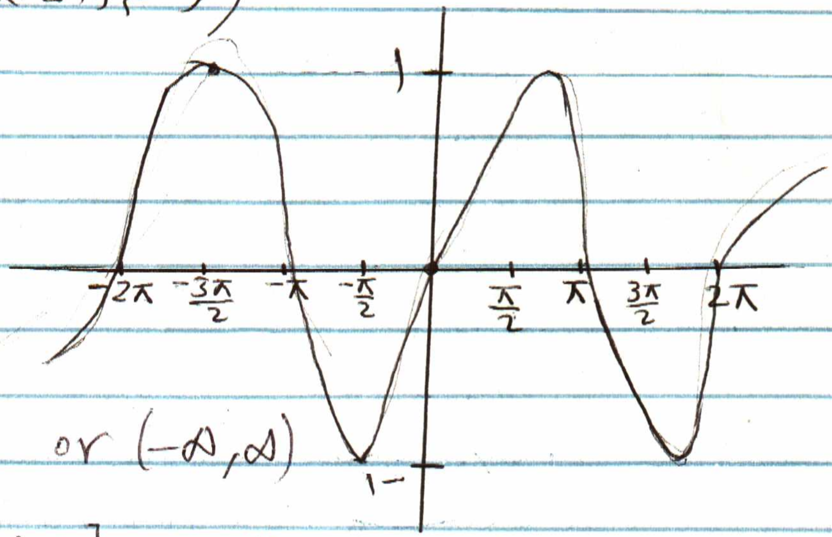


تابع لـ (3)

Graph of Trigonometric function

(رسم دالة sinx)

x	$y = \sin x$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0
$-\frac{\pi}{2}$	-1
$-\pi$	0

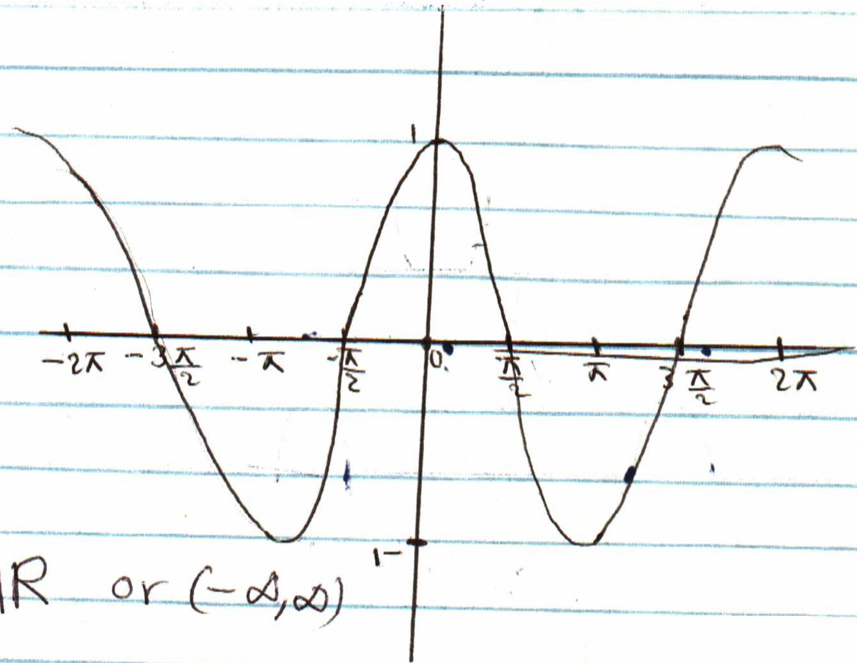


$D_f = R$ or $(-\infty, \infty)$

$R_f = [-1, 1]$, $-1 \leq y \leq 1$

(رسم دالة cosx)

x	$y = \cos x$
0	1
$\frac{\pi}{2}$	0
π	-1
$3\frac{\pi}{2}$	0
2π	1
$-\frac{\pi}{2}$	0

