الجامعة التكنولوجية
قسم الهندسة الكيمياوية
المرحلة الثالثة
الرياضيات التطبيقية
د.غاش مقبول
1. **Revision ordinary differential equations:**
   First and second order, simultaneous differential equations, Application for chemical engineering.
   
   (18hrs)

2. **Solution of differential equations by series:**
   Simple series, method of Froebins, Bessel’s equation. Application for chemical engineering.
   
   (18hrs)

3. **The Laplace transformation:**
   
   (18hrs)

4. **Partial Differential Equations:**
   
   (18hrs)

5. **Numerical Analysis:**
   
   (18hrs)
Applied Mathematics

Ref: Mathematical methods in chemistry, by Jenson & Jeffress.

Ordinary Differential Equations

Any equation containing differential coefficients is called a differential equation. Such equations can be divided into two main types—ordinary and partial, ordinary differential equations involving only one independent variable (and therefore only ordinary differential coefficients), and partial differential equations involving two or more independent variables (and therefore partial differential coefficients).

In general, therefore, any function of \( x/y \) and the derivatives of \( y \) up to any order such that:

\[
f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots) = 0 \quad \ldots (1)
\]

\( y \): dependent variable
\( x \): independent variable

For example:

\[
\frac{dy}{dx} = 3y, \quad \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - y = 5 \sin x \quad \text{third order}
\]

Ordinary Diff. Eqs.

\[
\text{while: } \quad y \frac{du}{dx} + \frac{dy}{dx} \frac{du}{dy} = u, \quad \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \quad \text{are partial diff. eqns.}
\]
Any differential equation of order \( n \) is said to be linear if it is linear in the dependent variable \( y \) and the derivatives 

\[
\frac{dy}{dx}, \frac{dy}{dx}, \ldots, \frac{dy}{dx^n}
\]

Regarding to order, 1st & 2nd equations are linear while 3rd eq. is non-linear. The solution of non-linear equations usually presents a difficult problem and for the most part therefore we shall only discuss the solution of linear equations.

First order equations

Equations of this type can, in general be written as:

\[
\frac{dy}{dx} = f(x,y) \quad (1)
\]

where \( f(x,y) \) is given function.

Types of solution

(a) Variables Separable

If \( f(x,y) = f(x)g(y) \) \quad (2)

where \( f(x) \) and \( g(y) \) are respectively functions of \( x \) only and \( y \) only, then equation (1) becomes:

\[
\frac{dy}{dx} = f(x)g(y) \quad (3)
\]

Since the variables \( x \) and \( y \) are now separate, we have integrating eq. (3).
which express $y$ implicitly in terms of $x$

**EX.1.** Solve the equation:

\[
\frac{dy}{dx} = \frac{y+1}{x-1}
\]

with boundary condition (B.C.), $y=1$ at $x=0$

\[
\int \frac{dy}{y+1} = \int \frac{dx}{x-1}
\]

or

\[
\ln(y+1) = \ln(x-1) + \ln C
\]

where $C$ is an arbitrary constant of integration. Hence,

\[
\frac{y+1}{x-1} = C
\]

Now if $y=1$ at $x=0$, then from eq. 6, we find:

\[
C = -2
\]

\[
\therefore \quad y = 2 (1-x) - 1
\]

**EX.2:** Find the concentration of mixing tank as a function of time where $q_1 = q_2$. Material balance:

Input - output = accumulation (steady-state) (Note for s-state system accumulation = 0).
Input: $q C_1$

Since $q = \frac{dC}{dt}$

Output: $q C_2$

Accumulation:

$$\frac{dC}{dt} = \frac{m^3}{sec} \cdot \frac{kg}{m^3} = \frac{kg}{sec}$$

$$\frac{dC}{dt} = \frac{m^3}{sec} \cdot \frac{kg}{m^3} = \frac{kg}{sec}$$

$$q C_1 - q C_2 = \frac{dC}{dt}$$  \(\text{Equation 1}\)

$$\frac{V}{q} \cdot \frac{dC}{dt} + C_2 = q C_1$$  \(\text{Equation 2}\)

Integrating the differential equation:

$$If \ C_1 = 0$$

Equation (2) becomes:

$$\frac{V}{q} \frac{dC}{dt} = -C_2$$  \(\text{Equation 3}\)

$$-\frac{dC}{C_2} = \frac{q}{V} \frac{dt}{dt}$$

$$-\ln C_2 = \frac{q}{V} t + K$$

This equation can be solved by using 13 C...
The homogeneous equation:

\[ \frac{f(x,y)}{g(x,y)} = \frac{f(x_0, y_0)}{g(x_0, y_0)} = (1) \]

where \( f(x,y) \) and \( g(x,y) \) are homogeneous functions of \( x \) and \( y \) of the same degree, say \( n \):

\[ \frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} = x^n \frac{f(y/x)}{g(y/x)} = \phi \left( \frac{y}{x} \right) = 2 \]

where \( \phi \) is a function of \( y/x \). This equation is usually called the homogeneous first order differential equation and may always be reduced to a variable separable equation by the substitution:

\[ y = vx \]

where \( v \) is a function of \( x \). For differentiating eq. (3), we have:

\[ \frac{dy}{dx} = v + x \frac{dv}{dx} = (4) \]

Regarding to eq. (2), becomes:

\[ v + x \frac{dv}{dx} = \phi (v) = (5) \]

\[ x \frac{dv}{dx} = \phi (v) - v \]

By separable method:

\[ \int \frac{dv}{\phi (v) - v} = \int \frac{dx}{x} \]
Solve the equation:

\[ 2xy \frac{dy}{dx} = x^2 + y^2 \]  \[ \text{(1)} \]

R.C.: \( y = 0 \) at \( x = 1 \)

Rewritten eq (1) with another form:

\[ \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \]

