Approximate Solution for Linear Time-Delayed Improper Integral Equation Using Orthogonal Polynomials

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ABSTRACT
In this paper we adopt the collocation method based on orthogonal polynomials (Laguerre, Hermite) to solve linear time delayed improper integral equation approximately. Some examples are given to illustrate the high accuracy and the efficiency of the proposed numerical techniques.

Keywords: Improper integral Equation, Time delayed, Orthogonal polynomials.

INTRODUCTION
The integral equation is called improper integral equation if one or both of its limits are infinite. Many problems of electromagnetic scattering problem boundary integral equation [3,12] lead to improper integral equation. Many researchers have developed the approximate method to solve improper integral equation using Galerkin method with Laguerre polynomials as a bases function [1] while Sloan [2] used quadrature methods for solving integral equation of the second kind over infinite intervals.

The general form of the improper integral equation is:-

\[ f(s) = g(s) + \int_{0}^{\infty} k(s,t)f(t)dt \]  \hspace{1cm} ... (1)

or

\[ f(s) = g(s) + \int_{-\infty}^{\infty} k(s,t)f(t)dt \]  \hspace{1cm} ... (2)

where \( g(s) \) is continuous function and the kernel \( K(s,t) \) might has singularity in the region \( D = \{(s,t) : 0 \leq s, t < \infty \} \cup \{(s,t) : -\infty < s, t < \infty \} \) and \( f(s) \) is to be determined

1- The linear time-delayed improper integral equation [2,4,11] (LT-DIIE)
The general form of (LT-DIIE) can be written as follows

$$f(s-\tau) = g(s) + \int_{0}^{\infty} k(s,t) f(t) dt \quad \cdots (3)$$

$$f(s-\tau) = g(s) + \int_{-\infty}^{\infty} k(s,t) f(t) dt \quad \cdots (4)$$

Where ($\tau>0$) is a appositive integral called time delayed

2-ORTHOGONAL BASES POLYNOMIALS [5]

Orthogonal polynomials and their properties have a major role in both pure and applied mathematics as well as in a numerical computation. Some important properties that will be needed throughout this paper, some of these properties which are:

2.1 Laguerre Polynomials [1,6,7]

The Laguerre polynomials denoted by $L_n(s)$ are important sets of orthogonal polynomial over the interval $[0,\infty)$.

Consider Laguerre base polynomials as \{L_0(s),L_1(s),..., L_n(s)\}

Where

$$L_n(s) = \sum_{m=0}^{n} \left(\frac{(-1)^m}{m!}\binom{n}{m}\right) s^m \quad \cdots (5)$$

With the following properties

$$(L_m(s),L_n(s))=\int_{0}^{\infty} e^{-s} L_m(s)L_n(s)ds=0 \quad m\neq n$$

And

$$\|L_m(s)\|=1 \quad m=0,1,2,...$$

2.2 Hermite polynomials [6,9,10]

The Hermite polynomials $H_n(s)$ are a set of orthogonal polynomials over the domain ($-\infty,\infty$). It is well known that the general form of Hermite polynomials is

$$H_n(s) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!} (2s)^{n-2k} \quad \cdots (6)$$

With the following properties

$$(H_m(s),H_n(s))=\int_{-\infty}^{\infty} e^{-s^2} H_m(s)H_n(s)ds=0 \quad m\neq n$$

And

$$\|H_m(s)\|=1 \quad m=0,1,2,...$$
3. COLLOCATION METHOD

Collocation method has been applying for along time and Kantorovich gave a
general scheme for defining and analyzing the collocation method to solve the
linear operator equations [7, part II].

To solve the equation approximately

\[ f(s - \tau) = g(s) + \int_0^\infty k(s, t) f(t) \, dt \quad s \in D = \{ [0, \infty) \cup (-\infty, \infty) \} \]

We usually choose a finite dimensional family of polynomials (Laguerre and
Hermite) that is believed to contain a function \( f_0(s) \) close to the exact solution \( f(s) \).

3.1 Solving (LT-DIIE) by Collocation Method With Aid of Laguerre
Polynomials

Consider the linear time delay improper integral equation

\[ f(s - \tau) = g(s) + \int_0^\infty k(s, t) f(t) \, dt \quad \ldots \text{(8)} \]

By approximating \( f(s) \) into linear combination of Laguerre polynomials

\[ f(s) = f_n(s) = \sum_{i=0}^n c_i L_i(s) \quad \ldots \text{(9)} \]

and substituting into equation (8), yield

\[ \sum_{i=0}^n c_i L_i(s - \tau) = g(s) + \int_0^\infty k(s, t) \sum_{i=0}^n c_i L_i(t) \, dt \quad \ldots \text{(10)} \]

For which we have the residue equation

\[ R_n(s) = \sum_{i=0}^n c_i (L_i(s - \tau) - \int_0^\infty K(s, t) L_i(t) \, dt) - g(s) \]

Let \( m_j(s) = \int_0^\infty K(s, t) L_i(t) \, dt \quad j=0, 1, 2, \ldots \)

So, in this method the collocation points \( s_0, s_1, s_2, \ldots, s_n \) are on the interval \([0, \infty)\) such

that \( s_j = \frac{j}{(1 + n)} \quad j=0, 1, \ldots, n \)

Hence we have \( R_n(s_j) = 0 \quad j=0, 1, \ldots, n \)

This leads to

\[ \sum_{i=0}^n c_i (L_i(s_j - \tau) - m_i(s_j)) = g(s_j) \quad \ldots \text{(11)} \]

