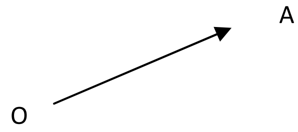
Chapter (1)

Vectors

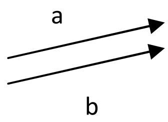
Concept of direction:

Represent vectors as directed line segments



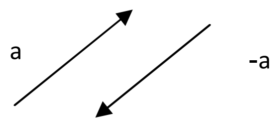
- Magnitude is length of line segment.
- Arrow is the direction from O to A.

Two vectors are equal if they have same magnitude and direction.



Negative vectors:

-a has same magnitude as a, but has opposite direction. So |-a| = |a|



Scalars and vectors:

A ***vector*** is a quantity that has a magnitude and a direction. For examples velocity, force, displacement and acceleration.

A ***scalar*** is a quantity that has magnitude only. Mass, time and volume are all examples.

A vector when written down, a symbol in bold type such as $\boldsymbol{V}$ or with an arrow, as $\vec{V}$ (i.e., magnitude plus direction), while V refers to the magnitude only.

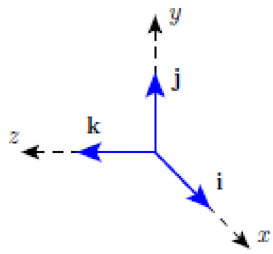
A ***unit vector*** is a vector whose magnitude is one.

A vector $\boldsymbol{V}$ parallel to the unite vector u can be expressed in the form

$\boldsymbol{V} = u\, V$ …….. (1)

If two vectors $\boldsymbol{V}$ and $\boldsymbol{V}'$ are parallel to each other, they may be written as $\boldsymbol{V} = u\, V$ and $\boldsymbol{V}' = u\, V'$, where the unit vector u is the same. Thus if $\lambda = V/\, V'$ we may write $\boldsymbol{V} = \lambda\, \boldsymbol{V}'$

In three dimensional space, vectors with a magnitude of **one** in the direction of the x-axis, y-axis and z-axis will be denoted by **i, j,** and **k** respectively.



Position Vector:

The position vector of any point is the directed line segment from the origin O(0,0,0) to the point and is given by the coordinates of the point. The position vector of P (3,4,1) is 3**i** + 4**j** + **k**, or (3,4,1).

Components of a Vector:

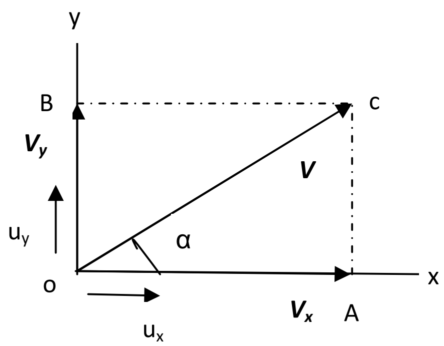
Any vector $\boldsymbol{V}$ can always be considered as the sum of two (or more) vectors, and the number of possibilities is infinite. Each set of vectors which, when added give $\boldsymbol{V}$ are called the components $\boldsymbol{V}$.

The once most commonly used are the rectangular components, i.e., the vector is expressed as the sum of two mutually perpendicular vectors.

Then, as we see from the figure

$\boldsymbol{V} = \boldsymbol{V_x} + \boldsymbol{V_y}$, with

$V_x = V\cos\alpha$ and $V_y = V\sin\alpha$ …….. (2)



Defining unit vectors $\boldsymbol{u_x}$ and $\boldsymbol{u_y}$ in the directions of the X- and Y-axes, we note that

$\boldsymbol{V_x} = \overrightarrow{OA} = \boldsymbol{u_x}\, V_x$, $\boldsymbol{V_y} = \overrightarrow{OB} = \boldsymbol{u_y}\, V_y$

Therefore we have

$\boldsymbol{V} = \boldsymbol{u_x}\, V_x + \boldsymbol{u_y}\, V_y$ …….. (3)

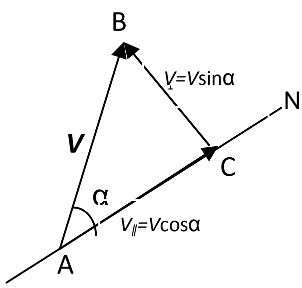
This equation expresses a vector in terms of its rectangular components in two dimensions. Substituting (2) in (3)

$\boldsymbol{V} = \boldsymbol{u_x}\, V\cos\alpha + \boldsymbol{u_y}\, V\sin\alpha$ …….. (4)

Note that the component of a vector in a particular direction is equal to the projection of the vector in that direction. From the figure

$V_\parallel = V\cos\alpha$

Also from this figure, we see that BC is tl component of $\boldsymbol{V}$ perpendicular to the chosen direction AN, and we can see that

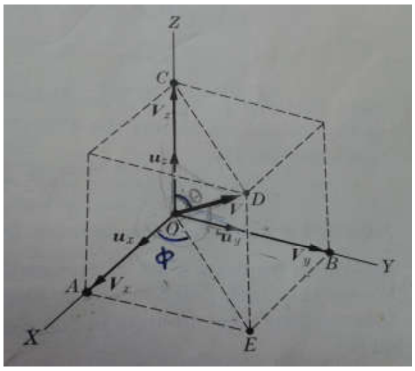


$V_\perp$ = BC= $V\sin\alpha$

Thus

$\boldsymbol{V} = V_\parallel + V_\perp$

There are three rectangular components in space: V_x, V_y, V_z



$$\left.\begin{aligned} V_x &= V \sin\theta \cos\phi \,, \\ V_y &= V \sin\theta \sin\phi \,, \\ V_z &= V \cos\theta \end{aligned}\right\} \qquad \text{........ (5)}$$

From which it follows, by direct computation, that

$$V^2 = V_x^2 + V_y^2 + V_z^2 \qquad \text{........ (6)}$$

Defining three unit vectors $\boldsymbol{u_x}$, $\boldsymbol{u_y}$, $\boldsymbol{u_z}$ parallel to the X-, Y-, and Z-axes, respectively, we have

$$\boldsymbol{V} = \boldsymbol{u_x} V_x + \boldsymbol{u_y} V_y + \boldsymbol{u_z} V_z \qquad \text{........ (7)}$$

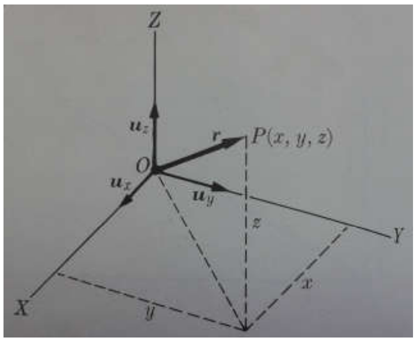
If we designate by α and β the angles the vector $\boldsymbol{V}$ makes with the X- and Y-axes, respectively, we also have, by similarity with the third of eq.(5),

$V_x = V \cos\alpha$, $V_y = V \cos\beta$

Using these two and $V_z = V \cos\theta$ in eq.(6), we obtain the relation

$\cos^2 \alpha + \cos^2 \beta + \cos^2 \theta = 1$

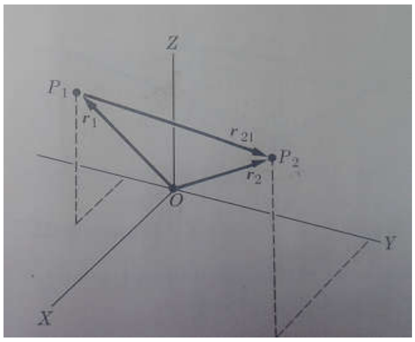
The quantities $\cos\alpha$, $\cos\beta$, and $\cos\theta$ are called the direction cosines of a vector.



An important case of a three-dimensional vector is the position vector $\mathbf{r} = \overrightarrow{OP}$ of a point P having coordinates (x, y, z). From this figure we see that

$$\mathbf{r} = \overrightarrow{OP} = \boldsymbol{u_x}\, x + \boldsymbol{u_y}\, y + \boldsymbol{u_z}\, z \qquad \text{........ (8)}$$

The relative position vector of two points P_1 and P_2 is $r_{21} = \overrightarrow{P1P2}$.



From the figure we note that $\overrightarrow{OP2} = \overrightarrow{OP1} + \overrightarrow{P1P2}$, so that

$$r_{21} = \overrightarrow{P1P2} = \overrightarrow{OP2} - \overrightarrow{OP1} = r_2 - r_1$$
$$= \boldsymbol{u_x}(x_2 - x_1) + \boldsymbol{u_y}(y_2 - y_1) + \boldsymbol{u_z}(z_2 - z_1) \qquad \text{..... (9)}$$

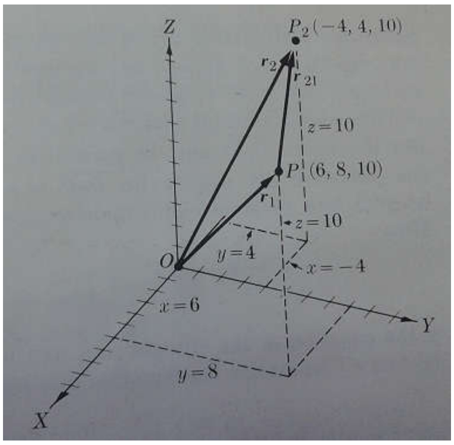
Note that $\overrightarrow{P2P1} = -\overrightarrow{P1P2}$. It should be observed that, by applying eq.(6) to eq.(9), we obtain the expression of analytic geometry for the distance between two points:

$$r_{21} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

***Example (1)*:** Find the distance between the two points (6,8,10) and (-4, 4, 10).

Solution:

We see from this figure that both points are in a plane parallel to the XY-plane, since they are both a distance (height) of 10 units in the Z-direction. From eq.(9), we find that the vector r_{21} is



$$r_{21} = \boldsymbol{u_x}(-4-6) + \boldsymbol{u_y}(4-8) + \boldsymbol{u_z}(10-10)$$
$$= \boldsymbol{u_x}(-10) + \boldsymbol{u_y}(-4) + \boldsymbol{u_z}(0)$$
$$= -\boldsymbol{u_x}(10) - \boldsymbol{u_y}(4)$$

Using eq.(6),we find that the magnitude is

$r^2_{21} = 100+16= 116$ or

$r_{21} = 10.77$ units

***Example (2)*:** Find the components of the vector that is 13 units long and makes an angle θ of 22.6° with the Z-axis, and whose projection in the XY-plane makes an angle ϕ of 37° with the +X-axis. Find also the angles with the X- and Y-axes.

Solution:

$V = 13$ units, $\theta = 22.6°$, $\cos\theta = 0.923$, $\sin\theta = 0.384$, $\phi = 37°$, $\cos\phi = 0.8$, $\sin\phi = 0.6$

Using eq.(5) yields

$V_x = 13(0.384)(0.8)= 4$ units

$V_y = 13(0.384)(0.6)= 3$ units

$V_z = 13(0.923)= 12$ units

In terms of eq.(7) we may write

$\boldsymbol{V} = \boldsymbol{u_x}\,(4) + \boldsymbol{u_y}\,(3) + \boldsymbol{u_z}\,(12)$

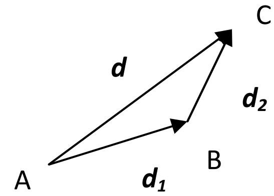
For the angles α and β that $\boldsymbol{V}$ makes with the X- and Y-axes, we have

$\cos\alpha = \frac{V_x}{V} = 0.308$ or $\alpha = 72.1°$

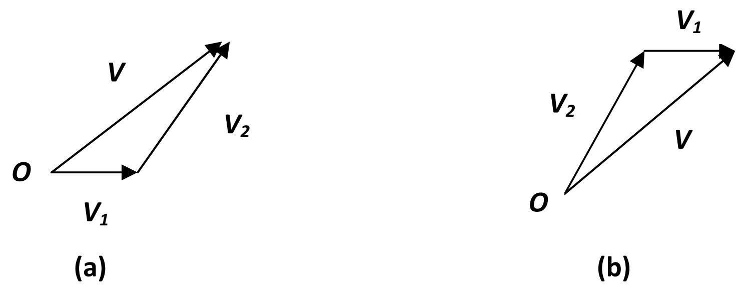
$\cos\beta = \frac{V_y}{V} = 0.231$ or $\beta = 77°$

Addition of Vectors

If a partical is displaced first from A to B represented by vector $\boldsymbol{d_1}$, and then from B to C, or $\boldsymbol{d_2}$, the result is equivalent to a single displacement from A to C or $\boldsymbol{d}$, which we write symbolically as $\boldsymbol{d} = \boldsymbol{d_1} + \boldsymbol{d_2}$.



We say that $\boldsymbol{V}$ is the sum of $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ if it obtained as indicated in this figure.



We can also see that the vector sum is commutative, the result being the same if the order in which the vectors are added is reversed, this is direct consequence of the geometry of the method.

$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2$ ……… (10)

To compute the magnitude of $\boldsymbol{V}$ we see from the figure that

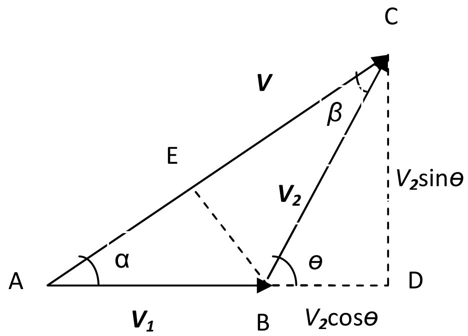
$(AC)^2 = (AD)^2 + (DC)^2$

But AD = AB + BD = $V_1 + V_2 \cos\theta$

And DC = $V_2 \sin\theta$

$V^2 = (\boldsymbol{V}_1 + \boldsymbol{V}_2 \cos\theta)^2 + (V_2 \sin\theta)^2$

$= V^2_1 + V^2_2 + 2\, V_1 V_2 \cos\theta$



Or $V = \sqrt{V_1^2 + V_2^2 + 2V_1V_2\cos\theta}$ ……… (11)

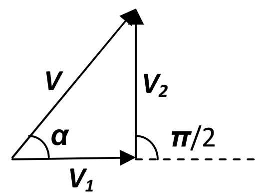
To determine the direction of $\boldsymbol{V}$, we need only find the angle α. From the figure we see that in triangle ACD, CD = AC sinα, and in triangle BDC, CD = BC sinθ.

Therefore $V\sin\alpha = V_2 \sin\theta$ or $\frac{V}{\sin\theta} = \frac{V_2}{\sin\alpha}$

Similarly, BE = $V_1 \sin\alpha = V_2 \sin\beta$ or $\frac{V_2}{\sin\alpha} = \frac{V_1}{\sin\beta}$

Combining both results, $\frac{V}{sin\theta} = \frac{V_1}{sin\beta} = \frac{V_2}{sin\alpha}$ ……… (12)

In the special case when $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ are perpendicular as shown in this figure, $\theta = \frac{1}{2}\pi$ and the following relations hold:



$V = \sqrt{V_1{}^2 + V_2{}^2}$, $\tan\alpha = \frac{V_2}{V_1}$ ……… (13)

The difference between two vectors is obtained by adding to the first the negative (or opposite) of the second.