Let \( u = \frac{y}{x} \) \( \rightarrow \) \( y = xu \) \( \Rightarrow \) \( \frac{dy}{dx} = x \frac{du}{dx} + u \frac{dx}{dx} \)

\[ \therefore \quad \frac{\frac{dy}{dx}}{x} = \frac{1}{2} \left( \frac{y}{x} + \frac{x}{u} \right) \]  \[ \text{(2)} \]

\[ 2x \frac{du}{dx} = \frac{1 - u^2}{u} \]  \[ \text{(2a)} \]

Therefrom:

\[ \int \frac{du}{1 - u^2} = \int \frac{dx}{x} \]  \[ \text{(3)} \]

\[ \therefore \quad -\ln(1 - u^2) = \ln x + \ln c \]

\[ \therefore \quad u^2 - x^2 = -\frac{x}{c} \]  \[ \text{(4)} \]

For R.C., \( y = 0 \) at \( x = 1 \)

\[ \therefore \quad c = 1 \]

\[ \therefore \quad x^2 - y^2 = x \]

Regarding to mixing tank problem (eq. 3):

\[ \frac{V}{Q} \frac{dc}{dt} + c_2 = c_1 \]  \[ \text{(5)} \]

If \( c_1 \) : constant

\[ \therefore \quad \frac{V}{Q} \frac{dc}{dt} = c_1 - c_2 \]  \[ \text{(6)} \]
\[ \frac{V}{q} \int \frac{dc_i}{c_1 - c_2} = \int dt \]

\[ -\ln(c_1 - c_2) = \frac{q}{4} t + k \]

where \( k \) : Integral constant

(C) The Exact Equation:

\[ F(x, y) = -\frac{P(x, y)}{Q(x, y)} \]

where \( P \) and \( Q \) are given functions of \( x \) and \( y \), then the general first order equation:

\[ \frac{dy}{dx} = F(x, y) \]

becomes:

\[ Q(x, y) \frac{dy}{dx} + P(x, y) = 0 \]

we have:

\[ \frac{du}{dx} = \left( \frac{du}{dy} \right) \frac{dy}{dx} + \frac{du}{dx} \]

Consequently, comparing eqs 3 & 4, we see that if:

\[ P(x, y) = \frac{du}{dx} \]

\[ Q(x, y) = \frac{du}{dy} \]

Then eq 3, maybe written as:

\[ \frac{du}{dx} = 0 \]

The solution of which is:

\[ u(x, y) = \text{constant} \]
Differentiating eq. (5) (and assuming $P$ and $Q$ have continuous first derivatives) we have:

\[
\frac{\partial P}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial Q}{\partial x}
\]  

(8)

Ex.: The equation

\[
(8y - x^2y) \frac{dy}{dx} + (x - x^2y) = 0
\]

is exact since $P(x,y) = x - x^2y$, $Q(x,y) = 8y - x^2y$ and

\[
\frac{\partial P}{\partial y} = -2xy = \frac{\partial Q}{\partial x}
\]

(9)

To find $u(x,y)$ we therefore use eq.(5) which gives:

\[
\frac{du}{dx} = P(x,y) = x - x^2y
\]

(3)

Integrating with respect to $x$, we have:

\[
u(x,y) = \frac{x^2}{2} (1 - 5) + \phi(y)
\]

(4)

where $\phi(y)$ is an arbitrary function of $y$.

Also \[
\frac{du}{dy} = Q(x,y) = 8y - x^2y
\]

(5)

By differentiating eq.(4) with respect to $y$ and comparing with eq.(5):

\[
8y - x^2y = -x^2y + \frac{d\phi}{dy}
\]

(6)

Hence

\[
\phi = 4y^2 + C
\]

(7)

where $C$ is an arbitrary constant of integration. Finally, equations (4) and (7) together give:

\[
u(x,y) = \frac{x^2}{2} (1 - 5) + 4y^2 + C
\]

(8)
and the solution of eq.(1) is therefore

\[ \frac{X^2}{2} (1 - \frac{1}{y^2}) + y^2 = \text{constant} \quad - (7) \]

The constant \( c \) being absorbed in the constant term on the right)

\[ \frac{\partial P(x,y)}{\partial y} \neq \frac{\partial Q(x,y)}{\partial x} \]

However, if eq.(1) is multiplied through by the factor \( y/x \) we find:

\[ 2y\ln x \frac{dy}{dx} + \frac{y^2}{x} = 0 \quad - (8) \]

which, since:

\[ \frac{\partial}{\partial y} \left( \frac{y^2}{x} \right) = \frac{\partial}{\partial x} (2y\ln x) \quad - (9) \]

is exact.

Finally, the solution as last example:

\[ \frac{\partial}{\partial y} \left( \frac{y^2}{x} \right) = \frac{\partial}{\partial x} (2y\ln x) \quad - (9) \]

The factor \( y/x \) used here to make eq.(1) exact is called an integrating factor.

In general if the equation:

\[ Q(x,y) \frac{dy}{dx} + P(x,y) = 0 \quad - (5) \]

is not exact, then there exists an integrating factor \( \mu(x,y) \) which
makes it exact, although the form of $\mu$ may be difficult to find. Suppose eqs. is multiplied through by $\mu(x,y)$ to give:

$$\mu(x,y) \frac{\partial Q(x,y)}{\partial y} + \mu(x,y) p(x,y) = 0 \quad (6)$$

Then (writing $P$, $Q$ and $\mu$ for $P(x,y)$, $Q(x,y)$ and $\mu(x,y)$ respectively) equation (6) is exact provided $\mu$ satisfies the partial diff. eq.

$$\frac{\partial}{\partial x} (\mu Q) = \frac{\partial}{\partial y} (\mu P) \quad \text{--- (7)}$$

or:

$$\mu \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) + Q \frac{\partial P}{\partial x} - P \frac{\partial Q}{\partial y} = 0 \quad \text{--- (8)}$$

Except when $P$ and $Q$ have exceptionally simple forms (case is usually difficult to solve). The determination of integrating factors is at this stage therefore largely a matter of trial and error.