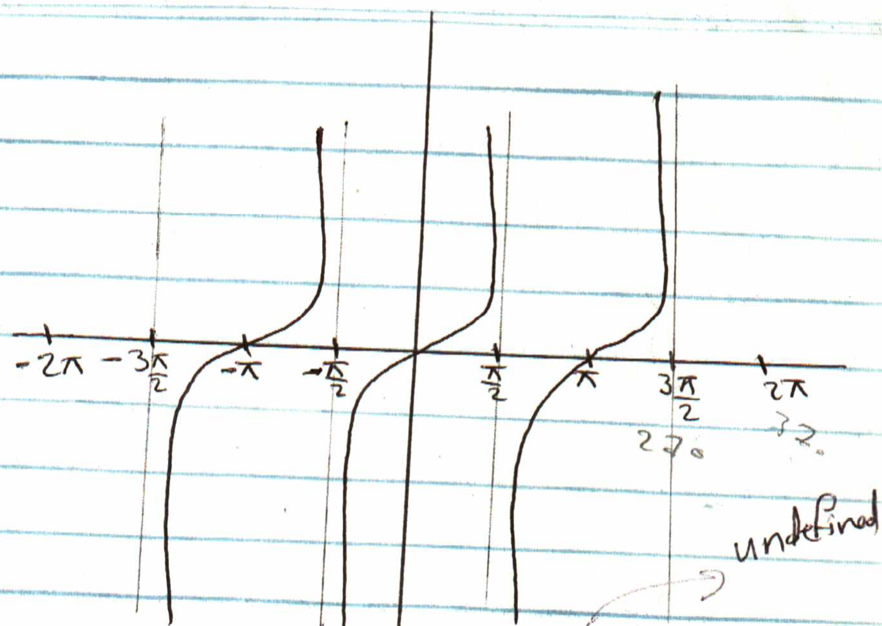


$D_f = \mathbb{R}$ or $(-\infty, \infty)$

$R_f = [-1, 1]$ $-1 \leq y \leq 1$

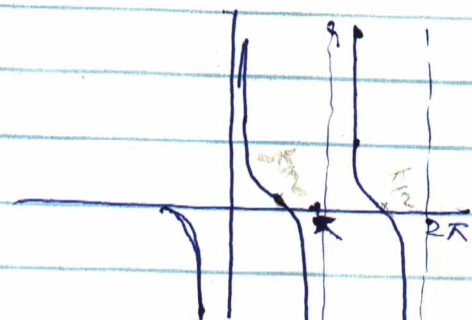
$y = \tan x$

x	$y=\tan x$
0	0
$\frac{\pi}{2}$	$\lim \frac{\sin x}{\cos x} = \infty$
π	0
$3\frac{\pi}{2}$	$-\infty$
2π	0
$\frac{-\pi}{2}$	$-\infty$
$-\pi$	0
$-3\frac{\pi}{2}$	∞
-2π	0



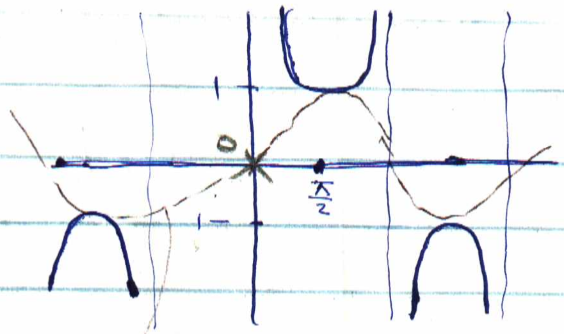
$D_f = \mathbb{R} / \{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}, \ldots\}$

$R_f = \mathbb{R}$



$D_f = R / \{0, \pi, 2\pi\}$ $\cot x = \frac{1}{\tan x}$

$R_f = \mathbb{R}$ or $(-\infty, \infty)$

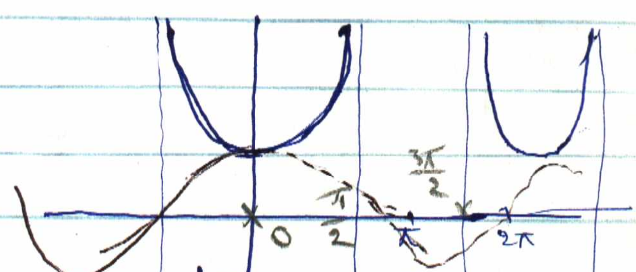


y=sinx

$\csc x = \frac{1}{\sin x}$

$D_f = \mathbb{R} / \{0, \pi, 2\pi\}$

$R_f = (-\infty, -1] \cup [1, \infty)$



y=cosx

H.W/Graph $y = f(x) = 3\cos x$

$\sec x = \frac{1}{\cos x}$

$D_f = R / \{\mp\frac{\pi}{2}, \mp\frac{3\pi}{2}\}$

$R_f = (-\infty, -1] \cup [1, \infty)$

Distance

Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ two points in the xy-plane.

Slope of line

slope of line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given as following :

$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Equation of line

Every line has an equation that can be written in general form $ax + by + c = 0$ [a, b, c constants $a, b \neq 0$] Called a Linear equation ».

To find the equation of line

[1] If the line passing through point (x_1, y_1) and has slope m then

$$\boxed{y - y_1 = m(x - x_1)}$$

Ex// Find the equation of line has slope $= \frac{-1}{4}$ and passing through $(5, -3)$

sol// $y - y_1 = m(x - x_1)$

$\Rightarrow y + 3 = \frac{-1}{4}(x - 5)$

$\Rightarrow y + 3 = -\frac{1}{4}x + \frac{5}{4} \Rightarrow y + \frac{1}{4}x + \frac{7}{4} = 0$

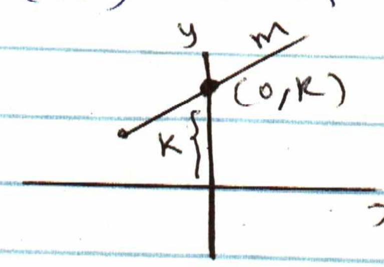
$\Rightarrow 4y + x + 7 = 0$

[2] If the line passing through two points (x_1, y_1), (x_2, y_2) then

$$\boxed{\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}}$$

[3] If the line intercept y-axis (k) and has slope $= m$, then

$$\boxed{y = mx + k}$$



Ex// Find m, k of $3y - 2x + 1 = 0$

sol// $3y = 2x - 1 \Rightarrow y = \frac{2}{3}x - \frac{1}{3}$

$\therefore m = \frac{2}{3}$, $k = \frac{-1}{3}$

Note: $ax+by+c=0$ (equation of line)