$\boldsymbol{D} = \boldsymbol{V_1} - \boldsymbol{V_2} = \boldsymbol{V_1} + (-\boldsymbol{V_2})$

Note that $\boldsymbol{V_2} - \boldsymbol{V_1} = -\boldsymbol{D}$

$\boldsymbol{D} = \sqrt{V_1^2 + V_2^2 + 2V_1V_2\cos(\pi - \theta)}$ or

$\boldsymbol{D} = \sqrt{V_1^2 + V_2^2 - 2V_1V_2\cos\theta}$ ……… (14)

<u>Example (3)</u>: Given two vectors: A is 6 units long and makes an angle of 36° with the positive X-axis, B is 7 units long and is in the direction of the negative X-axis. Find: a) the sum of the two vectors, b) the difference between the two vectors.

<u>Addition of Several Vectors:</u>

$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2 + \boldsymbol{V}_3 + \ldots$ ……… (15)

The case where all vectors are in one plane.

$\boldsymbol{V} = (\boldsymbol{u_x}\, V_{1x} + \boldsymbol{u_y}\, V_{1y}) + (\boldsymbol{u_x}\, V_{2x} + \boldsymbol{u_y}\, V_{2y}) + (\boldsymbol{u_x}\, V_{3x} + \boldsymbol{u_y}\, V_{3y}) + \ldots$

$= \boldsymbol{u_x}\,(V_{1x} + V_{2x} + V_{3x} + \ldots) + \boldsymbol{u_y}\,(V_{1y} + V_{2y} + V_{3y} + \ldots)$

$$\left.\begin{aligned} V_x &= V_{1x} + V_{2x} + V_{3x} + \ldots = \textstyle\sum_i V_{ix} = \sum_i V_i \cos\alpha_i \\ V_y &= V_{1y} + V_{2y} + V_{3y} + \ldots = \textstyle\sum_i V_{iy} = \sum_i V_i \sin\alpha_i \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (16)$$

Where α_i : is the angle V_i makes with the positive X-axis and $V_i \cos\alpha_i$ and $V_i \sin\alpha_i$ are the components of $\boldsymbol{V_i}$ along the X- and Y-axes.

<u>Example (4)</u>: Find the resultant of the sum of the following five vectors:

$\boldsymbol{V_1} = \boldsymbol{u_x}\,(4) + \boldsymbol{u_y}\,(-3)$ units, $\boldsymbol{V_2} = \boldsymbol{u_x}\,(-3) + \boldsymbol{u_y}\,(2)$ units,

$\boldsymbol{V_3} = \boldsymbol{u_x}\,(2) + \boldsymbol{u_y}\,(-6)$ units, $\boldsymbol{V_4} = \boldsymbol{u_x}\,(7) + \boldsymbol{u_y}\,(-8)$ units,

$\boldsymbol{V_5} = \boldsymbol{u_x}\,(9) + \boldsymbol{u_y}\,(1)$ units

Solution:

$V_x = 4 + (-3) + 2 + 7 + 9 = 19$ units

$V_y = -3 + 2 + (-6) + (-8) + 1 = -14$ units

$\boldsymbol{V} = \boldsymbol{u_x}\ (19) + \boldsymbol{u_y}\ (-14)$ units

$V = \sqrt{(19)^2 + (-14)^2} = 23.55$ units

$\tan\alpha = \frac{V_y}{V_x} = -0.738$ or $\alpha = -36.4^\circ$ the angle $\boldsymbol{V}$ makes with the X-axis.

Scalar Product:

The scalar product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, represented by the symbol $\boldsymbol{A} \cdot \boldsymbol{B}$, is defined as the scalar quantity obtained by finding the product of the magnitudes of $\boldsymbol{A}$ and $\boldsymbol{B}$ and the cosine of the angle between the two vectors

$\boldsymbol{A} \cdot \boldsymbol{B} = A\,B \cos\theta$ …………… (17)

$\boldsymbol{A} \cdot \boldsymbol{A} = A^2$ (when $\theta = 0$)

$\boldsymbol{A} \cdot \boldsymbol{B} = 0$ (when $\theta = \pi/2$)

The scalar product is commutative, $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$, since $\cos\theta$ is the same in both cases.

$\boldsymbol{C} \cdot (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{C} \cdot \boldsymbol{A} + \boldsymbol{C} \cdot \boldsymbol{B}$ …………… (18)

$\boldsymbol{u_x} \cdot \boldsymbol{u_x} = \boldsymbol{u_y} \cdot \boldsymbol{u_y} = \boldsymbol{u_z} \cdot \boldsymbol{u_z} = 1$ ($\theta = 0$)

$\boldsymbol{u_x} \cdot \boldsymbol{u_y} = \boldsymbol{u_y} \cdot \boldsymbol{u_z} = \boldsymbol{u_z} \cdot \boldsymbol{u_x} = 0$ ($\theta = \pi/2$) …………… (19)

$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$ ………… (20)

$A^2 = \boldsymbol{A} \cdot \boldsymbol{A} = A_x^2 + A_y^2 + A_z^2$

Example : Find the angle between the vectors $\boldsymbol{A} = 2\boldsymbol{u_x} + 3\boldsymbol{u_y} - \boldsymbol{u_z}$ and

$\boldsymbol{B} = -\boldsymbol{u_x} + \boldsymbol{u_y} + 2\boldsymbol{u_z}$

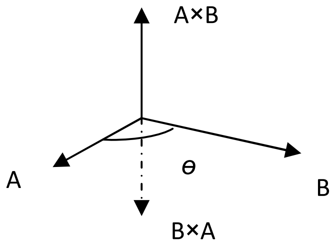
Solution: $\boldsymbol{A} \cdot \boldsymbol{B} = 2(-1) + 3(1) + (-1)(2) = -1$

$A = \sqrt{4+9+1} = \sqrt{14} = 3.74$ units, $B = \sqrt{1+1+4} = \sqrt{6} = 2.45$ units

$\boldsymbol{A} \cdot \boldsymbol{B} = A\,B\cos\theta$, $\cos\theta = \frac{\boldsymbol{A} \cdot \boldsymbol{B}}{A\,B} = -\frac{1}{9.17} = -0.109$, $\theta = 96.3^\circ$

Vector Product:

The vector product of two vectors ***A*** and ***B*** (***A***× ***B***) is defined as the vector perpendicular to the plane determined by ***A*** and ***B*** and in the direction of advance of a right-handed screw rotated from ***A*** to ***B***.



The magnitude of the vector product

$|\boldsymbol{A} \times \boldsymbol{B}| = A\, B \sin \theta$ ………… (21)

$\boldsymbol{A} \times \boldsymbol{B} = - \boldsymbol{B} \times \boldsymbol{A}$ ……… (22)

Because the sense of rotation of the screw is reversed when the order of the vectors is changed, so that the vector product is anticommutative.

$\boldsymbol{A} \times \boldsymbol{B} = 0$ (if θ=0 two vectors are parallel)

$|\boldsymbol{A} \times \boldsymbol{B}|$ = area of parallelogram

The vector product is distributive relative to the sum,

$\boldsymbol{C} \times (\boldsymbol{A} + \boldsymbol{B}) = \boldsymbol{C} \times \boldsymbol{A} + \boldsymbol{C} \times \boldsymbol{B}$ ……… (23)

$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{u_x}\,(A_y B_z - A_z B_y) + \boldsymbol{u_y}\,(A_z B_x - A_x B_z) + \boldsymbol{u_z}\,(A_x B_y - A_y B_x)$ ….. (24)

or

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{u_x} & \boldsymbol{u_y} & \boldsymbol{u_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{……… (25)}$$

For 2D

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1\, b_2 - a_2\, b_1$$

For 3D

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1\,(b_2\, c_3 - b_3\, c_2) + a_2\,(b_3\, c_1 - b_1\, c_3) + a_3\,(b_1\, c_2 - b_2\, c_1)$$

Example: Find the area of the parallelogram determined by the vectors

$\boldsymbol{A} = 2\boldsymbol{u_x} + 3\boldsymbol{u_y} - \boldsymbol{u_z}$ and $\boldsymbol{B} = -\boldsymbol{u_x} + \boldsymbol{u_y} + 2\boldsymbol{u_z}$

<u>Solution:</u>

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{u_x} & \boldsymbol{u_y} & \boldsymbol{u_z} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} = 7\boldsymbol{u_x} - 3\boldsymbol{u_y} + 5\boldsymbol{u_z}$$

Area = $|\boldsymbol{A} \times \boldsymbol{B}| = \sqrt{49 + 9 + 25} = 9.11$ units

Chapter (2)

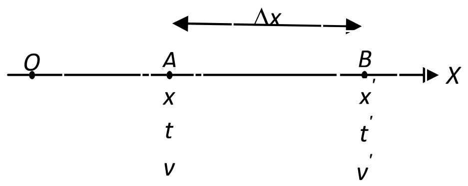
Kinematics

Rectilinear motion: Velocity

The motion of a body is rectilinear when its trajectory is a straight line. In principle, the displacement can be correlated with the time by means of a functional relation $x = f(t)$, x may be positive or negative.

At time (t) the object is at position A, with $OA = x$. At a later time (t') it is at B, with $OB = x'$. The average velocity between A and B is defined by:

$$v_{\text{ave}} = \frac{x' - x}{t' - t} = \frac{\Delta x}{\Delta t} \quad \ldots\ldots \ (1)$$



Δx: is the displacement of a particle and

Δt: is the elapsed time.

To determine the instantaneous velocity at a point A, this is equivalent to computing the limiting value of the fraction Δt when approaches zero.

$$v = \lim_{\Delta t \to 0} v_{ave} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$v = \frac{dx}{dt} \quad \ldots\ldots\ldots \ (2)$$

If we know $v = f(t)$, from eq.(2) we have $dx = v\, dt$,

$$\int_{x_\circ}^{x} dx = \int_{t_\circ}^{t} v dt$$

Where $x_\circ$ is the value of x at time $t_\circ$.

$$\int_{x_\circ}^{x} dx = x - x_\circ$$

$$x = x_\circ + \int_{t_\circ}^{t} v\, dt \quad \ldots\ldots\ldots \ (3)$$

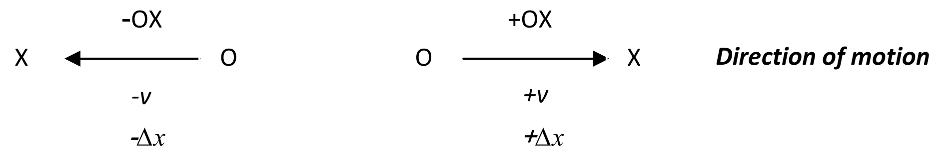
where $v\, dt$ represents the displacement of the body in the short time interval dt.

When we dividing the time interval $t - t_\circ$ into successive small intervals dt_1, dt_2, dt_3, ……., we find that

Displacement $= x - x_\circ = v_1\, dt_1 + v_2\, dt_2 + v_3\, dt_3 + \ldots\ldots$

$$= \sum_i v_i \, dt_i = \int_{t_\circ}^{t} v \, dt$$

The displacement Δx may be positive or negative depending on the motion of the particle is to the right or to the left, resulting in a positive or negative sign for the velocity.



In MKSC system of units, velocity is expressed by (m/sec). The concept of speed defined as ***distance/time***. It is always positive, i.e. *speed* = $| v |$.

The average speed does not have the same value as the average velocity. The displacement is computed by eq.(3) but the distance is obtained by $\int_{t_\circ}^{t} |v| \, dt$.

For example in going from city *A* to city *B*, which is 100mi east of *A*, a driver may first go to city *C*, which is 50mi west of *A*, and then turn back and go to *B*. The distance = 200 mi, but the displacement = 100 mi. If the motion takes place in 4 hr the average speed is

$$speed = \frac{distance}{time} = \frac{200}{4} = 50 \text{ mi/hr}$$

$$average\ velocity = \frac{\Delta x}{\Delta t} = \frac{100}{4} = 25 \text{ mi/hr}$$

Example: A particle moves along the *X*-axis in such a way that its position at any instant is given by $x = 5t^2 + 1$, where x in meters and t in seconds. Compute its average velocity in the time interval between **a)** 2s and 3s, **b)** 2s and 2.1s, **c)** 2s and 2.001s, **d)** 2s and 2.00001s. Also compute **e)** the instantaneous velocity at 2s.

Solution: $t_\circ = 2$s, $x = 5t^2 + 1$, $x_\circ = 5(2)^2 + 1 = 21$m, for each question

$\Delta x = x - x_\circ = x - 21$, $\Delta t = t - t_\circ = t - 2$

a) For $t = 3$s , $\Delta t = 1$s, $x = 5(3)^2 + 1 = 46$m, and $\Delta x = 46 - 21 = 25$m

$$v_{ave} = \frac{\Delta x}{\Delta t} = \frac{25}{1} = 25 \text{ m/s}$$

e) The instantaneous velocity at 2s. $v = \frac{dx}{dt} = \frac{d}{dt}(5t^2 + 1) = 10t = 20$ m/s

Rectilinear Motion: Acceleration

In general, the velocity of a body is a function of time. If the velocity remains constant, the motion is said to be uniform. The average acceleration between *A* and *B* is

$$a_{\text{ave}} = \frac{v' - v}{t' - t} = \frac{\Delta v}{\Delta t} \qquad \ldots\ldots \ (4)$$

Thus the average acceleration during a certain time interval is the change in velocity per unit time during the time interval (m/s^2).

The instantaneous acceleration is

$$a = \lim_{\Delta t \to 0} a_{ave} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$a = \frac{dv}{dt} \qquad \ldots\ldots\ldots \ (5)$$

In general, the acceleration varies during the motion. If the rectilinear motion has constant acceleration, the motion is said to be ***"uniformly accelerated"***.

If the velocity increases with time, the motion is said to be ***"accelerated"***, but if the velocity decreases with time, the motion is said ***"decelerated"***.

From eq. (5), we have

$$dv = a\, dt$$

$\int_{v_\circ}^{v} dv = \int_{t_\circ}^{t} a dt$, where $v_\circ$ is the velocity at the time $t_\circ$

$$\int_{v_\circ}^{v} dv = v - v_\circ$$

$$v = v_\circ + \int_{t_\circ}^{t} a\, dt \qquad \ldots\ldots\ldots \ (6)$$

($a\ dt$) gives the change in velocity during a short time interval dt. Thus, again dividing the time interval $t - t_\circ$ into small intervals dt_1, dt_2, dt_3, ……., we find that

Change in velocity = $v - v_\circ = a_1\, dt_1 + a_2\, dt_2 + a_3\, dt_3 + \ldots\ldots$

$$= \sum_i a_i\, dt_i = \int_{t_\circ}^{t} a\, dt$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$a = \frac{d^2x}{dt^2}$ …..… (7)

From eq. (5), $dv = a\ dt$

$v\ dv = a\ dt\ (\frac{dx}{dt}) = a\ dx$

$\int_{v_\circ}^{v} v\ dv = \int_{x_\circ}^{x} a dx$

$\frac{1}{2}v^2 - \frac{1}{2}v_\circ^2 = \int_{x_\circ}^{x} a dx$ …..… (8)

<u>Vector Representation of Velocity and Acceleration in Rectilinear Motion:</u>

If ***u*** is a unit vector in the positive direction of the X-axis, we may write in vector form

$\boldsymbol{v} = \boldsymbol{u}\ v = \boldsymbol{u}\ \frac{dx}{dt}$ and $\boldsymbol{a} = \boldsymbol{u}\ \frac{dv}{dt}$

Vectors ***v*** and ***a*** point along ***u*** or in opposite direction, depending on the signs of *dx/dt* and *dv/dt* respectively.