Example: The equation:

$$x \frac{dy}{dx} - y = 0 \quad \text{--- (9)}$$

is not exact. Suppose there exists an integrating factor $\mu$ such that

$$\mu x \frac{dy}{dx} - \mu y = 0 \quad \text{--- (10)}$$

is exact. Then:

$$\frac{\partial (\mu x)}{\partial x} = \frac{\partial (\mu y)}{\partial y} \quad \text{--- (11)}$$

or:

$$x \frac{\partial \mu}{\partial x} + y \frac{\partial \mu}{\partial y} + 2 \mu = 0 \quad \text{--- (12)}$$

Any solution of this partial differential equation will be a suitable integrating factor. It is easily verified that:

$$\mu(x,y) = e^{\int \frac{y}{x} \, dx}$$

satisfies (12).
\[
\mu = \frac{1}{x^2} \frac{1}{y^2} \frac{1}{x+y} \frac{1}{x^2+y^2} \frac{1}{x^2-xy}
\]

are all solutions of \( \text{eqn} 1 \), and therefore all qualify as integrating factors. For example, taking \( \mu = \frac{1}{x^2} \), \( \text{eqn} 1 \) becomes:

\[
\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = 0 \;
\]

which is exact. Since \( \text{eqn} 1 \), can now be written as \( \frac{dy}{dx} \frac{1}{x} = 0 \), the solution of \( \text{eqn} 1 \) is therefore:

\[
\frac{y}{x} = \text{constant} \;
\]

Similarly, taking \( \mu = \frac{1}{x^2-xy} \), \( \text{eqn} 1 \) becomes:

\[
\frac{x}{x^2-xy} \frac{dy}{dx} - \frac{y}{x^2-xy} = 0
\]

which again is exact. Since \( \text{eqn} 1 \) may be written as:

\[
\frac{dy}{dx} = \frac{1}{x^2-xy} \ln(x^2-xy) = 0
\]

we again find \( y/x = \text{constant} \) as \( \text{eqn} 1 \).

(d) The linear equation:

If

\[
F(xy) = \Omega(x) - P(x)y
\]

where \( P(x) \) and \( \Omega(x) \) are given functions of \( x \) (or constants, the general equation

then \( \frac{dy}{dx} = F(xy) \) becomes:

\[
\frac{dy}{dx} + P(x)y = \Omega(x)
\]

This equation (known as the general linear 1st order equation) may be solved with the help of an integrating factor.
\[ R(x) \frac{dy}{dx} + R(x) P(x) y = Q(x) R(x) \]  \[ \text{Eq. (3)} \]

where \( \text{Integrating factor } R(x) = e^{\int P(x) dx} \)

Finally,

\[ R(x) y = \int Q(x) \, R(x) \, dx \]

This is the general solution of eq. (2).

**Ex.:** Solve: \( \frac{dy}{dx} + \frac{3}{x} y = x^2 \)

- **B.** \( y = \frac{1}{2} \) when \( x = 1 \)

  \[ R = e^{\int \frac{3}{x} \, dx} = e^{\ln x^3} = x^3 \]

  \[ R = x^3 \]

  \[ \int x^3 \, dx = \frac{x^4}{4} + c \]

  \[ y = \frac{x^2}{6} + \frac{c}{x^3} \]

  For \( y = 1/6 \) when \( x = 1 \), we find \( c = 0 \) and hence

  the required solution is:

  \[ y = \frac{x^3}{6} \]
Ex: Liquid level system as shown in figure below: Find the equation which relates the head to time, with B.C of \( t=0 \) \( h=0 \).

\[
q_1 = \frac{h}{K} \]

\[ \text{M. B on the system:} \]

\[ \rho q_1 - \rho q_2 = \rho A \frac{dh}{dt} \] 

\[
\therefore q_1 - \frac{h}{K} = A \cdot \frac{dh}{dt} \]

\[ A \frac{dh}{dt} + \frac{h}{K} = q_1 \]

\[ \frac{dh}{dt} + \frac{h}{AK} = \frac{q_1}{A} \]

\[ \frac{dh}{dt} + \frac{h}{C} = \frac{q_1}{A} \]

\[ \text{Spd:} \]

\[ \frac{S^1}{e^\frac{v}{e}} \]

\[ \therefore R = e \]

\[ \boxed{h} = \frac{q_1}{e} \cdot e^\frac{v}{e} \cdot dt \]

\[ \therefore e^\frac{v}{e} \cdot h = \frac{q_1}{A} \cdot e^\frac{v}{e} + C \]

\[ \text{For} \; t=0 \; \rightarrow \; h=0 \]

\[ \therefore C = -\frac{e^\frac{v}{e}}{A} q_1 = -kv_1 \]

\[ \therefore e^\frac{v}{e} \cdot h = kv_1 \cdot e^\frac{v}{e} = kv_1 \]
\[ h = k q_1 \left( 1 - e^{-\frac{t}{T_c}} \right) \]

**Linear Equations**

The linear first order equation discussed in the last section is a special case of the general linear equation of order \( n \):

\[ a_0(x) \frac{dy}{dx} + a_1(x) \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x) \frac{dy}{dx} + a_n(x) y = f(x) \]

where \( a_0(x), a_1(x), \ldots, a_n(x) \) and \( f(x) \) are given functions of \( x \), and the constants. This equation is said to be homogeneous when \( f(x) = 0 \) and to be inhomogeneous when \( f(x) \neq 0 \). As we shall see later, it is useful in dealing with inhomogeneous equations of the type (1) to consider the corresponding homogeneous (or reduced) equation obtained by putting \( f(x) = 0 \). For example, the reduced equation corresponding to:

\[ x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 3y = \sin x \]

is:

\[ x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 3y = 0 \]

Solutions of the reduced equation (1) are:

\[ a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x) \frac{dy}{dx} + a_n(x) y = 0 \]

Suppose now \( y_1 \) and \( y_2 \) are two independent solutions of the linear combination:

\[ y = C_1y_1 + C_2y_2 \]

where \( C_1 \) and \( C_2 \) are arbitrary constants.
By substitution of \( y \) into (4) yields:

\[
\left( a_0(x) \frac{dy}{dx} + a_1(x) \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)y \right) + \left( a_0(x) \frac{dy_1}{dx} + \ldots + a_{n-1}(x) \frac{dy_1}{dx} + a_n(x)y_1 \right) = (6)
\]

Similarly, we may now show that if \( y_1, y_2, \ldots, y_n \) are independent solutions of (4), then the linear combination:

\[
y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n \tag{7}
\]

where \( C_1, C_2, \ldots, C_n \) are arbitrary constants, is also a solution.