$$y=-\frac{a}{b}x-\frac{c}{b}=mx+k$$

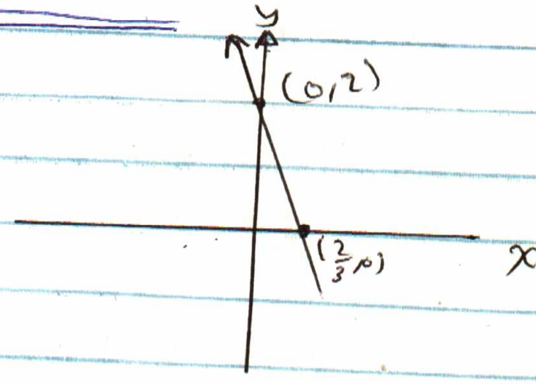
* Distance from point (x_1,y_1) and line $ax+by+c=0$ is given:

$$D=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

* Graph of line $ax+by+c=0$

Graph $y+3x-2=0$

x	y	
0	2	$(0,2)$
$\frac{2}{3}$	0	$(\frac{2}{3},0)$



الخطوط المتوازيه

الخطوط المتعامدة Perpendicular lines have slopes that are negative reciprocals.

بالسالب المقلوب ميول

Note:

A line parallel to $2x-3y=8$ will have a slope of $\frac{2}{3}$

So a perpendicular line has a slope $\frac{-3}{2}$

H.W

① Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 2$ and $g(x) = \sqrt{7-x}$?

② Find $f \circ g$ and $g \circ f$, where $f(x) = \frac{1-x}{3x}$ and $g(x) = \frac{1}{1+3x}$?

③ Sketch the graph of $f(x) = x^2 - 4$

$g(x) = -x^2 + 4$

then determine the domain and range of both functions

④ Find the equation of a line that has slope of $\frac{-2}{3}$ and a y-intercept of 4 ?

⑤ Find the equation of a line goes through the point $(-1, 6)$ and $(3, 2)$?

⑥ Find the equation of a line that is parallel to the x-axis and goes through the point $(-4, -7)$

⑦ Find the equation of a line that is perpendicular to $6y - 3x - 2 = 0$ and goes through the point $(4, -3)$?

⑧ If $f(x) = \sqrt{x}$, $g(x) = 2x + 1$, then find

(a) $(f \circ g)(3)$ (b) $(f \circ f)(x)$ (c) $(f \circ g \circ f)(x)$

Chapter 2

Integration

laws of integration

ثابت التكامل

① $\int 1\, dx = x + C$

② $\int k\, f(x)\, dx = k \int f(x)\, dx$

③ $\int [f(x) \mp g(x)]\, dx = \int f(x)\, dx \mp \int g(x)\, dx$

④ $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1$

⑤ $\int [f(x)]^n\, f'(x)\, dx = \dfrac{[f(x)]^{n+1}}{n+1} + C \quad \text{if } n \neq -1$

Example//: Evaluate

① $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}}\, dx = \dfrac{x^{3/2}}{\frac{3}{2}} + C$

② $\int (x^2+5)^2 x\, dx = \frac{1}{2} \int 2x\, (x^2+5)^2\, dx = \frac{1}{2}\, \dfrac{(x^2+5)^3}{3} + C$

③ $\int x \sqrt{2x^2+1}\, dx = \frac{1}{4} \int 4x \sqrt{2x^2+1}\, dx$

$= \frac{1}{4}\, \dfrac{(2x^2+1)^{\frac{3}{2}}}{3/2} + C$

Laws of Derivatives of Trigonometric Functions:-

① $(\sin(u))' = \cos(u) \cdot \frac{du}{dx}$

② $(\cos(u))' = -\sin(u) \cdot \frac{du}{dx}$

③ $(\tan(u))' = \sec^2(u) \cdot \frac{du}{dx}$

④ $(\cot(u))' = -\csc^2(u) \cdot \frac{du}{dx}$

⑤ $(\sec(u))' = \sec(u) \cdot \tan(u) \cdot \frac{du}{dx}$

⑥ $(\csc(u))' = -\csc(u) \cdot \cot(u) \cdot \frac{du}{dx}$.

Example

1) $\frac{d}{dx} \sin 2X = \cos \overset{u}{2X} \cdot \frac{d}{dx}(2X)$
$= 2\cos 2X$

2) $\frac{d}{dx} \sin X^5 = \cos X^5 \cdot \frac{d}{dx}(X^5)$
$= 5X^4 \cos X^5$

3) $\frac{d}{dx} \cos 3X = -\sin 3X \cdot \frac{d}{dx}(3X) = -3\sin 3X$

H.W

if $y = \frac{\sin X}{1+\cos(X)}$ find $\frac{dy}{dx}$

② If $x = x(t)$ and $y = y(t)$, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Ex// Find $\frac{d^2y}{dx^2}$ if $x = t - t^2$ and $y = t - t^3$