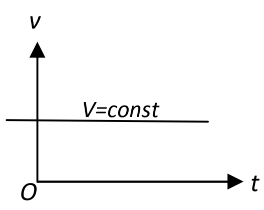
A simple rule is: If ***v*** and ***a*** have the same sign, the motion is accelerated, if the signs are opposite, the motion is retarded. There are three special cases of motion:

1) Uniform rectilinear motion (v is constant)

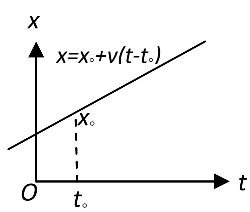
$a = \frac{dv}{dt} = 0$ (there is no acceleration), from eq. (3)

$x = x_\circ + \int_{t_\circ}^{t} v\ dt = x_\circ + v \int_{t_\circ}^{t} dt$

$x = x_\circ + v\ (t - t_\circ)$ …………. (9)



Graph of velocity



Graph of displacement

2) Uniformly accelerated rectilinear motion (a is constant), from eq. (6)

$v = v_\circ + \int_{t_\circ}^{t} a\ dt = v_\circ + a\int_{t_\circ}^{t} dt$

$v = v_\circ + a\,(t - t_\circ)$ ………… (10)

From eq. (3), we have

$x = x_\circ + \int_{t_\circ}^{t} v\,dt = x_\circ + \int_{t_\circ}^{t} [v_\circ + a(t - t_\circ)]\,dt = x_\circ + v_\circ \int_{t_\circ}^{t} dt + a\int_{t_\circ}^{t} (t - t_\circ)\,dt$

$x = x_\circ + v_\circ\,(t - t_\circ) + \frac{1}{2}\,a\,(t - t_\circ)^2$ ………… (11)

From eq. (8)

$\frac{1}{2}\,v^2 - \frac{1}{2}\,v_\circ^2 = a\int_{x_\circ}^{x} dx = a\,(x - x_\circ)$

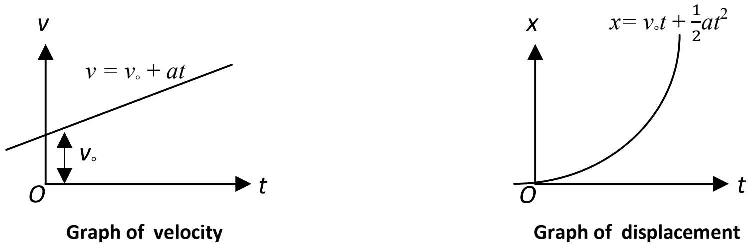
$v^2 = v_\circ^{\,2} + 2a\,(x - x_\circ)$ ………… (12)

3) ***Free vertical motion*** under the action of gravity, taking the upward direction as positive, $\boldsymbol{a = -g}$, the minus sign refer to the fact that the gravitational acceleration is downward. **g =9.8 m/s² = 32.2 ft/s²**. (The acceleration of gravity decreases when the distance above or below the earth's surface increases)**.**

When $t_\circ = 0$ and $x_\circ = 0$

$v = v_\circ + at$

$x = v_\circ\,t + \frac{1}{2}\,a\,t^{\,2}$



Example: A body moves along the X- axis according to the $x = 2t^3 + 5t^2 + 5$ where x in feet and t in seconds. Find **a)** v and a at any time, **b)** the position , v, a at t=2s and 3s, **c)** the average velocity and acceleration between t= 2s and t=3s.

Solution

a) $v = \frac{dx}{dt} = \frac{d}{dt}\,(2t^3 + 5t^2 + 5) = 6t^2 + 10t$ ft/s

$a = \frac{dv}{dt} = \frac{d}{dt}(6t^2 + 10t) = 12t + 10$ ft/s

b) At t =2s, x=41 ft, v= 44 ft/s, a= 34 ft/s^2

For t =3s, x=104 ft, v= 84 ft/s, a= 46 ft/s^2

c) Δt = 3-2=1s and from b) we have Δx= 63 ft, Δv= 40 ft/s

$v_{ave} = \frac{\Delta x}{\Delta t} = \frac{63}{1} = 63$ ft/s

$a_{ave} = \frac{\Delta v}{\Delta t} = \frac{40}{1} = 40$ ft/s^2

Example: The acceleration of a body moving along the X- axis is $a = (4x - 2)$ m/s^2, where x is in meters. Given $v_\circ = 10$ m/s at $x_\circ = 0$, find the velocity at any other position.

Solution

$\frac{1}{2}v^2 - \frac{1}{2}v^2_\circ = \int_0^x a\,dx$

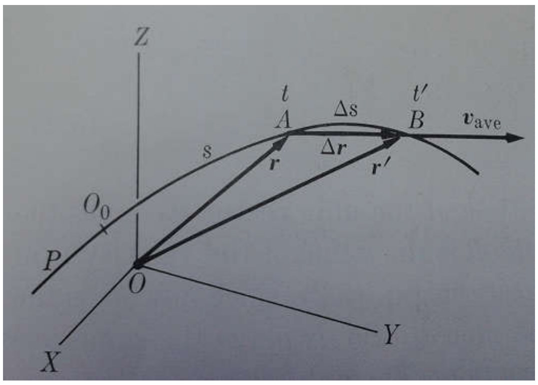
$\frac{1}{2}v^2 - \frac{1}{2}(10)^2 = \int_0^x (4x - 2)dx$

$v^2 = 100 + 2\,(2x^2 - 2x)_0^x = 4x^2 - 4x + 100$

$v = \sqrt{4x^2 - 4x + 100}$

Curvilinear Motion: Velocity

Let us consider a particle describing a curvilinear path P, as shown in the figure. At time t the particle is at point A, given by position vector



$\boldsymbol{r} = \overrightarrow{OA} = \boldsymbol{u_x}x + \boldsymbol{u_y}y + \boldsymbol{u_z}z$

At a later time t', the particle will be at B with

$\boldsymbol{r}' = \overrightarrow{OB} = \boldsymbol{u_x}x' + \boldsymbol{u_y}y' + \boldsymbol{u_z}z'$

Although the particle has moved along the arc $AB = \Delta s$, the displacement, which is a vector, is $\overrightarrow{AB} = \Delta r$.

$\boldsymbol{r}' = \boldsymbol{r} + \Delta\boldsymbol{r}$

$\overrightarrow{AB} = \Delta \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r} = \boldsymbol{u_x}\,(x' - x) + \boldsymbol{u_y}\,(y' - y) + \boldsymbol{u_z}\,(z' - z)$

$$= \boldsymbol{u_x}\,(\Delta x) + \boldsymbol{u_y}\,(\Delta y) + \boldsymbol{u_z}\,(\Delta z) \quad \ldots\ldots\ldots\ldots (13)$$

Where $\Delta x = \boldsymbol{x}' - x$, $\Delta y = \boldsymbol{y}' - y$, and $\Delta z = \boldsymbol{z}' - z$. The average velocity, also a vector, is defined by

$$\boldsymbol{v}_{\text{ave}} = \frac{\Delta \boldsymbol{r}}{\Delta t} \quad \ldots\ldots\ldots\ldots (14)$$

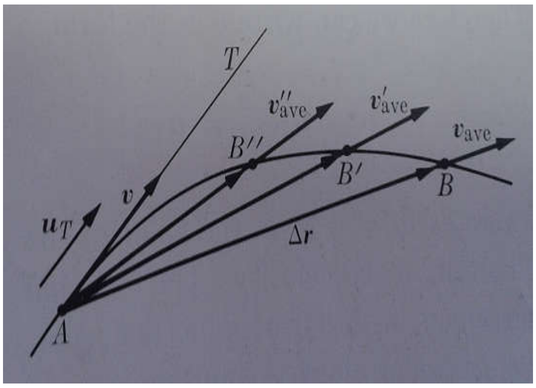
$$\boldsymbol{v}_{\text{ave}} = \boldsymbol{u_x}\frac{\Delta x}{\Delta t} + \boldsymbol{u_y}\frac{\Delta y}{\Delta t} + \boldsymbol{u_z}\frac{\Delta z}{\Delta t} \quad \ldots\ldots\ldots\ldots (15)$$

$\boldsymbol{v}_{\text{ave}}$ is represented by a vector parallel to the displacement $\overrightarrow{AB} = \Delta r$. The instantaneous velocity defined as

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \boldsymbol{v}_{ave} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} \quad \ldots\ldots\ldots\ldots (16)$$

when Δt approaches zero, point B approaches point A, during this process Δr changes in magnitude and direction.

In curvilinear motion, the instantaneous velocity is a vector tangent to the path.



$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} \quad \ldots\ldots (17)$$

$$\boldsymbol{v} = \boldsymbol{u_x}\frac{dx}{dt} + \boldsymbol{u_y}\frac{dy}{dt} + \boldsymbol{u_z}\frac{dz}{dt} \quad \ldots\ldots (18)$$

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt} \quad \ldots\ldots (19)$$

The magnitude of the velocity, called the speed,

$$v = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} \quad \ldots\ldots (20)$$

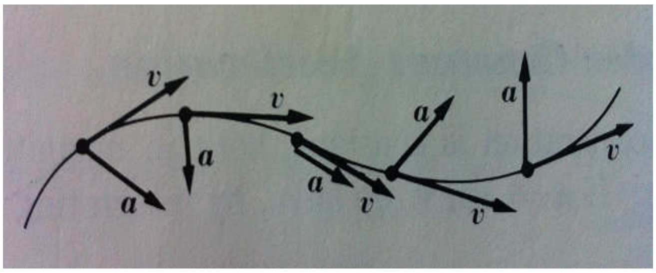
$s = O_oA$ gives the position of the particle as measured by the displacement along the curve, s may be positive or negative depending on which side of O_o the particle is. Δs = arc AB

$$\boldsymbol{v} = \boldsymbol{u_T}\frac{ds}{dt} = \boldsymbol{u_T}\, v \quad \ldots\ldots (21)$$

where $\frac{ds}{dt} = v$ gives the value of velocity, and the tangent unit vector $\boldsymbol{u_T}$ gives the direction of the velocity.

Curvilinear Motion: Acceleration

In curvilinear motion the velocity changes both in magnitude (because the particle may speed up or slow down) and in direction (because the velocity is tangent to the path which is bends continuously).



$\boldsymbol{a}_{\text{ave}} = \frac{\Delta \boldsymbol{v}}{\Delta t}$ .... (22)

$v = \boldsymbol{u_x}\, v_x + \boldsymbol{u_y}\, v_y + \boldsymbol{u_z}\, v_z$

$\Delta v = \boldsymbol{u_x}\, \Delta v_x + \boldsymbol{u_y}\, \Delta v_y + \boldsymbol{u_z}\, \Delta v_z$

$\boldsymbol{a}_{\text{ave}} = \boldsymbol{u_x} \frac{\Delta v_x}{\Delta t} + \boldsymbol{u_y} \frac{\Delta v_y}{\Delta t} + \boldsymbol{u_z} \frac{\Delta v_z}{\Delta t}$ .... (23)

The instantaneous acceleration is

$\boldsymbol{a} = \lim_{\Delta t \to 0} \boldsymbol{a}_{ave} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t}$ or $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$ .... (24)

Acceleration is always pointing toward the concavity of the curve, and is neither tangent nor perpendicular to the path.

$\boldsymbol{a} = \frac{d^2\boldsymbol{r}}{dt^2}$ .... (25)

$\boldsymbol{a} = \boldsymbol{u_x} \frac{dv_x}{dt} + \boldsymbol{u_y} \frac{dv_y}{dt} + \boldsymbol{u_z} \frac{dv_z}{dt}$ .... (26)

$a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$, $a_z = \frac{dv_z}{dt}$ .... (27) or

$a_x = \frac{d^2x}{dt^2}$, $a_y = \frac{d^2y}{dt^2}$, $a_z = \frac{d^2z}{dt^2}$ .... (28)

The magnitude of the acceleration is $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ .... (29)

Motion under constant acceleration

The case in which acceleration is constant, both in magnitude and direction, is of special importance. If $\boldsymbol{a}$ = const. we have

$\int_{v_\circ}^{v} d\boldsymbol{v} = \int_{t_\circ}^{t} \boldsymbol{a} dt = \boldsymbol{a} \int_{t_\circ}^{t} dt = \boldsymbol{a}\,(t - t_\circ)$ .... (30)

Where $v_\circ$ is the velocity at time $t_\circ$, since $\int_{v_\circ}^{v} d\boldsymbol{v} = \boldsymbol{v} - v_\circ$,

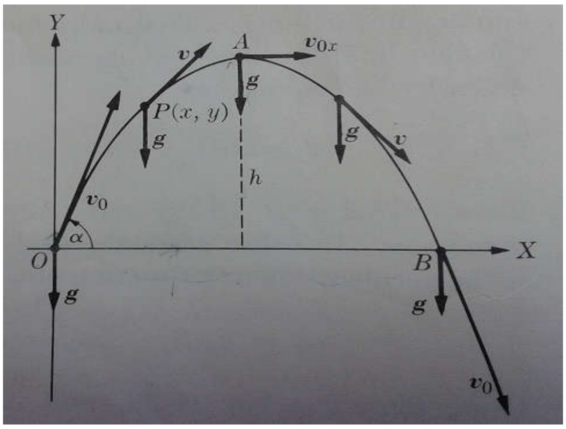
$\boldsymbol{v} = v_\circ + \boldsymbol{a}\,(t - t_\circ)$ .... (31)

If $\boldsymbol{r}_\circ$ gives the position at time $t_\circ$. Then

$\boldsymbol{r} = \boldsymbol{r}_\circ + \boldsymbol{v}_\circ\,(t - t_\circ) + \frac{1}{2}\,\boldsymbol{a}\,(t - t_\circ)^2$ …….. (32)

which gives the position of the particle at any time.

In rectilinear motion, both the velocity and the acceleration have the same (or opposite) direction.



$\boldsymbol{a} = \boldsymbol{g} = -\,\boldsymbol{u}_y\, g$ (acceleration of gravity)

$\boldsymbol{v}_\circ = \boldsymbol{u}_x\, v_{\circ x} + \boldsymbol{u}_y\, v_{\circ y}$, where

$v_{\circ x} = v_\circ \cos\alpha$, $v_{\circ y} = v_\circ \sin\alpha$ …….. (33)

$v_x = v_{\circ x}$, $v_y = v_{\circ y} - gt$ …….. (34)

$x = v_{\circ x}\, t$, $y = v_{\circ y}\, t - \frac{1}{2}\, gt^2$ …….. (35)

which gives the coordinates of the particle as functions of time. The time required for the projectile to reach the highest A is obtained by setting $v_y = 0$, at that point the velocity of the projectile is horizontal.