The general solution of the inhomogeneous equation (1) is the sum of the general solution of the reduced equation and any particular solution of the inhomogeneous equation:

\[
y = yc + yp \tag{8}
\]

General solution: complementary function + particular integral.

\[
y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n + y_p \tag{8}
\]

Linear homogeneous equations with constant coefficients:

\[
a_0 \frac{dy}{dx} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{4}
\]

for 2nd order equation \((n = 2)\):

\[
a_0 \frac{dy}{dx} + a_1 \frac{dy}{dx} + a_2 y = 0 \tag{2}
\]
A and B being arbitrary constants.

\[ y = Ax^{-2} + B e^{2x} \]

To solve the equation:

\[ y = \frac{x}{2} - x^{2} + 2x - 2 \]

\[ x^2 + 2x - 2 = 0 \]

\[ x = \frac{-2 \pm \sqrt{4 + 8}}{2} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3} \]

\[ y = \frac{-1 + \sqrt{3}}{2} \]

Suppose we try a solution:

\[ y = e^{x} \]

\[ y' = e^{x} \]

\[ y'' = e^{x} \]

\[ y'' = y \]

\[ (e^{x} - x) e^{x} = 0 \]

The solution:

\[ y = e^{x} \]

\[ y = (A + Ax) e^{x} \]

Suppose we try a solution:

\[ y = e^{x} \]

\[ y' = e^{x} \]

\[ y'' = e^{x} \]

\[ y'' = y \]

\[ (e^{x} - x) e^{x} = 0 \]

The solution:

\[ y = e^{x} \]
To solve the equation:
\[ \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0. \]

The auxiliary equation:
\[ m^2 + m + 1 = 0. \]
The roots:
\[ m = -\frac{1}{2} \pm i\left(\frac{\sqrt{3}}{2}\right) \] (imaginary roots)

The general solution:
\[ y = Ae^{-\frac{x}{2}} + Be^{-\frac{x}{2}} \left( \cos \frac{x\sqrt{3}}{2} + i \sin \frac{x\sqrt{3}}{2} \right) \]

Since:
\[ e^{ix} = \cos x + i \sin x, \]
so that eq(2) becomes:
\[ y = e^{-\frac{x}{2}} \left( E \cos \frac{x\sqrt{3}}{2} + F \sin \frac{x\sqrt{3}}{2} \right). \]

Where E and F are arbitrary constants.

To solve:
\[ \frac{d^3 y}{dx^3} = \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 2y = 0. \]
The auxiliary equation:
\[ m^3 - m^2 - m + 1 = 0. \]
\[ (m-1)(m^2 + 1) = 0. \]
\[ m_1 = 1, \quad m_2 = i, \quad m_3 = -i. \]

\[ y = (A + Bx)e^{-x} + Ce^{-x}. \]
A, B, and C being arbitrary constants.
Linear Inhomogeneous Constant Coefficient Equations:

\[
\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + \cdots + a_n y = f(x) \quad (1)
\]

where \( f(x) \) is a given function of \( x \), and \( a_1, a_2, \ldots, a_n \) are constants.

\( y' \) - Complementary Function + Particular Integral = General Solution

1. Complementary Function: \( f(x) = 0 \) and solved by the method of the last section (Homogeneous equations).
2. Find the particular integral.

For example,

\[
\frac{d^2y}{dx^2} + y = 3x
\]

\[
\frac{d^2y}{dx^2} + y = 0
\]

\( m^2 + 1 = 0 \)

\( m = i \) or \( m = -i \)

\( y_c = Ae^{ix} + Be^{-ix} \)

\( y_p = 3 \cos x + 3 \sin x \)

where \( A, B, E, F \) are constants.

The general solution:

\[
y = y_c + y_p = \sum \cos x + \sum \sin x + 3x
\]
methods for finding the particular solution.

(a) Method of undetermined coefficients:
This method is based on assuming a trial form for the particular integral.
The following rules were used:

(i) If \( f(x) = ax^k \) where \( a \) and \( b \) are constants, and if the auxiliary
equation has \( m = b \) as a root occurring \( k \) times,
\[
Y(x) = A x^k e^{bx}
\]
where \( A \) is a constant to be determined. (If \( m \neq b \), then \( A = 0 \)).

(ii) If \( f(x) = a \sin bx \) or \( a \cos bx \), where \( a \) and \( b \) are constants, and
if \( (m^2 + b^2) \) is a factor of the auxiliary equation of multiplicity \( k \),
then we take
\[
Y(x) = x^k (A \sin bx + B \cos bx)
\]
where \( A \) and \( B \) are constants to be determined. (If \( m^2 + b^2 \) is not a factor,
then \( k = 0 \)).

(iii) If \( f(x) = ax^k \) where \( a \) and \( s \) are constants, and if the
auxiliary equation has \( m = 0 \) as a root of multiplicity \( k \), then
we take
\[
Y(x) = x^k (A x^s + B x^{s-1} + \ldots + Px + Q)
\]
where \( A_1, B_1, \ldots, Q \) are constants to be determined. (If \( m = 0 \)
is not a root, then \( k = 0 \)).
EX: To find a particular integral of:
\[
\frac{dy}{dx} - 3 \frac{dy}{dx} + 2y = xe^x
\]

The auxiliary equation:
\[m^2 - 3m + 2 = 0\]

Roots: \(m = 1, m = 2\)

Since one of these roots is equal to the value of \(b(2)\), we must take \(k = 1\):

\[y(x) = xe^x\]

The general solution:
\[y = c_1 e^x + c_2 xe^x\]

where \(c_1, c_2\) are arbitrary constants.