Sol// $y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t}$

$$y'' = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{(1-2t)(-6t) - (1-3t^2)(-2)}{(1-2t)^2}}{(1-2t)}$$

$$\therefore y'' = \frac{2 - 6t + 6t^2}{(1-2t)^3}$$

Derivative of Trigonometric functions

(1) prove that if $y = f(x) = \sin x$, then $\frac{dy}{dx} = \cos x$

Sol// $f(x) = \sin x$, $f(x+\Delta x) = \sin(x+\Delta x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}$$

$$= \sin x \cdot \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} + \cos x \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}$$

$$= \sin x (0) + \cos x (1) = \cos x$$

Implicit Differentiation الاشتقاق الضمني

Example//: Find $\frac{d^2y}{dx^2}$ if $2x^3-3y^2=7$

sol// $6x^2-6yy'=0 \Rightarrow y'=\frac{x^2}{y}$

$$\Rightarrow y''=\frac{y(2x)-x^2y'}{y^2}=\frac{2xy-x^2(\frac{x^2}{y})}{y^2}$$

$$=\frac{\frac{2xy^2-x^4}{y}}{y^2}$$

$$y''=\frac{2xy^2-x^4}{y^3}$$

Chain Rule قانون السلسلة

① If $y=f(x)$ and $x=x(t)$ then $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$

Ex// Let $y=\sqrt{x^2+1}$, $x=\frac{1}{t}+t^2$, find $\frac{dy}{dt}$

sol// $\frac{dy}{dx}=\frac{1}{2}(x^2+1)^{-1/2}(2x)=\frac{x}{\sqrt{x^2+1}}=\frac{\frac{1}{t}+t^2}{\sqrt{(\frac{1}{t}+t^2)^2+1}}$

$\frac{dx}{dt}=-\frac{1}{t^2}+2t$

$$\therefore \frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=\left[\frac{\frac{1}{t}+t^2}{\sqrt{(\frac{1}{t}+t^2)^2+1}}\right]\cdot\left[-\frac{1}{t}+2t\right]$$

⑥ $[(f(x))^n]' = n(f(x))^{n-1} \cdot f'(x)$

⑦ $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$

Example/ Find y' or $\frac{dy}{dx}$

(i) $y = f(x) = x\sqrt{x^2-2}$

$y' = x\left[\frac{1}{2}(x^2-2)^{-\frac{1}{2}}(2x)\right] + \sqrt{x^2-2}$

(ii) $y = f(x) = \frac{x^2}{(x-1)^2}$

$y' = \frac{(x-1)^2(2x) - x^2(2)(x-1)}{(x-1)^4}$

Higher Derivatives :

$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$f'''(x) = \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$ and so on

$f^{(n)}(x) = \frac{d^ny}{dx^n}$ if n is apositive .

Example : Find y''' if $y = f(x) = (x-2)^3$

sol $y' = 3(x-2)^2$, $y'' = 6(x-2)$

$y''' = 6$

H.W $y = f(x) = \frac{x^2}{(x-1)^2}$ find $\frac{dy}{dx}$

The Derivative

The derivative of a function $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$ and defined by:-

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{, if the limit exists}$$

Example:- Use the definition to find $f'(x)$ if $f(x) = \sqrt{x}$

Sol// $f(x) = \sqrt{x}$, $f(x+\Delta x) = \sqrt{x+\Delta x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Laws of Derivatives

1. The derivative of a constant is zero.
2. $(x^n)' = n x^{n-1}$
3. $(c f(x))' = c f'(x)$, c is const
4. $(f(x) \mp g(x))' = f'(x) \mp g'(x)$
5. $(f(x) \cdot g(x))' = f(x) \cdot g'(x) + f'(x) \cdot g(x)$

H.W

Q1/ Find $\lim_{x \to 2} \frac{x-1}{x^2+2}$?

Q2/ Find $\lim_{x \to 1} \frac{x^3-1}{x-1}$?

Q3// Evaluate the following

(a) $\lim_{h \to 0} \frac{(h+3)^2-9}{h}$

(b) $\lim_{x \to 0} \frac{x^3-a^3}{x-a}$

$= (1)\cdot(2) = 2$

Continuity

A function $y = f(x)$ is said to be continuous at a point a if

(1) $f(a)$ exists

(2) $\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$

Example // Test the continuity of

(1) $f(x) = \frac{1}{x-2}$ at $a = 2$

Sol// $\because$ $f(x)$ does not exists at $a = 2$ then f is not conts.

Ex// $f(x) = \begin{cases} x & x < 1 \\ \frac{x}{2} & x \geq 1 \end{cases}$ at $a = 1$

Sol// $\because$ $\lim_{x \to 1^+} f(x) = \frac{1}{2} \neq \lim_{x \to 1^-} f(x) = 1$

$\therefore$ $\lim_{x \to 1}$ does not exists $\Rightarrow$ f is not continuous at $a = 1$

Examples : find

1) $\lim\limits_{x \to 3} \dfrac{x^2+2x+4}{x+2} = \dfrac{3^2+2(3)+4}{3+2} = \dfrac{19}{5}$

2) $\lim\limits_{x \to 2} \dfrac{x^3-8}{x^2-4} = \lim\limits_{x \to 2} \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$

$= \lim\limits_{x \to 2} \dfrac{(x^2+2x+4)}{(x+2)} = \dfrac{2^2+2(2)+4}{2+2} = \dfrac{12}{4}$