Equation (11) can be seen as a system of \( n+1 \) equations in \( n+1 \) unknown coefficients \( c_i \),
\( i=0, 1, \ldots, n \)

This system can easily be written in matrix formal as \( AC = G \)
where

\[
A = \begin{bmatrix}
L_0(s_0-\tau)-m(s_0) & L_0(s_0-\tau)-m(s_0) & \cdots & L_0(s_0-\tau)-m(s_0) \\
L_0(s_0-\tau)-m(s_0) & L_0(s_0-\tau)-m(s_0) & \cdots & L_0(s_0-\tau)-m(s_0) \\
M & M & \cdots & M \\
L_0(s_0-\tau)-m(s_0) & L_0(s_0-\tau)-m(s_0) & \cdots & L_0(s_0-\tau)-m(s_0)
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
g(s_0) \\
g(s_1) \\
g(s_2) \\
\vdots \\
g(s_n)
\end{bmatrix}
\]

By using Gauss elimination to determine the values \( c_i \) which satisfy equ(9).

### 3.2 Solving (LT-DIIE) Using Collocation Method With Aid of Hermite Polynomial

We will use Hermite polynomials as a bases function to approximate \( f(s) \) in the equation

\[
f(s - \tau) = g(s) + \int_{-\infty}^{\infty} k(s,t) f(t) dt \tag{12}
\]

Such that

\[
f(s) \equiv f_n(s) = \sum_{i=0}^{n} c_i H_i(s) \tag{13}
\]

Substituting equ(13) into equ(12) we get

\[
\sum_{i=0}^{n} c_i H_i(s-\tau) = g(s) + \int_{-\infty}^{\infty} k(s,t) H_i(t) dt \tag{14}
\]

So, we have the residue equation

\[
R_n(s) = \sum_{i=0}^{n} c_i (H_i(s-\tau)-\int_{-\infty}^{\infty} k(s,t) H_i(t) dt - g(s)) \tag{15}
\]

Let

\[
p_j(s) = \int_{-\infty}^{\infty} k(s,t) H_j(t) dt
\]

The collocation points \( s_0, s_1, \ldots, s_n \) on the interval \((-\infty, \infty)\) are

\[
s_j = \frac{j}{1+n} \quad j=0,1,\ldots,n
\]

Hence

\[
R_n(s_j) = 0 \quad j=0,1,\ldots,n
\]

This leads to

\[
\sum_{i=0}^{n} c_i (H_i(s_j-\tau)-p_j(s_j)) = g(s_j) \tag{16}
\]

This system can be written in matrix formal as \( BC=G \)

Where
By solving this system by using Gauss elimination to get the values of \( c_i \), which satisfy equ(13).

**NUMERICAL EXAMPLE**

**Example 1**
Consider the following delay infinite integral equation

\[
f(s-1) = (s-1)^2 \frac{-\sqrt{\pi}}{2} + \int_{-\infty}^{\infty} e^{st} f(t) dt
\]

with exact solution \( f(s) = s^2 \)

By using Hermite polynomials we get

\[
f(s) \equiv f_2(s) = c_0 H_0(s) + c_1 H_1(s) + c_2 H_2(s) = c_0 + 2c_1 s + (4s^2 - 2)c_2
\]

So

\[
R(s) = c_0 (1 - \sqrt{\pi}) + 2(s - 1)c_1 + (4s - 1)^2 - 2c_2
\]

Hence

\[
c_0 = \frac{1}{2}, \quad c_1 = 0, \quad c_2 = \frac{1}{4}
\]

The approximate solution is \( f_2(s) = s^2 \)

**Example 2**
Consider the following delay infinite integral equation

\[
f(s-1) = \cos(s-1) - \frac{1}{2} e^{-s} + \int_{0}^{\infty} e^{s-t} f(t) dt
\]

with exact solution \( f(s) = \cos(s) \)

using Laguerre polynomial

\[
f(s) \equiv f_n(s) = \sum_{i=0}^{n} c_i L_i(s)
\]

Substitute it in the above equation we get
Approximate Solution for Linear Time-Delayed Improper Integral Equation Using Orthogonal Polynomials

\[ \sum_{i=0}^{n} c_i L_i(s-1) = \cos(s-1) + \frac{1}{2} e^{-s} + \int_{0}^{\infty} e^{-s-t} \sum_{i=0}^{n} c_i L_i(t) dt \]

So

\[ R(s) = \sum_{i=0}^{n} c_i L_i(s-1) - \int_{0}^{\infty} e^{-s-t} \sum_{i=0}^{5} c_i L_i(t) dt \]

By applying the proposed algorithm the solution of eq(1) for different values of n for arbitrary final time the values of c_i displayed in tables(1).

Table (2) presents a comparison between the exact and numerical solution obtained by collocation method with aid of Laguerre polynomial for \( t \in [0,1] \) depending on least square error (L.S.E).

CONCLUSIONS

Collocation method as an approximate method for solving linear time-delayed improper integral equation using orthogonal polynomials was proposed. The method based on Laguerre and Hermite polynomials. From the numerical results in table(1) it is clear that using these functions to approximate the solution produce accurate results as n increases and the numerical solution convergent to the correct one as the length of series increase.

REFERENCES


Approximate Solution for Linear Time-Delayed Improper Integral Equation Using Orthogonal Polynomials


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Approximate Solution for Linear Time-Delayed Improper Integral Equation Using Orthogonal Polynomials

Table (2)

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