EX: To find a particular integral of:
\[
\frac{dy}{dx} + 4y = 3 \sin 2x
\]

The auxiliary equation is:
\[m^2 + 4 = 0\]

The solution (ii) \(b = 2\), we see that \(e^{ib}\) is a factor of \(m^2 + 4\).

\[K = 1\]

Then:

\[y(x) = x(A \sin 2x + B \cos 2x)\]

where \(A\) and \(B\) are constants to be found. Substituting \(x = 0\) into (3), we find \(A = 0\), \(B = -\frac{3}{4}\)

The particular integral is therefore:
\[y(x) = -\frac{3}{4} x \cos 2x\]
Then the general solution:
\[ y = \sum \cos 2x \cdot f \cdot \sin 2x - \frac{3}{4} x \cdot \cos 2x \]
where \( E, F \) are constants.

**EX:** To find a particular integral of:
\[ \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} = 6x + \sin x \]

The auxiliary equation:
\[ m^2 + 2m + 2 = 0 \]

Roots, \( m = i, -i \). The particular solution have the form as:

in (iii) and (iii):
- \( 6x \rightarrow Ax + B \)
- \( \sin x \rightarrow C \sin x + D \cos x \)

Let \( a, b, c, d \) and \( D \) constants.

\[ y(x) = Ax + B + C \sin x + D \cos x \]

Substitute eq (3) into (1) we find:
- \( A = 3/10 \), \( B = \frac{3}{10} \), \( C = \frac{1}{5} \), and \( D = \frac{1}{10} \)

which give the particular integral as:
\[ y(x) = 3x + \frac{1}{10} (2 \sin x + \cos x) \]

(b) Method of Variation of Constants:
- Method of Variation of Parameters.

For example, 2nd order equation:
\[ a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + ay = f(x) \]
Suppose the complementary function of this equation is:

$$y_3 = A_1 y_1 + A_2 y_2$$  \[ (2) \]

where $y_1$ and $y_2$ are independent solutions of the reduced equation

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$  \[ (5) \]

and $A_1, A_2$ are constants. We now replace these constants by function $V_1(x)$ and $V_2(x)$, and define a new function $Y(x)$ as:

$$Y(x) = V_1(x) y_1 + V_2(x) y_2$$  \[ (4) \]

The functions $V_1(x)$ and $V_2(x)$ are now to be found such that $Y(x)$ is a solution of (1).

It is convenient to take this condition to be:

$$V_1(x) y_1 + V_2(x) y_2 = 0 \quad \text{for all } x.$$  \[ (5) \]

Finally, \[ (5) \]

$$V_1(x) = -\frac{y_2 \phi(x)}{a_0 (y_1 y_2' - y_1' y_2)}$$  \[ (6) \]

and \[ (7) \]

$$V_2(x) = \frac{y_1 \phi(x)}{a_0 (y_1 y_2' - y_1' y_2)}$$

From which, \[ (8) \]

$$V_1(x) = -\frac{1}{a_0} \int_0^x \frac{y_2 \phi(x)}{y_1 y_2' - y_1' y_2} \, dx$$

and \[ (9) \]

$$V_2(x) = \frac{1}{a_0} \int_0^x \frac{y_1 \phi(x)}{y_1 y_2' - y_1' y_2} \, dx$$

The general solution is therefore the sum of $y(x)$ and $Y(x)$:

$$y = \left[ A_1 + V_1(x) y_1 \right] y_1 + \left[ A_2 + V_2(x) y_2 \right] y_2 + g$$
To solve
\[ \frac{dy}{dx^2} + 3 \frac{dy}{dx} + 2y = e^x - (1) \]

The reduced equation: \( y_1 = e^x \), \( y_2 = e^{-x} \)
The complementary function is therefore:
\( y = A_1 e^x + A_2 e^{-x} \) \( - (2) \)
The particular integral \( y_p(x) \) is:
\[ y_p(x) = v_1(x) e^x + v_2(x) e^{-x} \]
\[ v_1(x) = - \int \frac{e^x - e^x}{(e^x - e^{-x})} \, dx = \frac{x}{e} \]
\[ v_2(x) = \int \frac{e^x - e^x}{(e^x - e^{-x})} \, dx = - \frac{e^x}{4} \]
The general solution:
\[ y = (A_1 + \frac{x}{e}) e^x + (A_2 - \frac{2x}{4}) e^{-x} \]
\[ = A_1 e^x + A_2 e^{-x} + \frac{x}{e} e^x \]
where \( A_1 = A_1 - \frac{1}{e} \) and \( A_2 \) are arbitrary constants.

---

Solved: \( \Delta x \)

Heat loss vessel system heated \( M \) kg of water by using steam
Coil with saturated temp \( T_s \)
Find the temperature as a function of time
\( T = f(x) \)

B.C. at \( t = 0 \): \( T = T_0 \) (initial temp.)

\[ T(x) = \text{Heat gain} - \text{Heat loss} \]
Heat in - Heat out - Heat lost = accumulated Heat

\[ Q = 0 = M C P \frac{dT}{dt} \] - (1)

\[ h A(T_s - T) = M C P \frac{dT}{dt} \] - (2)

\[ \frac{h A}{M C P} (T_s - T) = \frac{dT}{dt} \] - (3)

\[ \frac{h A}{M C P} \int_{T_0}^{t} \frac{dT}{T_s - T} = \int_{T_0}^{t} \frac{dt}{t} \] - (4)

\[ \ln(T - T_s) = \frac{h A}{M C P} t + C \text{ control} \]

\[ \frac{R}{C_b} \quad t \rightarrow \infty \quad T = T_0 \]

\[-\ln(T_s - T_0) = 0 + C\]

\[ c = -\ln(T_s - T_0) \quad \text{substituted into Equation 4}\]

\[ -\ln(T_s - T) = \frac{h A}{M C P} t - \ln(T_s - T_0) \]

\[ \ln(T_s - T) - \ln(T_s - T_0) = \frac{h A}{M C P} t \]

\[ \ln(T_s - T) = \ln(T_s - T_0) - \frac{h A}{M C P} t \]

\[ \ln\left(\frac{T - T_0}{T_s - T_0}\right) = \frac{h A}{M C P} t \] - (6)
The D– operator

Symbol $D$ represent the differential operator $\frac{dy}{dx}$. Hence, $Dy = \frac{dy}{x}$

$D^2y = D(Dy) = \frac{d^2y}{dx^2} \quad D^3y = D(D^2y) = \frac{d^3y}{dx^3} \quad$ and so on.