$= 3$

3) $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{x} = 3 \lim\limits_{x \to 0} \dfrac{\sin(3x)}{3x} = 3(1) = 3$

4) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x\cos x}\right)$

$= \lim\limits_{x \to 0} \left(\dfrac{\sin x}{x}\right) \cdot \lim\limits_{x \to 0} \left(\dfrac{1}{\cos x}\right) = (1)(1)$

$= 1$

Notes

(1) $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$

(2) $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

(3) $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$

(4) $\lim\limits_{x \to -\infty} \dfrac{1}{x} = 0$

$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$ (لا يجوز
قيم غير معرفة)

Example : Evaluate

(1) $\lim\limits_{x \to \infty} \dfrac{x}{2x+4} = \lim\limits_{x \to \infty} \dfrac{\frac{x}{x}}{\frac{2x}{x} + \frac{4}{x}} = \lim\limits_{x \to \infty} \dfrac{1}{2 + \underbrace{\frac{4}{x}}_{(0)}} = \dfrac{1}{2}$

(2) $\lim\limits_{x \to 0} \dfrac{\tan 2x}{\sin x} = \lim\limits_{x \to 0} \dfrac{2x \tan 2x}{2x \sin x} = \left(\lim\limits_{x \to 0} \dfrac{\tan 2x}{2x}\right) \cdot \left(\lim\limits_{x \to 0} \dfrac{2x}{\sin x}\right)$

Limits

① The limit of $f(x)$ as X approaches a from the right is the number L if $\lim\limits_{x\to a^+} f(x) = L$.

② The limit of $f(x)$ as x approaches a from the left is the number L if $\lim\limits_{x\to a^-} f(x) = L$.

③ The number L is the limit of $f(x)$ as x approaches a, denoted by $\lim\limits_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

Theorem 1: If $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} g(x) = L_2$

then :-

(1) $\lim\limits_{x\to a} (f(x) \mp g(x)) = \lim\limits_{x\to a} f(x) \mp \lim\limits_{x\to a} g(x) = L$

(2) $\lim\limits_{x\to a} (f(x) \cdot g(x)) = [\lim\limits_{x\to a} f(x)] \cdot [\lim\limits_{x\to a} g(x)] = L_1 \cdot L$

(3) $\lim\limits_{x\to a} (k\, f(x)) = K \cdot (\lim\limits_{x\to a} f(x)) = k \cdot L_1$

k is Consta

(4) $\lim\limits_{x\to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$

Integration of Trigonometric functions :

1) $\int \cos u \, du = \sin u + C$

2) $\int \sin u \, du = -\cos u + C$

3) $\int \sec^2 u \, du = \tan u + C$

4) $\int \csc^2 u \, du = -\cot u + C$

5) $\int \sec u \tan u \, du = \sec u + C$

6) $\int \csc u \cot u \, du = -\csc u + C$

Example// Evaluate

1) $\int \tan x \sec^2 x \, dx = \frac{\tan^2 x}{2} + C$

2) $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx$

$$= \frac{1}{2} \int dx + \frac{1}{2} \cdot \frac{1}{2} \int 2 \cos 2x \, dx$$

$$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$$

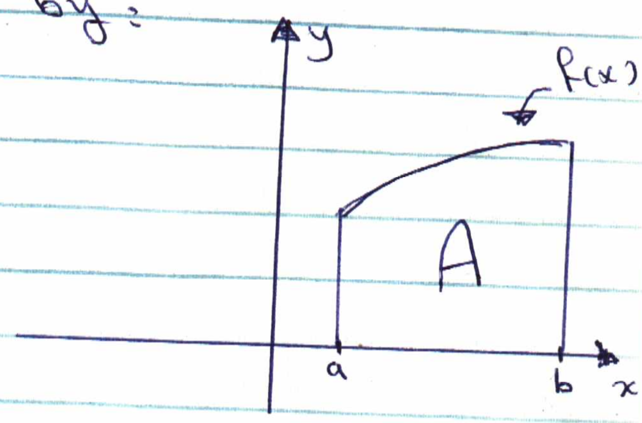
③ $\int \frac{\sin 2x}{\cos^2 2x} dx = \int ((\cos 2x))^{-2} \sin 2x \, dx$

$= \frac{-1}{2} \frac{((\cos 2x))^{-1}}{-1} + C$

Define Integral التكامل المحدد :-

The area under the Curve $y = f(x)$ from $x=a$ to $x=b$ is defined by:

$$\boxed{A = \int_a^b f(x)\, dx}$$



properties

(1) $\int_a^b k f(x)\, dx = k \int f(x)\, dx$, k is Constant

(2) $\int_a^b f(x)\, dx \geq 0$

(3) $\int_a^b f(x)\, dx = - \int_b^a f(x)\, dx$

(4) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$, where c is apoint in $[a,b]$

(5) If f is Continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$\int_a^b f(x)\, dx = F(x) \Big|_a^b = F(b) - F(a)$

Example ① Evaluate $\int_0^1 (x^2-2x+1)\,dx$

$$\int_0^1 (x^2-2x+1)\,dx = \left.\frac{x^3}{3} - x^2 + x\right|_0^1$$

$$= \frac{1}{3} - 1 + 1 = \boxed{\frac{1}{3}}$$

Example ②

: Find the area under $y = f(x) = \sin x$ from $x=0$ to $x=\pi$.