$t = \frac{v_{\circ y}}{g}$ or $t = \frac{v_\circ \sin\alpha}{g}$ …….. (36)

The maximum height h is obtained by

$h = \frac{v_\circ^2\, sin^2\alpha}{2g}$ …….. (37)

The time required for the projectile to return to ground level at B, called the ***time of flight***, by making $y = 0$ in eq.(35)

$t = \frac{2\, v_\circ \sin\alpha}{g}$ (the time is twice the value given by eq.(36))

The range R = OB

$R = v_{\circ x}\,\frac{2v_\circ\, sin\alpha}{g} = \frac{2v_\circ^2\, sin\alpha\ cos\alpha}{g}$ or

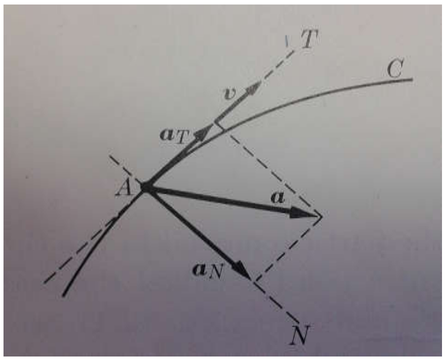
$R = \frac{v_\circ^2\, sin2\alpha}{g}$ …….. (38)

The range is a max. for $\alpha = 45^\circ$

Example: A gun fires a bullet with a velocity of 200 m/s at an angle of 40° with the ground. Find the velocity and position of the bullet after 20s. Also find the range and the time required for the bullet to return to ground.

Tangential and Normal Components of Acceleration

At time t the particle is at A with velocity v and acceleration $\boldsymbol{a}$. Since $\boldsymbol{a}$ is pointing toward the concave side of the path, we may decompose it into a tangential component $\boldsymbol{a_T}$ parallel to the tangent AT (tangential acceleration) and a normal component $\boldsymbol{a_N}$ parallel to the normal AN (normal acceleration).



Change in magnitude of velocity is related to $\boldsymbol{a_T}$

Change in direction of velocity is related to $\boldsymbol{a_N}$

$$\boldsymbol{a} = \boldsymbol{u_T}\frac{dv}{dt} + \boldsymbol{u_N}\frac{\boldsymbol{v^2}}{\boldsymbol{\rho}} \qquad \text{........ (39)}$$

ρ: the radius of curvature

$$a_T = \frac{dv}{dt} \quad , \quad a_N = \frac{\boldsymbol{v^2}}{\boldsymbol{\rho}} \qquad \text{........ (40)}$$

The magnitude of the acceleration at point A is:

$$a = \sqrt{a_T^2 + a_N^2} \qquad \text{........ (41)}$$

If the curvilinear motion is ***uniform*** (magnitude of v const.)

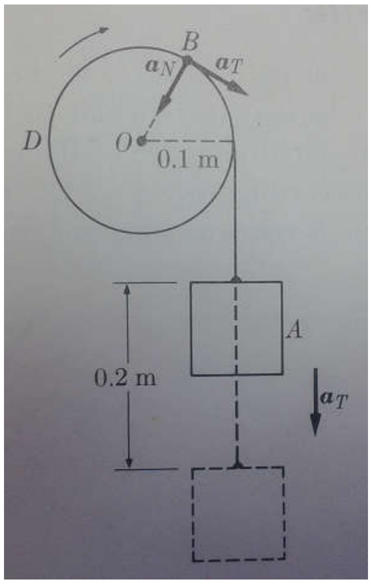
v = constant ➡ $\boldsymbol{a_T} = 0$

If the motion is rectilinear (the direction of v does not change)

ρ = infinity ➡ $\boldsymbol{a_N} = 0$

Example A disk D is rotating freely about its horizontal axis. A cord is wrapped around the outer circumference of the disk, and a body A, attached to the cord, falls under the action of gravity. The motion of A is 0.04 m/s, and 2s later A has fallen 0.2 m. Find the tangential and normal accelerations, at any instant of any point on the rim of the disk.

Solution



$x = v_\circ\, t + \frac{1}{2}\, a\, t^{2}$, $v_\circ = 0.04$ m/s

$x = 0.04\, t + \frac{1}{2}\, a\, t^{2}$ m

$t = 2$s, $x = 0.2$m ➡ $a = 0.06$ m/s^2

$x = 0.04\, t + 0.03t^{2}$ m

The velocity at A

$v = \frac{dx}{dt} = 0.04 + 0.06t$ m/s

$a_T = \frac{dv}{dt} = 0.06$ m/s^2

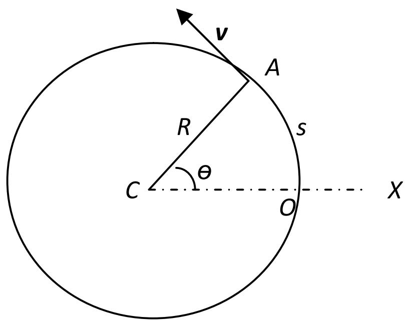
$\rho = 0.1$ m

$a_N = \frac{\boldsymbol{v^2}}{\boldsymbol{\rho}} = \frac{(0.04+0.06t)^2}{0.1} = 0.016 + 0.048t + 0.036t^2$ m/s^2

The total acceleration of point B is $a = \sqrt{a_T^2 + a_N^2}$

Circular Motion: Angular Velocity

For circular motion, the velocity $\boldsymbol{v}$ being tangent to the circle, is perpendicular to the radius R = CA.



$s = R\theta$

$v = \frac{ds}{dt} = R\,\frac{d\theta}{dt}$ …….. (42)

$\omega = \frac{d\theta}{dt}$ …….. (43)

is called the ***angular velocity*** and is equal to the time rate of change of the angle. (rad s^{-1} or simply s^{-1}).

$v = \omega R$ ……. (44)

In the case of ***uniform circular motion*** (ω = constant) ➡ the motion is periodic.

The period P is the time required for a complete turn or revolution and the frequency γ is the number of revolution per unit time.

$P = \frac{t}{n} = \frac{time}{no.of\ revolution}$ (s)

$\gamma = \frac{n}{t}$ (Hz) or (s^{-1}) ➡ $\gamma = \frac{1}{P}$ ……. (45)

$\theta = \omega t$ or $\omega = \frac{\theta}{t}$ ……. (46)

For a complete revolution, $t = P$, $\theta = 2\pi$

$\omega = \frac{2\pi}{P} = 2\pi\gamma$ ……. (47)

<u>Circular Motion: Angular Acceleration</u>

The angular acceleration is defined by

$\alpha = \frac{d\omega}{dt}$ ……. (48)

$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ ……. (49)

$a_T = \frac{dv}{dt} = R\frac{d\omega}{dt} = R\alpha$ ……. (50) (tangential acceleration)

$a_N = \frac{v^2}{R} = \omega^2 R$ ……. (51) (normal acceleration)

In uniform circular motion (α= 0) there is no a_T, but there is still a_N due to the change in the direction of the velocity.

The total acceleration is given by $a = a_T + a_N$

The magnitude of the total acceleration is given by $a = \sqrt{a_T^2 + a_N^2}$

Chapter (3)

Force and Momentum

The law of inertia

- A free particle is one that is not subject to any interaction.

There are some particles which may be considered free, either because:

- ✓ They are sufficiently far away from others for their interactions to be negligible
- ✓ The interactions with the other particles cancel, giving a zero net interaction.

The law of inertia states that ***"a free particle always moves with constant velocity, or without acceleration"***. That is, a free particle either moves in a straight line with constant speed or is at rest (zero velocity). This statement is also called ***"Newton's first law"***.

We assume that the motion of the particle is relative to an observer who is himself a free particle or system, who is not subject to interactions with the rest of the world. Such an observer is called an *inertial observer* and the frame of reference he uses is called an *inertial frame of reference*.

We assume that inertial frames of reference are not rotating, because the existence of rotations would imply that there are accelerations (or changes in velocity due to changes in direction) and therefore that there are interactions.

Linear momentum

The mass it is a number we attach to each particle or body and that it is obtained by comparing the body with a standard body, using the principle of an equal arm balance. The linear momentum of a particle is defined as the product of its mass and its velocity.

$$\boldsymbol{P} = m\boldsymbol{v} \qquad \ldots\ldots\ldots\ (1) \qquad (\text{m Kg s}^{1})$$

Linear momentum is a vector quantity, and it has the same direction as the velocity.

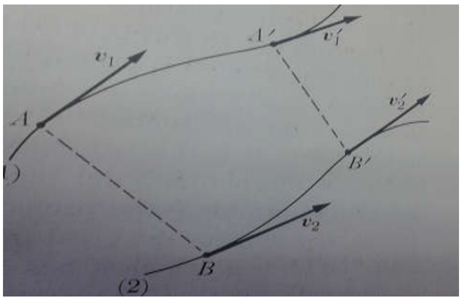
We can restate the law of inertia by saying that ***"a free particle always moves with constant momentum".***

Principle of Conservation of Momentum

At (t) $\boldsymbol{P} = \boldsymbol{P_1} + \boldsymbol{P_2} = m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2}$ ………. (2)

At (t') $\boldsymbol{P}' = \boldsymbol{P_1}' + \boldsymbol{P_2}' = m_1\boldsymbol{v_1}' + m_2\boldsymbol{v_2}'$ ………. (3)

(we assume that the masses of the particles are independent of their states of motion).



$\boldsymbol{P} = \boldsymbol{P}'$ (the total momentum of a system composed of two particles which are subject only to their mutual interaction remains constant). This result constitutes the principle of the conservation of momentum, or ***"the total momentum of an isolated system of particles is constant"***.

$\boldsymbol{P} = \sum_i \boldsymbol{P_i} = \boldsymbol{P_1} + \boldsymbol{P_2} + \boldsymbol{P_3} + \ldots\ldots = \text{const.}$ ………. (4)

For the case of two particles

$\boldsymbol{P_1} + \boldsymbol{P_2} = \text{const.}$ ………. (5)

$\boldsymbol{P_1} + \boldsymbol{P_2} = \boldsymbol{P_1}' + \boldsymbol{P_2}'$ ………. (6)

$\boldsymbol{P_1}' - \boldsymbol{P_1} = \boldsymbol{P_2} - \boldsymbol{P_2}' = -(\boldsymbol{P_2}' - \boldsymbol{P_2})$ ………. (7)

Or $\Delta\boldsymbol{P_1} = -\Delta\boldsymbol{P_2}$ ………. (8)

This result indicates that for two interacting particles, the change in momentum of one particle in a certain time interval is equal and opposite to the change in momentum of the other during the same time interval. ***"an interaction produces an exchange of momentum"***.

Example : A gun whose mass is 0.8 Kg fires a bullet whose mass is 0.016 Kg with a velocity of 700 ms^{-1}. Compute the velocity of the guns recoil.

Solution:

A gun and the bullet are at rest initially and their total momentum is zero. After the explosion the bullet is moving forward with a momentum

$P_1 = m_1v_1 = 0.016 \times 700 = 11.2$ m Kg s^{-1}

The gun must then recoil with an equal but opposite momentum.

$P_2 = 11.2$ m Kg s^{-1} $= m_2 v_2$

$v_2 = \frac{11.2}{0.8} = 14$ m s^{-1}

Redefinition of Mass

Using eq.(1) of momentum, and assuming that the mass of a particle is constant, we can express the change in momentum of the particle in a time Δt as

$\Delta \boldsymbol{p} = \Delta(m\boldsymbol{v}) = m\, \Delta \boldsymbol{v}$

$m_1 \Delta v_1 = - m_2 \Delta v_2$

$\frac{m_2}{m_1} = \frac{|\Delta \boldsymbol{v_1}|}{|\Delta \boldsymbol{v_2}|}$ ………. (9)

which indicates that the ratio of the masses of the particles is inversely proportional to the magnitude of the changes of velocity.

Newton's second and third law

$\Delta t = t^{'} - t$

$\frac{\Delta \boldsymbol{P_1}}{\Delta t} = - \frac{\Delta \boldsymbol{P_2}}{\Delta t}$ ………. (10)

If we make Δt very small

$\frac{d\boldsymbol{P_1}}{dt} = - \frac{d\boldsymbol{P_2}}{dt}$ ………. (11)

So that the instantaneous rates of vector change of momentum of the particles at any time t, are equal and opposite. The force acting on a particle is

$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt}$ ………. (12)

"Force is equal to the time rate of change of momentum of a given particle, which in turn is due to the interaction of the particle with other particles". ***(Newton's second law of motion).***

If the particle is free, P = const. ⟶ $\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt} = 0$, we can say that no force acts on a free particle.

$\boldsymbol{F_1} = - \boldsymbol{F_2}$ ………. (13)

Where $F_1 = dP_1/dt$ is the force on particle 1 due to its interaction with particle 2, and $F_2 = dP_2/dt$ is the force on particle 2 due to its interaction with particle 1. *"when two particles interact, the force on one particle is equal and opposite to the force on the other"* . This is ***(Newton's third law of motion)*** or ***the law of action and reaction***.

$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt} = \frac{d(m\boldsymbol{v})}{dt}$ ……… (14)

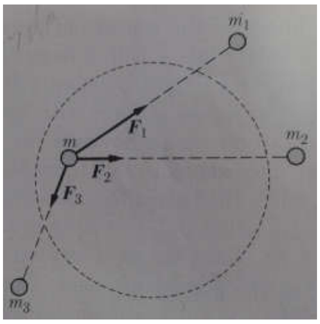
If m is constant

$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a}$ ……… (15)

In this case, the force has the same direction as the acceleration. If the force = const. ➡ $\boldsymbol{a} = \frac{\boldsymbol{F}}{m}$ is also const. and the motion is uniformly accelerated.

The force of gravitational attraction of the earth, called weight, is

$\boldsymbol{W} = mg$ ……… (16)



If particle m interacts with particles m_1, m_2, m_3, …..,
each one produces a change in the momentum of m
that is characterized by the respective forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$,…. The total rate of change of momentum of particle m is

$\frac{d\boldsymbol{P}}{dt} = \boldsymbol{F_1} + \boldsymbol{F_2} + \boldsymbol{F_3} + \ldots = \boldsymbol{F}$ (resultant force)

<u>Units of Force</u>

In MKSC system the force is measured in (m kg s^{-2}), a unit that called a *Newton* (N) which defined as the force that is applied to a body whose mass is one kg to produce an acceleration of 1 m s^{-2}.

dyne = cm g s^{-2}, N = 10^5 dynes

<u>Example (1)</u>: A car whose mass is 1000 kg moves uphill along a street inclined 20°. Determine the force which the motor must produce if the car is to move **a)** with uniform motion, **b)** with an acceleration of 0.2 m s^{-2}. Find also in each case the force exerted on the car by the street.