From the rules of differentiation we now see:

(i) $D[f(x) + g(x)] = Df(x) + Dg(x)$

where $f(x)$ and $g(x)$ are differentiable functions.

(ii) $D[cf(x)] = cDf(x)$, where $c$ is a constant.

(iii) $D^n[f(x)] = D[D^{n-1}f(x)]$ = $D^n f(x)$, where $m$ and $n$ are positive integers.

The general nth order linear differential equation with constant coefficients:

$$\frac{d^ny}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2 \frac{dy}{dx} + a_1 y = f(x) \quad (i')$$

$$(a_0 D + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n y = f(x) \quad (ii')$$

or, in symbolic form, as

$$F(D)y = f(x)$$

where:

$$F(D) = a_0 D + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n \quad (\text{polynomial operator in } D)$$

Now since $D$ behave as an algebraic quantity so also must $F(D)$ (which is only a linear combination of powers of $D$). Hence $F(D)$ may be factorised in the same way as algebraic expressions. For example,

$(D^2 + 4D + 3)y$ may be written as $(D+3)(D+1)y$

$(D^2 + 4D + 3) = (D+3)(D+1)$

$(D+3)(D+1)y = (D+1)(D+3)y$
we note here that the order of writing the factors of eq. (3) is important when \( a, a_1, a_2 \) are not constants but functions of \( x \). For example:

\[
(D + 2x)(D + 1)y = D^2 + 2x Dy + Dy + 2xy
\]

whereas:

\[
(D + 1)(D + 2x)y = D' + D(2xy) + Dy + 2xy
\]

Hence,

\[
(D + 2x)(D + 1)y \neq (D + 1)(D + 2x)y
\]

We now prove three useful theorems on the polynomial operator \( F(D) \) defined by

\[
y(x) \equiv e^{\int_{a}^{x} u \, dt}
\]

**Theorem 1:** If \( k \) is a constant,

\[
F(D) e^{kx} = F(k) e^{kx}
\]

\[
= (a_k e^{kx} + a_{k-1} e^{kx} + \cdots + a_1 e^{kx} + a_0) e^{kx}
\]

For example,

\[
(6D^2 + 3D + 2) e^{x} = (6x^2 + 3x + 2) e^{x} = 6s^2 e^{x}
\]

**Theorem 2:** If \( k \) is a constant and \( V(x) \) is an arbitrary function of \( x \),

\[
F(D) \sum_{n=0}^{\infty} \frac{k x^n}{n!} V(x)
\]

\[
= \sum_{n=0}^{\infty} \frac{k x^n}{n!} F(D + k) V(x)
\]

\[
= \sum_{n=0}^{\infty} \frac{k x^n}{n!} e^{D^2 + kx} V(x)
\]

\[
= \sum_{n=0}^{\infty} \frac{k x^n}{n!} e^{kx} V(x)
\]

For example:

\[
(6D^2 + 4D + 1) e^{x} = e^{x} \sum_{n=0}^{\infty} \frac{(n+1)(2n+1)}{2} x^n
\]

\[
= e^{x} \sum_{n=0}^{\infty} \frac{2x^n}{2} (D^2 + x^2) V(x)
\]

\[
= e^{x} (2D^2 + x^2) V(x)
\]

**Theorem 3:** If \( k \) is a constant,

\[
F(D^2) \sin kx = F(-k^2) \sin kx
\]

\[
F(D^2) \cos kx = F(-k^2) \cos kx
\]
For example:

\[
(D^4 + 3D^2 + 1) \sin 2x = \sum (-2^4) + 3(-2^2) - 13 \sin 2x = 3 \sin 2x
\]

\[
(D^2 - 2D) \cos 2x = \sum (2^2) - 2(-2) \cos 2x = 2 \cos 2x.
\]

**Inverse operator**

\[
D^n = \frac{1}{D}
\]

\[
D(D^n y) = y
\]

\[
\frac{d^n}{dx^n} y = D^n y
\]

**D-operator method for particular integrals:**

Consider the linear first order constant coefficient equation:

\[
(D-k) y = f(x)
\]

where \( k \) is a constant and \( f(x) \) a given function. The complementary function of this equation is:

\[
y = Ae^{kx}
\]

where \( A \) is constant. To find a particular integral \( y_p(x) \) we now assume the form:

\[
y(x) = V(x) e^{kx}
\]

Then by theorem 2:

\[
(D-k) y(x) = (D-k) \int e^{kx} V(x) \, dx = e^{kx} \int e^{kx} V(x) \, dx
\]

Then:

\[
y_p(x) = e^{kx} \int e^{-kx} f(y) \, dy
\]

Therefore, the general solution of equation is:

\[
y = A e^{kx} + e^{kx} \int e^{-kx} f(y) \, dy - \frac{1}{k} \int \frac{f(x)}{e^{-kx}} \, dy
\]

For \( k \neq 0 \),

\[
y = A e^{kx} + \int e^{kx} f(y) \, dy - \frac{1}{k} \int f(x) \, dy
\]
\[ \frac{d^3 y}{dx^3} \left( D^2 - 1 \right) y = x^2 \]

\[ y'' = e^x \]

The equation:

\[ \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad (1) \]

where \( a_0, a_1, \ldots, a_n \) are given constants, \( f(x) \) is an unspecified function of \( x \), is usually referred to as Euler's equation.