Sol// $A = \int_a^b f(x)\,dx = \int_0^{\pi} \sin x\,dx = \left.-\cos x\right|_0^{\pi}$

$$= -\cos\pi + \cos(0)$$

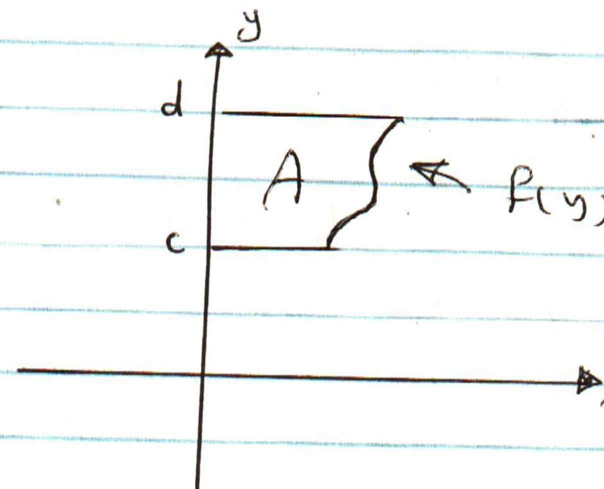
$$= \boxed{2}$$

H.W Find the area under $y = f(x) = \cos x$ from $x=0$ to $x = \frac{\pi}{2}$

(حول محور Y)

Remark :

$$A = \int_c^d f(y)\,dy$$



Example: Calculate the area under the curve $x = f(y) = 6 - y - y^2$

Sol // $0 = 6 - y - y^2$ حدود التكامل

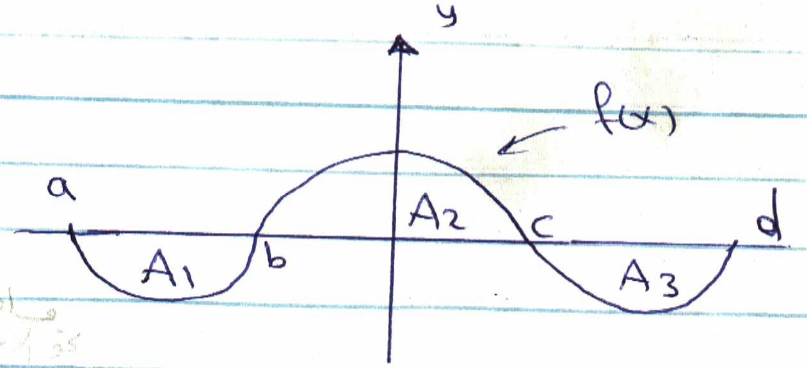
$= (3+y)(2-y) \Rightarrow y = -3$ or $y = 2$

$$A = \int_c^d f(y)\,dy = \int_{-3}^{2} (6 - y - y^2)\,dy$$

$$= 6y - \frac{y^2}{2} - \frac{y^3}{3} \Big|_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right) = \boxed{\frac{125}{6}}$$

Remark



$$A = \int_a^d f(x)\,d(x)$$

$$= |A_1| + A_2 + |A_3|$$

Ex // Find the area between $y = x^3 - 4x$ and the x-axis?

Sol // $0 = x^3 - 4x \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x-2)(x+2) = 0$

i.e $x = 0$ or $x = 2$ or $x = -2$

$$A_1 = \int_{-2}^{0} f(x)\,dx = \int_{-2}^{0} (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 \Big|_{-2}^{0} = 4$$

$$A_2 = \int_0^2 (x^3 - 4x)\,dx = \frac{x^4}{4} - 2x^2 \Big|_0^2 = -4$$

$$\therefore A = |A_1| + |A_2| = 4 + 4 = \boxed{8}$$

Chapter 3

Inverse of Trigonometric functions

① $y = f(x) = \sin^{-1}x$, $D_f = \{x \mid -1 \le x \le 1\}$ $[-1,1]$, $R_f = \{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$

② $y = f(x) = \cos^{-1}x$, $D_f = \{x \mid -1 \le x \le 1\}$, $R_f = \{y \mid 0 \le y \le \pi\}$

③ $y = f(x) = \tan^{-1}x$, $D_f = \mathbb{R}$, $R_f = \{y \mid -\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$

④ $y = f(x) = \cot^{-1}x$, $D_f = \mathbb{R}$, $R_f = \{y \mid 0 \le y \le \pi\}$

⑤ $y = f(x) = \sec^{-1}x$, $D_f = \{x \mid x \ge 1, x \le -1\}$ $R/(-1,1)$, $R_f = \{y \mid y \ne n(\frac{\pi}{2})\}$ $[0,\pi]/\frac{\pi}{2}$