Solution: **a)** $F - mg \sin\alpha = ma$ or $F = m(a + g \sin\alpha)$

$N - mg \cos\alpha = 0 \longrightarrow N = mg \cos\alpha = 9210$ N

For uniform motion $a = 0$, $\longrightarrow F = mg \sin\alpha = 3350$ N

b) For $a = 0.2$ m s^{-2} **,** $F = 3550$ N

Example (2): A particle of mass 10 kg, subject to a force $F = (120\,t + 40)$N, moves in a straight line. At time $t = 0$ the particle is at $x_\circ = 5$m, with a velocity $v_\circ = 6$ m s^{-1}. Find its velocity and position at any later time.

Solution:

$F = ma$

$120\,t + 40 = 10\,a \longrightarrow a = (12\,t + 4)$ m s^{-2}

For rectilinear motion $a = dv/dt = 12\,t + 4$

$$\int_6^v dv = \int_0^t (12t + 4)dt$$

$v = (6t^2 + 4t + 6)$ m s^{-1}

$v = dx/dt$

$$\int_5^x dx = \int_0^t v dt = \int_0^t (6t^2 + 4t + 6)dt$$

$x = (2t^3 + 2t^2 + 6t + 5)$ m

Example (3): Determine the acceleration with which the masses m and m' of the figure move . Assume that the wheel can rotate freely around O and disregard any possible effects due to the mass of the wheel.

Solution: Both masses move with the same acceleration $\boldsymbol{a}$.

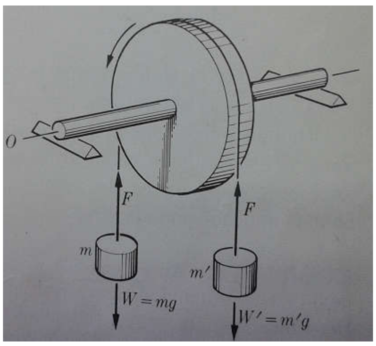
The downward motion of m with $\boldsymbol{a}$ is

$Mg - F = m\boldsymbol{a}$ and the upward motion of m' with the same $\boldsymbol{a}$ is

$F - m'g = m'\boldsymbol{a}$

$$\boldsymbol{a} = \frac{m - m'}{m + m'} g$$

The tension in the string is $F = \dfrac{2m\,m'}{m + m'} g$



Frictional Forces:

Whenever there are two bodies in contact, there is a resistance which opposes the relative motion of the two bodies. The loss of momentum when two bodies are contact is indicate of a force opposing the motion, which called ***sliding friction***. It is due to the interaction between the molecules of the two bodies, referred to as ***cohesion*** or ***adhesion*** depending on whether the two bodies are of the same or different materials. This phenomenon is depends on the condition and nature of the surface, the relative velocity,……

$\boldsymbol{F}_f$ = sliding friction = $f\,N$ ………. (17)

f : coefficient of friction

N: normal force pressing one body against the other

Unit vector $\boldsymbol{u}_v = \frac{\boldsymbol{v}}{v} = \frac{velocity\ vector}{velocity\ magnitude}$

$\boldsymbol{F}_f = -\,\boldsymbol{u}_v\, f\, N$

$m\boldsymbol{a} = \boldsymbol{F} - \boldsymbol{u}_v\, f\, N$

There are two kinds of coefficient of friction:

1) *Static coefficient of friction* (f_s) when multiplied by the normal force, gives the minimum force required to set in relative motion two bodies that are initially in contact and at relative rest.
2) *Kinetic coefficient of friction* (f_k) when multiplied by the normal force, gives the force required to maintain the two bodies in uniform relative motion.

$f_s > f_k$

If $N = W = mg \Longrightarrow ma = F - fmg$, or $F = m\,(a + fg)$

a: acceleration of the body. (***a***=0 for uniform motion), $F = fN$

Example: A body whose mass is 0.8kg is on a plane inclined 30°. What force must be applied on the body so that it moves ***a)*** uphill and ***b)*** downhill? In both cases assume that the body moves with uniform motion and with an acceleration of 0.1 ms^{-2}. The coefficient of sliding friction with the plane is 0.3.

Frictional Forces in Fluids

F_f = fluid friction = $-k\eta v$ ………. (18)

The coefficient k depends on the shape of the body, in the case of a sphere of radius R

$K = 6\pi R$ ………. (19) a relation known as ***Stokes law***.

η: coefficient depends on the internal friction of the fluid, is also called ***viscosity*** and η is called the ***coefficient of viscosity*** (N s m^{-2}) or (m^{-1} kg s^{-1}) or (cm^{-1} g s^{-1}), called ***poise***.

$1\ m^{-1}\ kg\ s^{-1} = (10^2\ cm)^{-1}\ (10^3\ g)\ s^{-1} = 10\ cm^{-1}\ g\ s^{-1} = 10P$

η of liquids decreases as temperature increase

η of gases increases with temperature increase

When a body moves through a viscous fluid under action of a force F, the eq. of motion is

$ma = F - k\eta v$ ………. (20)

Assuming the force F constant, ➡ ***a*** =0, and there is no further increase in v, the fluid friction being exactly balanced by the applied force. The particle continues moving in the direction of the force with a constant velocity, called ***limiting or terminal velocity*** (v_L).

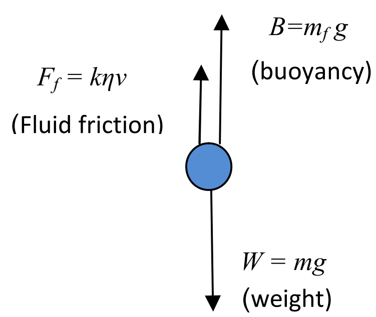
$v_L = \frac{F}{k\eta}$ ………. (21)

In free fall, $F = mg$

$v_L = \frac{mg}{k\eta}$ ………. (22)

If m_f is the mass of fluid displaced by the body,

$B = -m_f g$ (upward buoyant force)



The net downward force will be

$mg - m_f g = (m - m_f)\ g$

$v_L = \frac{(m-m_f)g}{k\eta}$ ………. (23)

Example: Find the limiting velocity of a raindrop. Assume a diameter of 10^{-3}m, the density of air relative to water is 1.3×10^{-3}, $\eta = 1.81\times 10^{-5}$ N s m^{-2} and $\rho=10^3$kg m^{-3}.

Systems with Variable Mass

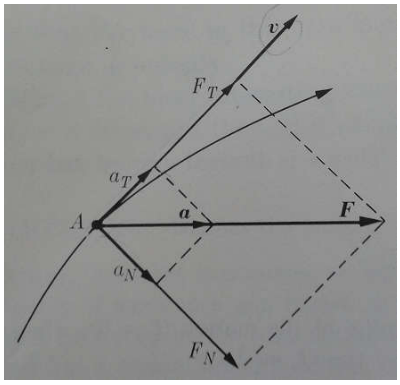
In certain cases the mass is variable. The simplest example is a raindrop, if the mass of the drop is m when it is moving with velocity v, and that moisture, whose velocity is $v_\circ$, condenses on the drop at the rate dm/dt. The total rate of change of momentum is the sum of *(mdv/dt)* corresponding to the acceleration of the drop, and *(dm/dt) (v-v∘)*, corresponding to the rate of the gain of momentum of the moisture. The equation of motion of the drop is

$$F = m\frac{dv}{dt} + \frac{dm}{dt}(v - v_\circ)$$

Curvilinear Motion

If the force has the same direction as the velocity, the motion is in a straight line. To produce curvilinear motion, the resultant force must be at an angle with respect to the velocity, so that the acceleration has a component perpendicular to the velocity which will account for the change in the direction of the motion. If the mass is constant, the force is parallel to the acceleration.

$F = ma$



$F_T = m\,a_T$ or $F_T = m\frac{dv}{dt}$ ………. (24)

F_T: tangential force

The normal or centripetal force

$F_N = m\,a_N$ or $F_N = \frac{mv^2}{\rho}$ ………. (25)

ρ : is the radius of curvature of the path.

The F_N is always pointing toward the center of curvature of the trajectory. The F_T is responsible for the change in the magnitude of the velocity, and the F_N is responsible for the change in the direction of the velocity.

If $F_T = 0$ $a_T = 0$ (the motion is uniform circular motion)

If $F_N = 0$ $a_N = 0$ (the motion is rectilinear)

If the particle case of circular motion, ρ is the radius R of the circle and $v = wR$

$F_N = mw^2R$ ………. (26)

For uniform circular motion the only acceleration is a_N, which can be written in vector form

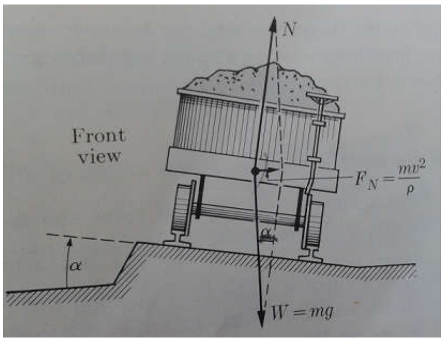
$a = w \times v$

$F = ma = mw \times v = w \times (mv)$

$P = mv$

$F = w \times P$ ………. (27)

The railroad tracks and highways are banked at curves to produce F_N required by a vehicle moving along a curve. Their resultant F_N must be enough to produce the F_N.



$F_N = \frac{mv^2}{\rho}$, from the figure

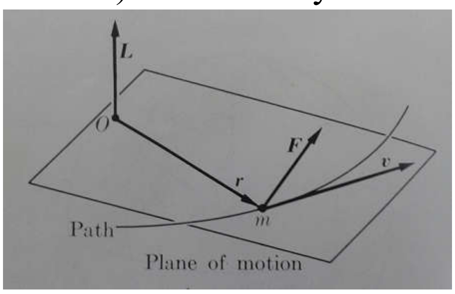
$\tan\alpha = \frac{F_N}{W} = \frac{v^2}{\rho g}$ ………. (28)

α : the angle of banking

Example: A mass suspended from a fixed point by a string of length L is made to rotate around the vertical with angular velocity *w*. Find the angle of the string with the vertical. This arrangement is called a conical pendulum.

Angular Momentum

The angular momentum around point *O* of a particle of mass *m* moving with velocity *v* (and therefore having momentum $P = mv$) is defined by the vector product



$$\left.\begin{aligned} L &= r \times p \\ L &= m\, r \times v \end{aligned}\right\} \quad ……….\ (29)$$

L is a vector ⊥ to the plane determined by *r* and *v*. The ***L*** of the particle in general changes in magnitude and direction while the particle moves.

If a particle moves in a plane, and the point *O* lies in the plane, the direction of *L* remains the same, that is ⊥ to the plane, since both *r* and *v* are in the plane.

In the case of circular motion, when *O* is the center of the circle, the vectors r and v are ⊥, and $v = wr$,

$\boldsymbol{L} = mrv = mr^2w$ ………. (30)

The direction of ***L*** is the same as that of *w*

$\boldsymbol{L} = mr^2w$ ………. (31)

If the plane motion is not circular, but curvilinear, the relation will be more difficult.

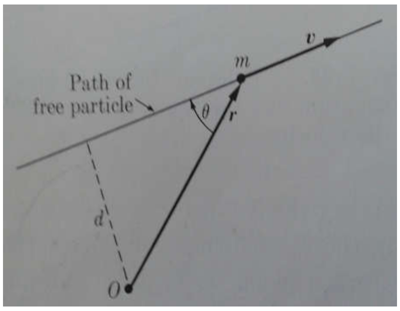
The time rate of change of the angular momentum of a particle is equal to the torque of the force applied to it.

$\frac{dL}{dt} = \boldsymbol{T}$ ………. (32)

This equation is correct only if ***L*** and ***T*** are measured relative to the same point.

Central Forces

If the torque on a particle is zero ($\boldsymbol{T} = r \times F = 0$), dL/dt =0 or ***L*** = constant vector. This condition is fulfilled if $F = 0$ (the particle is free).



$L = mvr \sin\theta = mvd$

$d = r \sin\theta$

This quantity remains a constant because all factors involved are also constant, since the path of the free particle is in a straight line and the velocity does not change.

$r \times F = 0$ is also fulfilled if *F* is parallel to r, if the direction of *F* passes through the point *O*, this force is called a *central force*.

Therefore, when a body moves under the action of a central force its angular momentum remains constant, ***"When the force is central, the angular momentum relative to the center of force is a constant of motion and conversely"***.

Chapter (4)

Work and Energy

Work:

$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt}$, $\int_{P_\circ}^{P} d\boldsymbol{p} = \int_{t_\circ}^{t} \boldsymbol{F} dt$

$\boldsymbol{p} - \boldsymbol{p}_\circ = \int_{t_\circ}^{t} \boldsymbol{F} dt = \boldsymbol{I}$ ………. (1)

$\boldsymbol{I} = \int_{t_\circ}^{t} \boldsymbol{F} dt$,is called the impulse. "the change in momentum of the particle is equal to the impulse"

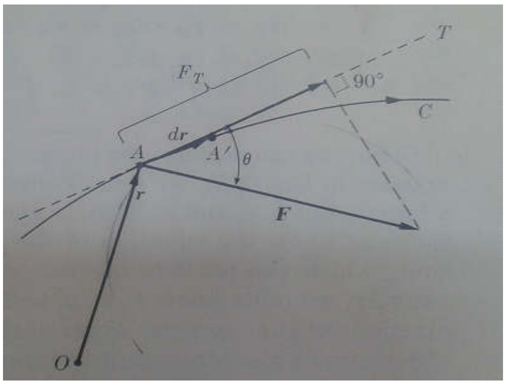
$\boldsymbol{p} = mv$

$mv - mv_\circ = \boldsymbol{I}$, or $v = v_\circ + \frac{1}{m}\boldsymbol{I}$

$v = \frac{dr}{dt}$

$\int_{r_\circ}^{r} d\boldsymbol{r} = \int_{t_\circ}^{t} (v_\circ + \frac{1}{m}\boldsymbol{I})\, dt$ or $\boldsymbol{r} = \boldsymbol{r}_\circ + v_\circ\, t + \frac{1}{m} \int_{t_\circ}^{t} \boldsymbol{I} dt$

If A particle moving along a curve C under the action of a force $\boldsymbol{F}$, the displacement $\overrightarrow{AA'}$ = d$\boldsymbol{r}$. The work done by the force $\boldsymbol{F}$ during that displacement is defined by the scalar product



$dW = \boldsymbol{F}.\boldsymbol{dr}$ ………. (2)

The magnitude of $d\boldsymbol{r}$ is ds

$dW = Fds\cos\theta$ ………. (3)

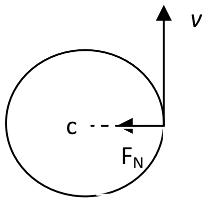
θ: is the angle between the direction of the force $\boldsymbol{F}$ and the displacement $\boldsymbol{dr}$.

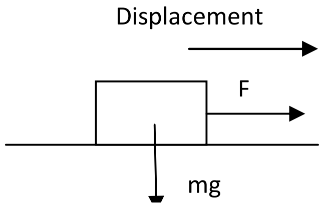
$F\cos\theta$: is the component F_T of the force along the tangent to the path

$dW = F_T\, ds$ ………. (4)

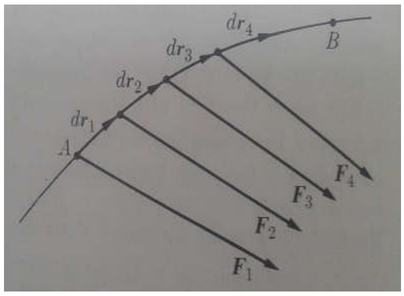
"work is equal to the displacement times the component of the force along the displacement".