The general solution of this equation:

\[ X = e^x \quad (2) \]

\[ \frac{dy}{dx} = \frac{d^2 y}{dx^2} \frac{dx}{dt} = \frac{d}{dt} \frac{dy}{dx} \quad (3) \]

\[ X \frac{d^2 y}{dx^2} = \frac{dy}{dx} = \frac{d}{dt} \frac{dy}{dx} \quad (4) \]

and

\[ X^n \frac{d^n y}{dx^n} = D^n (D_{n-1} \cdots D_0 y) = (D^n - y_{n+1}) \quad (5) \]

where \( D^n \) is the operator \( \frac{d^n}{dt^n} \).

For example, using these results the equation:

\[ x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + 2y = x^2 \]

becomes

\[ \frac{d^2 y}{dt^2} + \frac{dy}{dt} + 2y = e^t \]

This may be solved using one of the earlier methods.
The general linear second order equation

The Euler equation discussed in the last section is a special case of the general linear second order equation:

\[ \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x) y = f(x) \]  

where \( p(x) \), \( q(x) \) and \( f(x) \) are given functions of \( x \). Usually this type of equation has to be solved by series approximation methods; otherwise we may proceed either by means of a substitution or by knowing one solution. The following examples illustrate these approaches.

**Ex. 1.**

\[ \frac{d^2y}{dx^2} + (4x - \frac{1}{x}) \frac{dy}{dx} + 4x^2 y = 0 \]  

Substituting \( x = \frac{1}{z} \):

\[ \frac{d^2y}{dz^2} + 2 \frac{dy}{dz} + y = 0 \]  

which has the solution:

\[ y = (A + Bz) e^{-z} \]  

Hence, the general solution is:

\[ y = (A + Bx^2) e^{-x^2} \]

**Ex. 2.**

\[ x^2 \frac{d^2y}{dx^2} - x(x+1) \frac{dy}{dx} + (x+2) y = 0 \]

One solution of this equation is \( y_1 = x \), we may find another solution by writing \( y_2 = x u(x) \) and solving for \( u(x) \):

Then \( \frac{du}{dx} - \frac{du}{dx^2} = 0 \)

by integration, gives:

\[ u = Ae^{x^2} + B \]

where \( A \) and \( B \) are constants.
The general solution of \( y'(x) \) is:

\[
y' = x(Ae^x + B) + cy
\]

where \( B \) is an arbitrary constant.

Simultaneous Equations

The general solution of simultaneous equations with constant coefficients in two or more dependent variables may be found by solving for each dependent variable separately.

**Example:** To solve the equations:

\[
\begin{align*}
\frac{dx}{dt} + 2y + 3x &= 0 \quad (1) \\
3x + \frac{dy}{dt} - 2y &= 0 \quad (2)
\end{align*}
\]

we first write the operator \( \frac{d}{dt} \) as \( D \) to give:

\[
\begin{align*}
(D + 3)x + 2y &= 0 \quad (D - 1) \\
3x + (D - 2)y &= 0
\end{align*}
\]

\[
(\begin{array}{l}
(D + 3)x + 2(D - 1)y = 0 \\
6x + 2(D - 2)y = 0
\end{array})
\]

By subtract eq. (1) from (4) gives:

\[
(D^2 + D - 6)x - 6x = 0
\]

or

\[
(D^2 + D - 12) x = 0
\]

The solution found to be: \( x(t) = ae^{2t} + be^{-4t} \)

Substitute into equation (2) to find \( y(t) \):

The solution is:

\[
y(t) = -3ae^{2t} + \frac{1}{2}be^{-4t}
\]
To solve:
\[
\frac{dx}{dt} + y = t^3 
\]
\[
\frac{dy}{dt} - x = t 
\]

By differentiating eqn 1 with respect to \( t \):
\[
\frac{dy}{dt} = 3t^2 - \frac{dx}{dt} 
\]

By inserting eqn 3 into eqn 2:
\[
\frac{d^2x}{dt^2} + x = 3t^2 - t 
\]

This equation may be solved by methods already discussed, the final solutions are:
\[
x = A \cos t + B \sin t + 3t^2 - t - 6 \\
y = A \sin t - B \cos t + t^2 - 6t + 1 
\]

Generally, for simultaneous equations:
\[
\frac{dX_1}{dt} = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 \\
\frac{dX_2}{dt} = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 \\
\frac{dX_3}{dt} = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 \\
\frac{dX_n}{dt} = a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 
\]
where the coefficients are zero.

We can write these simultaneous equations in matrix form as:

\[ \frac{dx}{dt} = Ax \]

where,

\[ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]  and \[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

The solution of (1) is easily verified to be:

\[ X = e^{At} X(0) \]

where \( X(0) \) is the column matrix of the values of \( x_1, x_2 \) and \( x_3 \) at \( t = 0 \) (assumed given), and,

\[ e^{At} = I + At + \frac{A^2}{2!} t^2 + \frac{A^3}{3!} t^3 \]
The Laplace Transformation

The Laplace transform \( \mathcal{L}\{f(t)\} \) of a function \( f(t) \) where \( t \geq 0 \) is defined by the integral:

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt
\]

where the variable \( t \) is assumed to be real. It is sometimes convenient to adopt the notation:

\[
\hat{f}(s) = \mathcal{L}\{f(t)\}
\]

where \( \mathcal{L} \) represents symbolically the Laplace transform.

\[
\mathcal{L}\{k f(t)\} = k \hat{f}(s) \quad \text{for \( k \) constant.}
\]

\[
\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}
\]

\( \alpha \) and \( \beta \) are constants; \( f(t), g(t) \) are functions of \( t \).

\( k > 0 \).