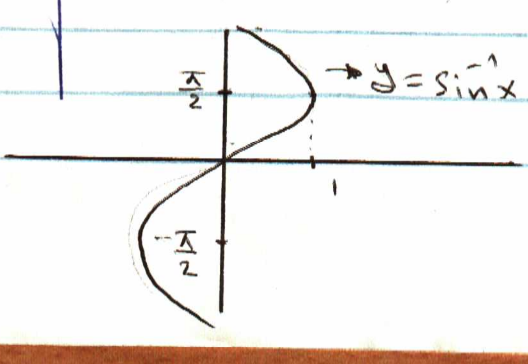
⑥ $y = f(x) = \csc^{-1}x$, $D_f = \{x \mid x \ge 1, x \le -1\}$ $R/(-1,1)$, $R_f = \{y \mid 0 \le |y| \le \frac{\pi}{2}\}$ $[-\frac{\pi}{2}, \frac{\pi}{2}]/0$

Remark: $\sin^{-1}x \ne \frac{1}{\sin x} = (\sin x)^{-1}$

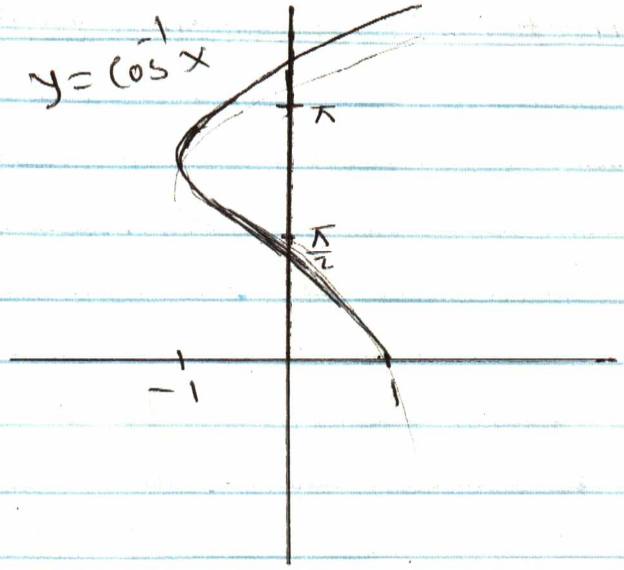
$$\sin(\sin^{-1}x) = \sin^{-1}(\sin x) = x$$

Graph of $y = \sin^{-1}x$

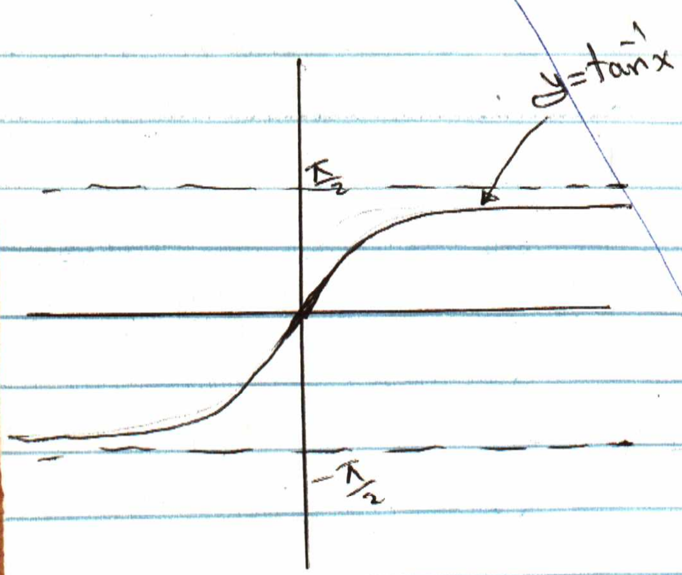
x	$y = \sin^{-1}x$
1	$y = \sin^{-1}1 \Rightarrow 1 = \sin y \Rightarrow \sin\frac{\pi}{2} = 1 \Rightarrow \frac{\pi}{2} = \sin^{-1}1$ $(1, \frac{\pi}{2})$
0	$y = \sin^{-1}0 \Rightarrow 0 = \sin y \Rightarrow \sin 0 = 0 \Rightarrow 0 = \sin^{-1}0$ $(0,0)$
-1	$y = \sin^{-1}(-1) \Rightarrow -1 = \sin y \Rightarrow \sin(-\frac{\pi}{2}) = -1 \Rightarrow \frac{\pi}{2} = \sin^{-1}(-1)$ $(-1, -\frac{\pi}{2})$



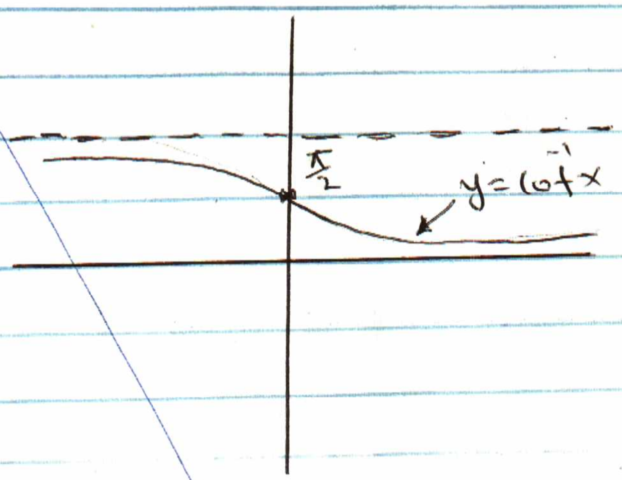
Graphs :