If a force is $\perp$ to the displacement ($\theta = 90^\circ$), the work done by the force is zero.





The total work done on the particle when moving from A to B is the sum of all the infinitesimal works done during successive displacements.



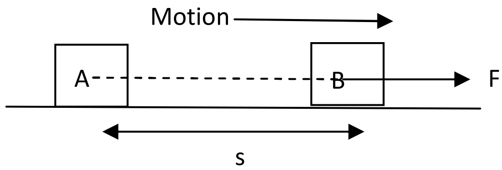
$W = \boldsymbol{F}_1 \cdot \mathrm{d}\boldsymbol{r}_1 + \boldsymbol{F}_2 \cdot \mathrm{d}\boldsymbol{r}_2 + \boldsymbol{F}_3 \cdot \mathrm{d}\boldsymbol{r}_3 + \ldots$

or

$$W = \int_A^B \boldsymbol{F} . d\boldsymbol{r} = \int_A^B F_T \; ds \qquad \ldots\ldots\ldots. (5)$$

When the force is constant in magnitude and direction and the body moves in a straight line in the direction of the force. Then $F_T = F$

$$W = \int_A^B F . ds = F \int_A^B ds = F\, s \qquad \ldots\ldots\ldots. (6)$$

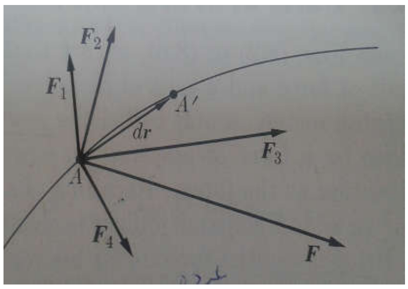


work = force × distance

If F_x, F_y and F_z are the rectangular components of $\boldsymbol{F}$ and dx, dy and dz are the rectangular components of $d\boldsymbol{r}$,

$$W = \int_A^B (F_x \; dx + \; F_y \, dy + \; F_z \, dz) \ldots\ldots\ldots. (7)$$

When the particle is acted on by several forces $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$, …., the work done by each force during the displacement $\overrightarrow{AA'} = d\boldsymbol{r}$, is $dW_1 = \boldsymbol{F}_1 . d\boldsymbol{r}$, $dW_2 = \boldsymbol{F}_2 . d\boldsymbol{r}$, $dW_3 = \boldsymbol{F}_3 . d\boldsymbol{r}$



The total work

$dW = dW_1 + dW_2 + dW_3 + \ldots\ldots$

$dW = \boldsymbol{F}_1 . d\boldsymbol{r} + \boldsymbol{F}_2 . d\boldsymbol{r} + \boldsymbol{F}_3 . d\boldsymbol{r} + \ldots\ldots$

$\quad = (\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \ldots\ldots) . d\boldsymbol{r}$

$\quad = \boldsymbol{F} . d\boldsymbol{r} \qquad \ldots\ldots\ldots. (8)$

$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \ldots\ldots.$ is the resultant force.

Example: A ball whose mass is 0.1kg is allowed to fall from a height of 2m and after hitting the floor, it bounce back up to a height of 1.8m. Determine the impulse it received from gravity while it was falling and the impulse it received when it struck the floor.

Solution:

$v^2 = v_\circ^2 + 2a\,(x - x_\circ)$, $x_\circ = v_\circ = 0,\ a = g,\ x = h_1 = 2m$

$v_1 = \sqrt{2gh_1} = 6.26$m/s the velocity of the particle when it arrives at the floor.

$v_1 = -\boldsymbol{u}_y\,6.26$m/s (since v_1 is directed down ward)

$p_\circ = m\,v_\circ = 0$

$m\,v_1 - 0 = -\boldsymbol{u}_y\,0.626$ kg m/s impulse due to gravity

$h_2 = 1.8m$, the velocity when the ball bounces back

$v_2 = \sqrt{2gh_2} = 5.94$ m/s, or $v_2 = \boldsymbol{u}_y\,5.94$ m/s (moving upward)

$p_2 - p_1 = m\,v_2 - m\,v_1 = \boldsymbol{u}_y 1.221$ kg m/s (impulse)

Power

The instantaneous power is defined by

$P = \frac{dW}{dt}$ ………. (9) " *the work done per unit time during a very small time interval dt*"

$P = \boldsymbol{F}.\,\frac{d\boldsymbol{r}}{dt} = \boldsymbol{F}\,\boldsymbol{.}\,\boldsymbol{v}$ ………. (10) *"force times velocity"*

The average power during a time interval *t* is given by $P_{ave} = \frac{W}{t}$

Units of Work and Power

In MKSC system, work is expressed in (N m) it called joule (J). One joule is the work done by a force of one Newton when it moves a particle one meter in the same direction as the force. N = m kg s^{-2}, J = N m = m^2 kg s^{-2}.

In the (cgs) system, work is expressed in (dynes cm) it called an erg.

1N = 10^5 dyn, 1m = 10^2 cm, 1J = 10^5 dyn × 10^2 cm = 10^7 ergs

In British system, the unit of work is *foot-pound* (ft-Ib). In MKSC system power is expressed in J/s called watt (W). One watt is the power of a machine that does work at the rate of one joule every second.

1 KW = 10^3 W, 1 MW = 10^6 W

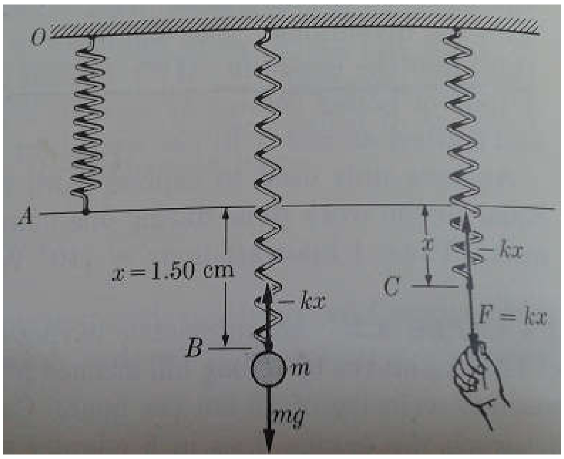
A unit of power is the horsepower (hp) = 550 ft-Ib per s = 746 W

Another unit of work is Kilowatt-hour

1 Kilowatt-hour = (10^3 W) (3.6×10^3 s) = 3.6×10^6 J

Example: A car having a mass of 1200kg moves up along hill inclined 5° with a constant velocity of 36 km per hour. Calculate the work the engine does in 5 minutes and the power developed by it. Neglect all frictional effects.

Example: Calculate the work required to expand the spring a distance of 2 cm without acceleration. It is known that when a body whose mass is 4 kg is hung from the spring, the springs length increases by 1.5 cm.



Solution:

$F = kx$

Weight mg = 39.2 N, $x = 1.5 \times 10^{-2}$ m

$mg = kx$, $k = \frac{39.2}{1.5 \times 10^{-2}} = 2.61 \times 10^3$ N m^{-1}

$W = \int_0^x F\,dx = \int_0^x kx\,dx = \frac{1}{2} k x^2$

For x = 2 cm, $W = 5.22 \times 10^{-1}$ J

Example: A force $F = 6t$ N acts on a particle whose mass is 2 kg. If the particle starts from rest, find the work done by the force during the first 2s.

Solution:

$F = ma$, $a = \frac{F}{m} = 3t$ m s^{-2}

$v = v_\circ + \int_{t_\circ}^{t} a dt$, $v_\circ = 0$

$v = \int_0^t a dt = \int_0^t 3t dt = 1.5\, t^2$ m s^{-1}

$x = x_\circ + \int_{t_\circ}^{t} v dt$, $x_\circ = 0$

$x = \int_0^t 1.5t^2 \, dt = 0.5\, t^3$ m

To find the work

a) $t = (x/0.5)^{1/3} = 1.26\, x^{1/3}$, $F = 6t = 7.56\, x^{1/3}$ N

$W = \int_0^x 7.56\, x^{1/3} \, dx = 5.67\, x^{4/3}$

For t = 2s , $x = 0.5\,(2)^3$ = 4m, W= 36 J

or ***b)*** $x = 0.5\, t^3$, $dx = 1.5\, t^2\, dt$, W = $\int F dx$

$W = \int_0^t (6t)(1.5\, t^2\, dt) = 2.25\, t^4$ J

For t = 2s, W = 36 J

Kinetic Energy

$F_T = m \frac{dv}{dt}$

$F_T\, ds = m \frac{dv}{dt} ds = m\, dv\, \frac{ds}{dt} = mv\, dv$

$W = \int_A^B F_T \, ds = \int_A^B mv\, dv = \frac{1}{2} m\, v_B^2 - \frac{1}{2} m\, v_A^2$ ………. (11)

The kinetic energy given by

$E_k = \frac{1}{2} m\, v^2$ or $E_k = \frac{p^2}{2m}$ ………. (12)

$W = E_{k,B} - E_{k,A}$ ………. (13) (J or ergs) or eV

"the work done on a particle is equal to the change in its kinetic energy"

electron volt (eV) = 1.6021×10^{-19} J

1 MeV = 10^6 eV

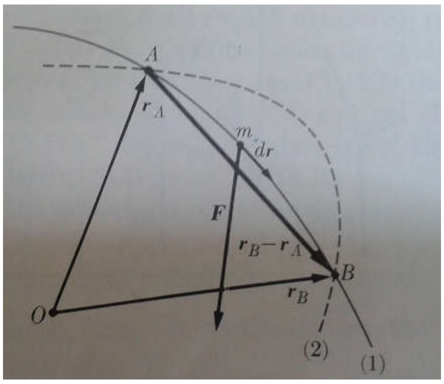
The difference between the impulse I and the work W is

$I = \int_{t_\circ}^{t} F dt$ (it is a time integral, we must know the force as a function of time)

But $W = \int_A^B F_T \, ds$ (work it is a space integral, the force function of position)

Work of a Force Constant in Magnitude and Direction

Consider a particle m moving under the action of a force $\boldsymbol{F}$ which is constant in magnitude and direction. The work of $\boldsymbol{F}$ when a particle moves from A to B is: (path 1)



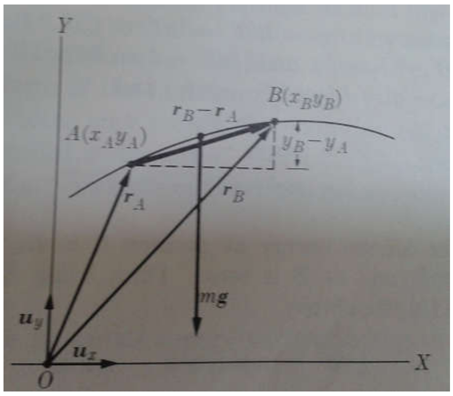
$W = \int_A^B F.dr = F.\int_A^B dr = F.(r_B - r_A)$ ………. (14)

The work in this case is independent of the path joining points A and B.

If the particle moves along path (2), the work will be the same because the vector difference $r_B - r_A = \overrightarrow{AB}$ is still the same.

$W = F \,.\, r_B$ - $F \,.\, r_A$ ………. (15)

In the case of the work done by the force of gravity



$F = mg = -\boldsymbol{u}_y\, mg$, $r = \boldsymbol{u}_x x + \boldsymbol{u}_y y$

$r_B - r_A = \boldsymbol{u}_x\,(x_B - x_A) + \boldsymbol{u}_y\,(y_B - y_A)$

$W = -mg\,(y_B - y_A) = mg\,y_A - mg\,y_B$ ………. (16)

The work independent on the path, and depends only on the difference $y_B - y_A$.

Potential Energy

A force is conservative if its dependence on the position vector $\boldsymbol{r}$ or on the coordinates x, y, z of the particle is such that the work W can always be expressed as the difference between a quantity $E_p(x, y, z)$ evaluated at the initial and at the final points.

The quantity $E_p(x, y, z)$ is called the potential energy and is a function of the coordinates of the particles.

If $\boldsymbol{F}$ is a conservative force

$W = \int_A^B F.dr = E_{p,A} - E_{p,B}$ ………. (17)

"Potential energy is a function of the coordinates such that the difference between its value at the initial and final position is equal to the work done on the particle to move it from the initial to the final position"

For example, the force of gravity is conservative and the potential energy due to gravity is:

$E_p = mgy$ ………. (18)

Similarly, the potential energy corresponding to a constant force is

$E_p = - F \, . \, r$ ………. (19)

"The work done by conservative forces is independent of the path"

If the path is closed, the final point coincides with the initial point (*A* and *B* are the same point), then $E_{p,A} = E_{p,B}$, and the work is zero ($W = 0$)

$W_o = \oint F . dr = 0$ ………. (20)

$F \, . \, dr = Fds \cos\theta$ ………. (21)

θ: is the angle between the force and the displacement

$F \cos\theta = - \frac{dE_p}{ds}$ ………. (22)

$F \cos\theta$ is the component of the force in the direction of the displacement *ds*.

When a vector is such that its component in any direction is equal to the directional derivative of a function in that direction, the vector is called the *gradient* of the function.

$F = - \text{grad}\, E_p$

$F_x = - \frac{\partial E_p}{\partial x}$, $F_y = - \frac{\partial E_p}{\partial y}$, $F_z = - \frac{\partial E_p}{\partial z}$ ………. (23)

$F = - \text{grad}\, E_p = - \boldsymbol{u}_x \frac{\partial E_p}{\partial x} - \boldsymbol{u}_y \frac{\partial E_p}{\partial y} - \boldsymbol{u}_z \frac{\partial E_p}{\partial z}$ ………. (24)

The potential energy $E_p(x, y, z)$ is a function of all three variables *x, y,* and *z*.

Conservation of energy of a particle

When a force acting on a particle is conservative, $E_{K,B} - E_{K,A} = E_{p,A} - E_{p,B}$ or

$(E_K + E_p)_B = (E_K + E_p)_A$ ………. (25)

$(E_K + E_p)$ is called the *total energy* of the particle (E), the total energy of a particle is equal to the sum of its kinetic energy and its potential energy.

$E = E_K + E_p = \frac{1}{2} mv^2 + E_p(x,y,z)$ ………. (26)

"When the forces are conservative the total energy E of the particle remains constant"

$E = E_K + E_p = const.$ ………. (27)

"the energy of the particle is conserved"

In the case of a falling body, $E_p = mgy,$ the conservation of energy gives:

$E = \frac{1}{2} mv^2 + mgy =$ const. ………. (28)

If initially the particle is at height $y_\circ$ and its velocity is zero,

the total energy = $\frac{1}{2} mv^2 + mgy = mgy_\circ$,

$v^2 = 2g(y_\circ - y) = 2gh, \; h = y_\circ - y$

Nonconservative Forces

There are some forces are not conservative, such as sliding friction which is always opposite the displacement. Similarly, fluid friction opposes the velocity, and depends on velocity but not on position. For example a particle falling through a fluid. Calling E_p the potential energy corresponding to the conservative forces and W' the work done by the non conservative forces, the total work done on the particle when moving from A to B is $W = E_{p,A} - E_{p,B} + W'$

$E_{K,B} - E_{K,A} = E_{p,A} - E_{p,B} + W'$

$(E_K + E_p)_B - (E_K + E_p)_A = W'$ ………. (30)

In this case $(E_K + E_p)$ does not remain constant but decreases if W' is negative and increases if W' is positive.