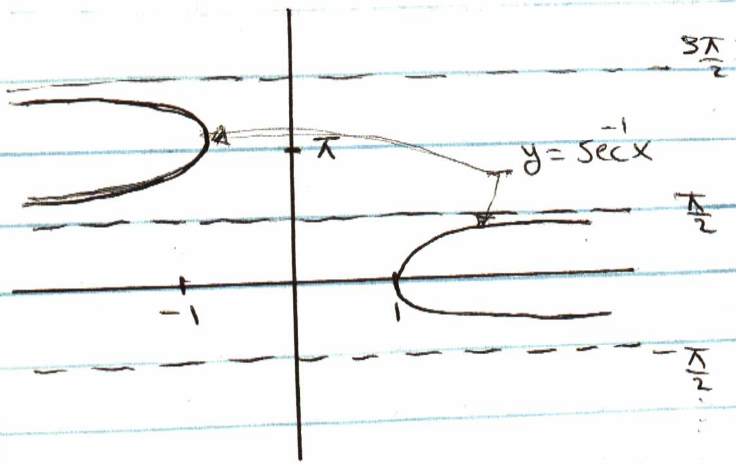
y = cos⁻¹x
π
π/2
-1
1



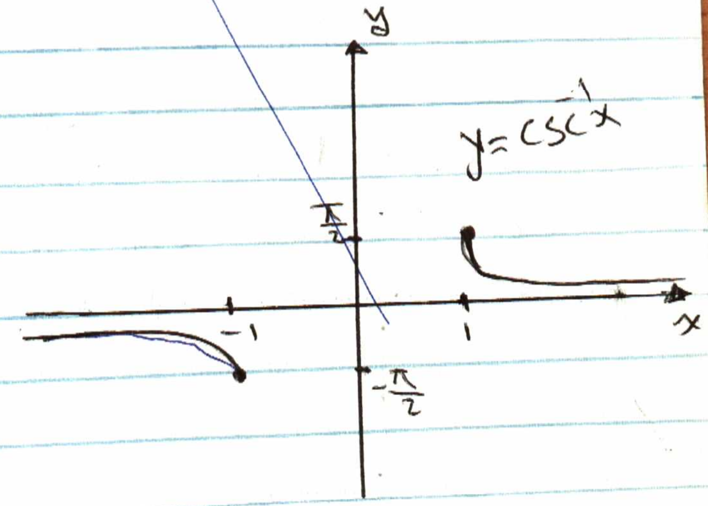
y = tan⁻¹x
π/2
-π/2



π/2
y = cot⁻¹x



3π/2
π
y = sec⁻¹x
π/2
-1
1
-π/2



y
y = csc⁻¹x
π/2
-1
1
x
-π/2

Laws of Derivative (Inverse of Trigonometric function)

(1) $y = \sin^{-1} u \Rightarrow \begin{cases} y' = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx} \\ y' = \dfrac{-1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx} \end{cases}$

$y = \cos^{-1} u \Rightarrow$

(2) $y = \tan^{-1} u \Rightarrow \begin{cases} y' = \dfrac{1}{1+u^2} \dfrac{du}{dx} \\ y' = \dfrac{-1}{1+u^2} \dfrac{du}{dx} \end{cases}$

$y = \cot^{-1} u \Rightarrow$

(3) $y = \sec^{-1} u \Rightarrow \begin{cases} y' = \dfrac{1}{|u|\sqrt{u^2-1}} \cdot \dfrac{du}{dx} \\ y' = \dfrac{-1}{|u|\sqrt{u^2-1}} \cdot \dfrac{du}{dx} \end{cases}$

$y = \csc^{-1} u \Rightarrow$

Laws of Integration

(1) $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \begin{cases} \sin^{-1} u \\ -\cos^{-1} u \end{cases} + C$

(2) $\displaystyle\int \frac{du}{1+u^2} = \begin{cases} \tan^{-1} u \\ -\cot^{-1} u \end{cases} + C$

(3) $\displaystyle\int \frac{du}{|u|\sqrt{u^2-1}} = \begin{cases} \sec^{-1} u \\ -\csc^{-1} u \end{cases} + C$

Example : Find $\frac{dy}{dx}$ if

① $y = \cos^{-1}(2x) \Longrightarrow y' = \frac{-2}{\sqrt{1-(2x)^2}}$

② $y = \tan^{-1}(x^2) \Longrightarrow y' = \frac{2x}{1+x^4}$

③ $y = x \sin^{-1}(3x)$: $y' = x\left[\frac{3}{\sqrt{1-(3x)^2}}\right] + \sin^{-1}(3x)$

Examples : Evaluate

① $\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}x \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$ $\quad 45^\circ$

② $\int \frac{x^2 dx}{1+x^6} = \frac{1}{3}\int \frac{3x^2}{1+(x^3)^2}dx = \frac{1}{3}\tan^{-1}x^3 + C$

③ $\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int \frac{3\,dx}{\sqrt{1-9x^2}} = \frac{1}{3}\sin^{-1}(3x) + C$

Properties

① $\sin^{-1}(-x) = -\sin^{-1}x$

② $\cos^{-1}(-x) = \pi - \cos^{-1}x$

③ $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

④ $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

⑤ $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$

⑥ $\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$

⑦ $\sec^{-1}(-x) = \pi - \sec^{-1}x$