Chapter (5)

Oscillatory Motion

Kinematics of Simple Harmonic Motion (SHM):

A particle moving along the X- axis has simple harmonic motion when its displacement x relative to the origin of the coordinate system is given as a function of time by the relation:

$x = A \sin(\omega t + \alpha)$ ....(1)

$(\omega t + \alpha)$: is the phase, α: is the initial phase (its value for $t = 0$).

The displacement of the particle varies between $x = -A$ and $x = +A$ (the sine or cosine function (-1 to 1)).

A: maximum displacement from the origin (amplitude).

SHM is periodic, and its periodic is $P = \frac{2\pi}{\omega}$.

The frequency γ is equal to the number of complete oscillations per unit time. $\gamma = \frac{1}{P}$

ω: angular frequency

$\omega = \frac{2\pi}{P} = 2\pi\gamma$ ....(2)

$v = \frac{dx}{dt} = \omega A \cos(\omega t + \alpha)$ ....(3)

$\alpha = \frac{dv}{dt} = -\omega^2 A \sin(\omega t + \alpha) = -\omega^2 x$ ....(4)

Example: Discuss the motion of a particle of mass m on which an oscillating force $F = F_\circ \sin\omega t$ is acting.

Force and Energy in SHM

Applying the equation of motion $F = ma$, we have

$F = - m\,\omega^2 x = - kx$ …………(5)

$k = m\,\omega^2$ or $\omega = \sqrt{k/m}$ …………(6)

This indicates that in SHM the force is proportional to the displacement, and opposed to it. Thus the force is always pointing toward the origin O. This is the point of equilibrium. The force given by eq. (5) is the type of force that appears when one deforms an elastic body such as a spring, the constant $k = m\,\omega^2$ sometimes called the elastic constant, represents the force required to displace the particle one unit of distance.

$P = 2\pi \sqrt{\frac{m}{k}}$, $\gamma = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ …………(7)

Which express the period and frequency of a SHM.

The kinetic energy of the particle is

$E_k = \frac{1}{2} mv^2 = \frac{1}{2} m\,\omega^2 A^2 \cos^2 (\omega t + \alpha)$ …………(8)

$\cos^2\theta = 1 - \sin^2\theta$

$E_k = \frac{1}{2} m\,\omega^2 A^2 [\,1 - \sin^2 (\omega t + \alpha)\,] = \frac{1}{2} m\,\omega^2 (A^2 - x^2)$ …………(9)

E_k is maximum at the center ($x = 0$) and zero at the extremes of oscillation ($x = \pm A$).

$F = - \frac{dE_p}{dx}$, where E_p : the potential energy

$dE_p / dx = kx$

$\int_0^{E_p} dE_p = \int_0^{x} kx\,dx$

$E_p = \frac{1}{2} k x^2 = \frac{1}{2} m\,\omega^2 x^2$ …………(10)

The total energy of the SHM

$E = E_k + E_p = \frac{1}{2} m\,\omega^2 A^2 = \frac{1}{2} k A^2$ …………(11)

Which is a constant quantity, since the force is conservative.

This figure show E_p represented by a parabola.

Basic equation of SHM

$\boldsymbol{F} = ma = - kx$

In rectilinear motion $\boldsymbol{a} = d^2x / dt^2$

$m\frac{d^2x}{dt^2} = - kx$ or $m\frac{d^2x}{dt^2} + kx = 0$

$\omega^2 = k / m$

$\frac{d^2x}{dt^2} + \omega^2 x = 0$ …………(12)

This is a differential eq. whose solution are known to be sine or cosine functions of ωt.

$x = A \sin (\omega t + \alpha)$ is the general solution of eq. (12).

Therefore, we verify that an attractive force proportional to the displacement produces SHM.

The Simple Pendulum

The particle moves in an arc of a circle of radius $l = OA$. The tangential component of the resultant force from the figure is:

$F_T = - mg \sin\theta$

(-) because it is opposed to the displacement $S = CA$

$F_T = m\, a_T$

$a_T = l\, d^2\theta / dt^2$ tangential acceleration

$ml\frac{d^2\theta}{dt^2} = - mg \sin\theta$ *or*

$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$ …………(13) (the eq. of the tangential motion)

If the angle θ is small, (amplitude of the oscillation is small), $\sin\theta \sim \theta$

$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$

The angular motion of the pendulum is SHM with $\omega^2 = g / l$.

The angle $\theta = \theta_\circ \sin(\omega t + \alpha)$

$P = 2\pi / \omega = \; 2\pi\sqrt{\frac{l}{g}}$ ………….(14)

The general formula for the period is

$P = 2\pi\sqrt{l/g}\,(1+\frac{1}{4}\, sin^2\, \frac{1}{2}\, \theta_\circ + \frac{9}{64}\, sin^4\, \frac{1}{2}\, \theta_\circ + \ldots\ldots)$

θ: the amplitude in radians

$P_\circ = 2\pi\sqrt{l/g}$ (for very small amplitude)

$P = 2\pi\sqrt{l/g}\,(1 + \frac{1}{16}\, \theta_\circ^2)$ (for small amplitude $\sin\frac{\theta_\circ}{2} \sim \frac{\theta_\circ}{2}$)

Superposition of Two SHM: Same Direction, Same Frequency:

The displacement of the particle produced by each SHM is given by:

$x_1 = A_1 \sin(\omega t + \alpha_1) = OP_1$

$x_2 = A_2 \sin(\omega t + \alpha_2) = OP_2$

$x = x_1 + x_2 = OP$ the result

$= A_1 \sin(\omega t + \alpha_1) + A_2 \sin(\omega t + \alpha_2)$

$x = OP = A \sin(\omega t + \alpha)$ ………….(15)

$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$ ………….(16)

$\delta = \alpha_2 - \alpha_1$ (the angle between $\overrightarrow{OP1'}$ and $\overrightarrow{OP2'}$)

ω: angular frequency

α: initial phase

$A \cos\alpha = A_1 \cos\alpha_1 + A_2 \cos\alpha_2$

$A \sin\alpha = A_1 \sin\alpha_1 + A_2 \sin\alpha_2$

$tan\ \alpha = \frac{A1\ sin\ \alpha 1\ +\ A2\ sin\ \alpha 2}{A1\ cos\ \alpha 1\ +\ A2\ cos\ \alpha 2}$ ………….(17)

- ❖ If $\alpha_2 = \alpha_1$, then $\delta = 0$ (the two motions are in ***phase*** and their rotating vectors are parallel)

$A = A_1 + A_2$, $\alpha = \alpha_1$ …………(18)

- ❖ If $\alpha_2 = \alpha_1 + \pi$, then $\delta = \pi$ (the two SHM are in ***opposition*** and their rotating vectors are anti parallel and interfere by attenuation)

- If $A_1 > A_2$
  $A = A_1 - A_2$, $\alpha = \alpha_1$ …………(19)
- If $A_1 = A_2$, (the two SHM completely ***cancel*** each other)

- ❖ If $\alpha_2 = \alpha_1 + \pi/2$, then $\delta = \pi/2$ (the two SHM are in ***quadrature*** and their rotating vectors are ***perpendicular***)

$A = \sqrt{A_1^2 + A_2^2}$ …………(20)

$\alpha = \alpha_1 + arc\ tan \frac{A_2}{A_1}$ …………(21)

Example: A particle is subjected simultaneously to two SHM of the same frequency and direction. Their equations are $x_1 = 10sin\ (2t + \pi/4)$ and $x_2 = 6sin\ (2t + 2\pi/3)$. Find the resultant motion.

Superposition of Two SHM: Same Direction, Different Frequency:

If $\alpha_1 = 0$ and $\alpha_2 = 0$

$x_1 = A_1 \sin \omega_1 t$, $x_2 = A_2 \sin \omega_2 t$ …………(22)

$\omega_1 t - \omega_2 t = (\omega_1 - \omega_2) t$ the angle between the rotating vectors $\overrightarrow{OP1'}$ and $\overrightarrow{OP2'}$ (is not constant), $\overrightarrow{OP'}$ does not have constant length and angular velocity

$x = x_1 + x_2$, is not simple harmonic

$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos(\omega 1 - \omega 2)\, t}$ …………(23)

And it oscillates between the values $A = A_1 + A_2$ [when$(\omega_1 - \omega_2)\, t = 2n\pi$] and $A = /A_1 - A_2/$ [when$(\omega_1 - \omega_2)\, t = 2n\pi + \pi$] the amplitude is modulated. The frequency of the amplitude is:

$\gamma = (\omega_1 - \omega_2) / 2\pi = \gamma_1 - \gamma_2$ ....(24)

If $A_1 = A_2$

$x = x_1 + x_2 = A_1 (\sin \omega_1 t + \sin \omega_2 t)$

$= 2 A_1 \cos \frac{1}{2} (\omega_1 - \omega_2) t \sin \frac{1}{2} (\omega_1 + \omega_2) t$ ....(25)

$\frac{1}{2} (\omega_1 + \omega_2)$ angular frequency

$A = 2A_1 \cos \frac{1}{2} (\omega_1 - \omega_2) t$ ....(26)

<u>Superposition of Two SHM: Perpendicular Directions:</u>

If a particle moves in a plane that its two coordinate *x* and *y* oscillate with SHM. In the case in which the two motions have the same frequency, so that the initial phase for the motion along the *X*- axis is zero.

$x = A \sin \omega t$ ....(27)

$y = B \sin (\omega t + \delta)$ ....(28)

δ: the phase difference between the *x* and *y* oscillations.

We have assumed that the amplitudes *A and B* are different. The path of the particle is limited by the lines $x = \pm A$ and $y = \pm B$.

- If the two motions are in phase, $\delta = 0$ and

$y = B \sin \omega t$

$y = (\frac{B}{A}) x$ ....(29) (eq. of the straight line)

The motion is SH, with amplitude $\sqrt{A^2 + B^2}$, and the displacement along the line

$r = \sqrt{x^2 + y^2} = \sqrt{A^2 + B^2} \sin \omega t$ ....(30)

- If the two motions are in opposition, $\delta = \pi$ and

$y = - B \sin \omega t$

$y = - (\frac{B}{A}) x$ ....(31) (eq. Of the straight line)

The motion is SH, with amplitude $\sqrt{A^2 + B^2}$. Therefore we say that when $\delta = 0$ *or* π, the interference of two perpendicular SHM of the same frequency results in ***rectilinear polarization***.

When $\delta = \pi/2$, the motion are in quadrature, and

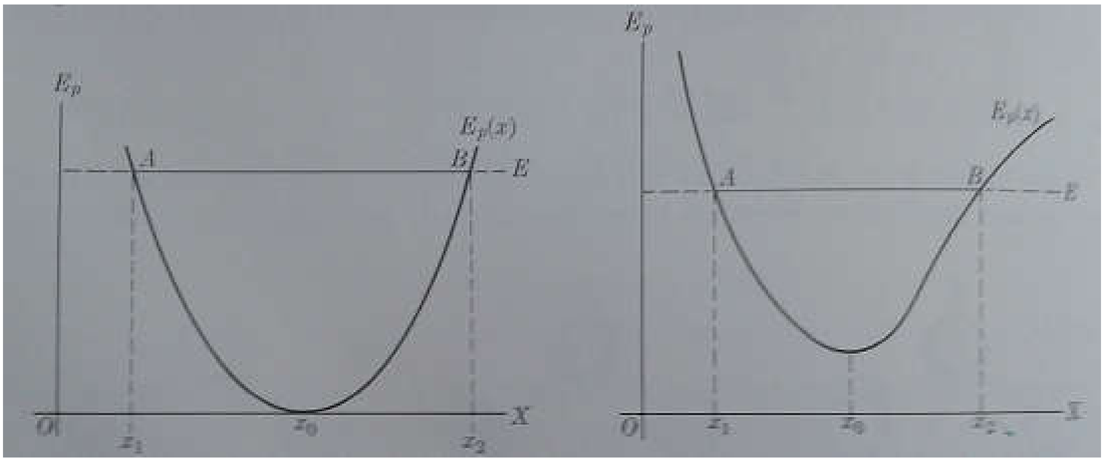
$y = B \sin(\omega t + \pi/2) = B \cos \omega t$ …………(32)

$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ (eq. of the ellipse)

An harmonic Oscillations

SHM is generated by a force $F = -kx$, corresponding to a potential energy $E_p = \frac{1}{2} k x^2$, when x is measured from the equilibrium position *O.* When the equilibrium position is at $x_\circ$ instead of the origin, then $E_p = \frac{1}{2} k (x - x_\circ)^2$. The graph of E_p is a parabola.

If the E_p is not a parabola but has a well-defined minimum at $x_\circ$ the equilibrium position. This is result in ***an harmonic oscillatory motion***.



Harmonic oscillator with equilibrium position at $x_\circ$ An harmonic oscillator with equilibrium position at $x_\circ$

Damped Oscillations

In SHM the oscillations have constant amplitude. A vibrating body such as a spring or a pendulum oscillates with an amplitude that gradually decrease and eventually stops (the oscillatory motion is damped).

The elastic force is $F = -kx$, there is another force opposed to the velocity $F' = -\lambda v$, λ: is a constant, v: is the velocity.

The resultant force on the body is $F + F'$. The eq. of motion is

$ma = -kx - \lambda v$

$ma + kx + \lambda v = 0$ …………(33)

$v = dx/dt, \quad a = d^2x/dt^2$

$m\, d^2x/dt^2 + \lambda\, dx/dt + kx = 0$ …………(34)

The solution of this eq. is

$x = A\, e^{-\gamma t} \sin(\omega t + \alpha)$ …………(35)

A and α are arbitrary constants.

λ and ω are fixed constants.

$\gamma = \lambda / 2m, \quad \omega = \sqrt{\frac{k}{m} - \gamma^2} = \sqrt{\omega_\circ^2 - \gamma^2}$ and $\omega_\circ = \sqrt{k/m}$

$\omega_\circ$: is the natural angular frequency without damping

The effect of damping is:

1) Decrease the frequency of the oscillations.
2) The amplitude of the oscillation is not constant, and is given by $A' = Ae^{-\gamma t}$ (the amplitude decreases as t increase).

In the case of a pendulum, $\omega = \sqrt{\frac{g}{l} - \gamma^2}$, and the period $P = \frac{2\pi}{\omega}$

Forced Oscillations:

The forced vibration of an oscillator, is (the vibrations which result when we apply an external oscillatory force to a particle subject to an elastic force).

Let $F = F_\circ \cos \omega_f t$, be the oscillating applied force.

ω_f: its angular frequency.

Assuming that the particle is subject also to an elastic force $-kx$,and a damping force $-\lambda v$. The eq. of motion is:

$ma = -kx - \lambda v + F_\circ \cos \omega_f t$

or

$ma + kx + \lambda v = F_\circ \cos \omega_f t$

$m\, d^2x/dt^2 + \lambda\, dx/dt + kx = F_\circ \cos \omega_f t$ …………(36)

In this case, the particle will oscillate with neither its free undamped angular frequency $\omega_\circ = \sqrt{k/m}$, nor the damped angular frequency , $\omega = \sqrt{\frac{k}{m} - \gamma^2}$

Instead, the particle is forced to oscillate with the angular frequency ω_f of the applied force. The solution of eq. (36) is:

$x = A \sin(\omega_f t - \alpha)$ …………(37)

A: is the amplitude of the forced oscillation.

The velocity amplitude is $v_\circ = \omega_f A$.

When $\omega_f = \omega_\circ = \sqrt{k/m}$, the $v_\circ$ and E_k of the oscillator are maximum ***(energy resonance).***

(energy resonance occurs when the frequency of the applied force is equal to the natural frequency of the oscillator without damping).

Chapter (6)

Dynamics of a rigid body

Angular momentum of a rigid body

A rigid body is a body in which the distances between all its component particles remain fixed under the application of a force or torque, therefore it is conserves its shape during its motion (translation and rotation motion).

Particle A_i describes a circle of radius

$R_i = A_i B_i$ with a velocity $v_i = \omega \times r_i$

r_i: the position vector relative to the

origin O.

$v_i = \omega\, r_i \sin \theta_i = \omega\, R_i$

The angular momentum of particle A_i relative to the O

$L_i = m_i\, r_i \times v_i$

The total angular momentum of the rotating body along the rotation axis Z is

$L_z = L_{1z} + L_{2z} + L_{3z} + \ldots\ldots = \sum_i L_{iz}$ ……….. (1)

$I = m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^2 + \ldots\ldots\ldots = \sum_i m_i R_i^2$ ……….. (2)

I: the moment of inertia of the body (m^2 . kg)

$L_z = I\, \omega$ ……….. (3)

Calculation of the Moment of Inertia

A rigid body is composed of a very large number of particles, so that

I $= \sum_i m_i R_i^2 = \int R^2\, dm$ ……….. (4)

$dm = \rho\, dV$

$I = \int \rho\, R^2\, dV$ ……….. (5)

ρ: is the density of the body

If the body is homogeneous, its density is constant.

$I = \rho \int R^2 \, dV$

$R^2 = x^2 + y^2$

The moment of inertia around the Z-axis is:

$I_z = \int \rho \, (x^2 + y^2) dV$ ……….. (6)

If the body is a thin plate the moment of inertia relative to the X- and Y-axes written as $I_x = \int \rho \, y^2 dV$ and $I_y = \int \rho \, x^2 dV$ (the Z-coordinate is essentially zero).

$I_z = I_x + I_y$ ……….. (7)

If a is the separation between the axes

Z and Z_c

$I = I_c + M a^2$ ……….. (8)

(Steiner's theorem)

Where I and I_c are the moments of inertia of the body relative to Z and Z_c,

M: the mass of the body

The radius of gyration of a body is K

$I = M K^2$ or $K = \sqrt{I/M}$ ……….. (9)

K: It represents the distance from the axis at which all the mass could be concentrated without changing the moment of inertia.

$I = \frac{1}{2} M R^2$ (for a homogeneous disk)

$K^2 = \frac{1}{2} R^2$

For a homogeneous thin rod

$I_c = \frac{1}{12} M L^2$ ……….. (10)

<u>Equation of Motion for Rotation of a Rigid Body</u>

$\frac{dL}{dt} = \boldsymbol{T}$ ………. (11)

$L = \sum_i L_i$ is the total angular momentum

$\boldsymbol{T} = \sum_i T_i$ is the total torque due to the external forces.

$\boldsymbol{L} = I\omega$

$\frac{d(\boldsymbol{I\omega})}{dt} = \boldsymbol{T}$ ………. (12)

$I\,\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{T}$ or $I\alpha = \boldsymbol{T}$ ………. (13)

$\alpha = \frac{d\boldsymbol{\omega}}{dt}$ is the angular acceleration of the rigid body.

If $\boldsymbol{T} = 0$, then $I\omega$ = const. That is, a rigid body rotating around a principle axis moves with constant angular velocity when no external torques are applied. (The law of inertia for rotational motion).

When the axis of rotation does not have a point fixed in an inertial system, we must compute the angular momentum and the torque relative to the center of mass of the body.

$\frac{dL_{CM}}{dt} = \boldsymbol{T_{CM}}$ ………. (14)

If the rotation is around a principle axis

$I_C\,(d\omega/dt) = \boldsymbol{T_{CM}}$

Example: A disk of radius 0.5m and mass 20kg can rotate freely around a fixed horizontal axis passing through its center. A force of 9.8N is applied by pulling a string wound around the edge of the disk. Find the angular acceleration of the disk and its angular velocity after 2s.

Solution: $I = \frac{1}{2} M R^2$

$\boldsymbol{T} = FR = \boldsymbol{I}\alpha = \frac{1}{2} M R^2\, \alpha$

$F = \frac{1}{2} M R\, \alpha \longrightarrow \alpha = \frac{2F}{MR} = \frac{2\times 9.8}{20\times 0.5} = 1.96 \text{ rad/s}^2$

$\omega = \alpha t = 1.96\times 2 = 3.92$ rad/s

To find F'

$F + Mg - 2F' = 0$

$2F' = F + Mg = 9.8 + (20\times 9.8) = 205.8$ N $\longrightarrow F' = 102.9$ N

Kinetic Energy of Rotation

$E_k = \sum_i \frac{1}{2}\ m_i v_i^2$

$v_i = \omega\ R_i$

R_i: is the distance of the particle to the axis of rotation.

$E_k = \sum_i \frac{1}{2}\ m_i v_i^2 = \sum_i \frac{1}{2}\ m_i R_i^2\ \omega^2 = \frac{1}{2}\ (\sum_i m_i R_i^2\ )\ \omega^2$

$I = \sum_i m_i\ R_i^2$ (moment of inertia)

$E_k = \frac{1}{2}\ I\omega^2$ ………. (15)

When the rotation is about a principle axis ($L = I\omega$)

$E_k = \frac{L^2}{2I}$ ………. (16)

If the rigid body rotates about an axis passing through its center of mass and at the same time has translational motion relative to the observer. The kinetic energy of a body in an inertial frame of reference is

$E_k = \frac{1}{2}\ M\ v_{CM}^2 + E_{k,\mathrm{CM}}$ ………. (17)

M: is the total mass

v_{CM} : is the velocity of the center of mass

$E_{k,\mathrm{CM}}$: is the internal kinetic energy relative to the center of mass.

$\frac{1}{2}\ M\ v_{CM}^2$ is the translational kinetic energy

$E_{k,\mathrm{CM}}$ is the rotational kinetic energy relative to the center of mass ($\frac{1}{2}\ I\omega^2$)

$E_k = \frac{1}{2}\ M\ v_{CM}^2 + \frac{1}{2}\ I_c\omega^2$ ………. (18)

I_C : the moment of inertia relative to the axis of rotation passing through the center of mass.

The total energy of the body is:

$E = \frac{1}{2}\ M\ v_{CM}^2 + \frac{1}{2}\ I_c\omega^2 + E_p = \text{const.}$

If the body falling under the action of gravity, $E_p = Mgy$

y: the height of the CM of the body relative to a horizontal reference plane.

$E = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_c\omega^2 + Mgy$ = const.

Example

A sphere, a cylinder, and a ring, all with the same radius, roll down along an inclined plane starting at a height y◦. Find in each case the velocity when they arrive at the base of the plane.

Solution

At the starting point B, the total energy $E = Mgy_◦$

At any intermediate position, the translational velocity v and the body is rotating with angular velocity ω, ($v = \omega R$)

$E = \frac{1}{2} M v^2 + \frac{1}{2} I_c\omega^2 + Mgy = \frac{1}{2} M v^2 + \frac{1}{2} (\frac{I_c v^2}{R^2}) + Mgy$

$I_c = MK^2$ (K: is the radius of gyration)

$E = \frac{1}{2} M \left(1 + \frac{K^2}{R^2}\right) v^2 + Mgy$

Or by using the initial energy $E = Mgy_◦$

$v^2 = \frac{2g\,(y_◦ - y)}{1+(\frac{K^2}{R^2})}$

If instead of a rolling rigid body, we had a body which slid down the plane, $v^2 = 2g\,(y_◦ - y)$

$K^2/R^2 = \frac{2}{5}$ for the sphere

$K^2/R^2 = \frac{1}{2}$ for the disk

$K^2/R^2 = 1$ for the ring

Physical Pendulum

The physical pendulum is any rigid body that can oscillate freely around a horizontal axis under the action of gravity.

The period of oscillation:

$$P = 2\pi\sqrt{\frac{K^2}{gb}} \quad \ldots\ldots\ldots (19)$$

*b:*is the distance from the center of mass *C* to the axis of oscillation.

K: the radius of gyration of the body relative to the axis ZZ'.

$l = K^2/b$ the length of the equivalent simple pendulum

or

$$P = 2\pi\sqrt{\frac{2R}{g}} \quad \ldots\ldots\ldots (20)$$

R: the radius of the ring.

Chapter (7)

Structure of Martter

Particles:

Three of the fundamental particles are especially important: *electrons, protons,* and *neutrons*.

Electrons are negatively charged, protons are positively charged, and neutrons are uncharged or neutral particles. *p* and *n* have the same mass (1.67×10^{-27} Kg), *e* mass (9.1×10^{-31} Kg).

Two interesting features for particles:

1) Many particles have a transient life, decaying in a very short time into other particles.
2) When two particles collide, they may be destroyed and new particles may be created.

Atoms:

The *p* and *n* clustered in a very small central region called (nucleus), which has a size of the order of 10^{-15}m. The *e* move about the nucleus in a region of the order of10^{-10}m. (10^{5} times larger than the nucleus).

Atomic number (Z): number of electrons in an atom or of protons in the nucleus.

Z **=** no. of *e* = no. of *p*

Mass number (A): number of protons and neutrons in the nucleus.

A = no. of p + no. of n

no. of $n = A - Z$

- ❖ All atoms with the same Z belong to the same atomic species (same chemical element), ex: if $Z = 1$, H_2 atom, $Z = 6$, C atom.

Isotopes: atoms with the same Z may have different A (no. of n). ex: ^1H, ^2H, ^3H (all have Z =1, one p).

Ions: an atom may gain or lose some electrons, becoming negatively or positively charged. ex: H^+, Cu^{+2}, P^{-3}.

Molecules:

Atoms from aggregates called *molecules*. When a molecule is formed, the atoms lose their identity.

A molecules: is a system composed of several nuclei and a group of electrons moving about the nuclei in such a way that a stable configuration results.

The forces that hold a molecule together are of electromagnetic origin.

Atoms in molecules are arranged in regular patterns characteristic of each molecule.

Matter in Bulk:

There are three physical states or phases for matter (gases, liquids, and solids).

- The regular arrangement of the atoms or groups of atoms is one of the most important features of solids, that is the structure of solids exhibits a regularity or periodicity constituting is called *a crystal lattice*.
- The physical properties of solids are directly related to the nature and geometrical arrangement of the units which compose the lattice.
- Some solids do not show this regular arrangement of the atoms or molecules, called (*amorphous*) such as glass.
- Another state of matter called (*plasma*) consisting of a gaseous mixture of positive and negative ions (or charged particles).
- When a gas is heated to a very high temperature, it becomes a plasma.

Interactions:

The particles in an atom interact among themselves in such away as to produce a stable configuration.

Atoms in turn interact to produce molecules, and molecules interact to form bodies.

Chapter (8)

Elastic

Stress:

The stress *S* at the section is the ratio of the force *F* to the area *A*.

$Stress = \frac{F}{A}$ ……….. (1)

The stress is called *a tensile stress*. (each portion pulls on the other), and it is also a *normal stress* (because the distributed force perpendicular to the area).

1 Pa (Pascal) = 1 N/m^2

1) The *normal stress* is defined as the ratio of the component $F_\perp$ to the area A'.

$Normal\ stress = \frac{F_\perp}{A'}$ ……….. (2)

2) The *tangential stress* is the ratio of the component $F_\parallel$ to the area A'.

$Tangential\ (shear)\ stress = \frac{F_\parallel}{A'}$ ……….. (3)

3) A bar subjected to pushes at its ends, is said to be in compression. The stress on the broken section also a normal stress but is now a *compressive stress*.

Strain:

Strain is refers to the relative change in dimensions or shape of a body that is subjected to stress.

1) The *tensile strain* in the bar is defined as the ratio of the increase in length to the original length.

$Tensile\ strain = \frac{l - l_\circ}{l_\circ} = \frac{\Delta l}{l_\circ}$ ……….. (4)

$l_\circ$: natural length

Δl: elongation

The *compressive strain* is the ratio of the decrease in length to the original length.

2) The strain produced by a hydrostatic pressure, called a *volume strain*, defined as the ratio of the change in volume ΔV to the original volume *V*.

   $Volume\ strain = \frac{\Delta V}{V}$ ……….. (5)

3) $Shear\ strain = \frac{x}{h} = tan\ \Phi$ ……….. (6)

   x: displacement of the corner.

   h: transverse dimension.

   Φ measure in radius.

Elastic modulus:

The ratio of stress to strain is called an *elastic modulus* of the material.

1) The ratio of tensile stress to tensile strain for a given material, equals the ratio of compressive stress to compressive strain. This ratio is called *Young's modulus*.

   $Y = \frac{tensile\ stress}{tensile\ strain} = \frac{compressive\ \ stress}{compressive\ \ strain}$

   $Y = \frac{F_{\perp}/A}{\Delta l/l_\circ} = \frac{l_\circ\ F_{\perp}}{A\ \Delta l}$ ……….. (7)

When a material elongates under tensile stress, the dimensions perpendicular to the direction of stress become shorter by an amount proportional to the fractional change in length.

$\frac{\Delta w}{w_\circ} = -\ \sigma\ \frac{\Delta l}{l_\circ}$ ……….. (8)

σ: dimensionless constant (***Poisson's ratio***)

$w_\circ$: original width

Δw: change in width

2) The modulus relating a hydrostatic pressure to the volume strain is called ***the bulk modulus*** (is the negative ratio of a change in pressure *dP* to the volume strain *dV/V*)

$$\beta = -\frac{dP}{dV/V} = -V\frac{dP}{dV} \qquad \text{………..} (9)$$

The reciprocal of the bulk modulus is called the *compressibility k*

$$k = \frac{1}{\beta} = -\frac{1}{V}\frac{dV}{dP} \qquad \text{………..} (10)$$

3) The shear modulus *S* is the ratio of the shear stress to the shear strain

$$S = \frac{F_{\parallel}/A}{x/h} = \frac{h}{A}\frac{F_{\parallel}}{x} = \frac{F_{\parallel}/A}{\emptyset} \qquad \text{………..} (11)$$