Simple Transforms

Using the definition (eq. (1)) we obtain the following transforms:

\[
L.T.: \int_0^\infty e^{-st} f(t) \, dt
\]

a) If \( f(t) = \alpha \) for \( t > 0 / \alpha \text{ (constant)} \):

\[
L.T. f(\alpha) = \frac{\alpha}{s}
\]

\( s > 0 \)

b) If \( f(t) = \alpha e^{kt} \text{ where} \alpha \text{ are real constants} \):

\[
L.T. f(t) = \frac{\alpha}{s - k}
\]

and for \( f(t) = k e^{-at} \rightarrow \hat{f}(s) = \frac{k}{s + a} \)
(c) If \( f(t) = A \sin wt \), where \( A, w \) are constants.

\[ f(s) = \frac{Aw}{s^2 + w^2} \quad s > 0 \]

(d) If \( f(t) = A \cos wt \)

\[ f(s) = \frac{AS}{s^2 + w^2} \quad s > 0 \]

(e) If \( f(x) = x^n e^{\pm w} \) where \( n \geq 1 \) or \( n < 0 \) then

\[ f(s) = \frac{n!}{s^{n+1}} \quad s > 0 \]

specific case:

\[ f(t) = e^{\frac{t}{2}} \]

\[ f(s) = \frac{1}{s + \sqrt{2}} \] (How?) \[ f'(s) = \frac{1}{s + \sqrt{2}} \]

(f) If \( f(x) = A \sinh w1x \) where \( A, w1 \) real constants.

\[ f(s) = \frac{Aw1}{s^2 - w^2} \quad s > w \]

(g) If \( f(x) = A \cosh w1x \) then

\[ f(s) = \frac{As}{s^2 - w^2} \quad s > w \]

another case:

\[ f(x) = e^{-a} \]

\[ f(s) = \frac{n!}{(s + a)^{n+1}} \quad s > -a \]

\[ f(x) = e^{a} \sin bx = \frac{b}{(s + a)^2 + b^2} \quad s > -a \]

\[ f(x) = e^{a} \cos bx = \frac{s + a}{(s + a)^2 + b^2} \quad s > -a \]
Inverse Transform

\[ L \left\{ f(t) \right\} = f(s) \]

\[ f(t) = L^{-1} \left\{ f(s) \right\} = \mathcal{L}^{-1} \left\{ f(s) \right\} \]

\[ LL^{-1} = L^{-1}L = 1 \]

**Ex. 1:**

\[ L^{-1} \left( \frac{k}{s} \right) = k \]

\[ L^{-1} \left( \frac{s}{s^2 + \omega^2} \right) = \cos \omega t \]

(Partial fractions)

(a) \( \sin \omega t \)

**Ex. 2:** To find \( L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\} \), where \( a \) and \( b \) are constants.

\[ \frac{1}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b} \quad (1) \]

Comparing the coefficients of powers of \( s \) on each side of equation (1),

\[ B = -A = \frac{1}{a-b} \quad \Rightarrow \quad A(s+b) = 1 \quad \Rightarrow \quad \frac{A}{s+a} = \frac{1}{s+a} \rightarrow A = \frac{1}{b-a} \]

\[ B(s+a) = 1 \quad \Rightarrow \quad B(a-b) = 1 \quad \Rightarrow \quad B = \frac{1}{b-a} \]

\[ L^{-1} \left\{ \frac{1}{(s+a)(s+b)} \right\} = \frac{1}{b-a} e^{-at} - \frac{1}{b-a} e^{-bt} = \frac{1}{b-a} \left( e^{at} - e^{bt} \right) \]
\[
\begin{align*}
S &= \frac{1}{b^2-a^2} \left( \frac{S}{s^2+a^2} - \frac{S}{s^2+b^2} \right) \quad b \neq a,
\end{align*}
\]

\[
A = \frac{a}{s^2+b^2}, \quad B = \frac{b}{s^2+a^2}
\]

\[
L^{-1} \left[ \frac{1}{s^2(s^2+2a^2)} \right] = \frac{A}{s^2} + \frac{B}{s^2+a^2}
\]

\[
A = -B = \frac{1}{2a^2}
\]

\[
A(s^2-a^2) = 1, \quad s \neq 0
\]

\[
B(s^2+2a^2) = 1, \quad s \neq 0
\]

\[
A \cdot 2 : 1 \rightarrow A = \frac{a}{b}
\]

\[
B \cdot 2 : 1 \rightarrow B = \frac{a}{b}
\]

\[
L^{-1} \left[ \frac{\frac{1}{s^2+a^2}}{s^2+b^2} \right] = \frac{1}{s^2} \frac{a}{s^2+b^2}
\]

\[
\frac{1}{a^2} - \frac{1}{a} L^{-1} \frac{1}{s^2} \frac{a}{s^2+b^2} = \frac{e^{-\frac{a}{s}}}{a^2} - \frac{1}{a^3} \sin(\frac{a}{s})
\]
(b) Repeated Roots:

\[
f(s) = \frac{A_1}{s + a} + \frac{A_n}{s + b} + \frac{C_1}{(s + b)^2} + \frac{C_2}{(s + b)^3}
\]

\[
f(1) = \frac{A_1}{s + a} + \frac{A_n}{s + b} + \frac{C_1}{s + b} + \frac{C_2}{(s + b)^2}
\]

Al, A_n, and C_2 are found as previous method.

\[
C_2 = \frac{d}{ds} \left[ \frac{f(s)}{(s + a)(s + b)(s + c)} \right]_{s = -c}
\]

\[
C_1 = \frac{1}{2!} \frac{d^2}{ds^2} \left[ \frac{f(s)}{(s + a)(s + b)(s + c)} \right]_{s = -c}
\]

Ex. \[ f(s) = \frac{s}{(s + 1)(s + 2)^2} \]

\[
\frac{s}{(s + 1)(s + 2)^2} = \frac{A_1}{s + 1} + \frac{C_1}{s + 2} + \frac{C_2}{(s + 2)^2}
\]

\[
A_1 = \left[ \frac{s}{(s + 2)^2} \right]_{s = -1} = -\frac{1}{1} = -1
\]

\[
C_2 = \left[ \frac{s}{s + 1} \right]_{s = -2} = 2
\]

\[
C_1 = \frac{d}{ds} \left[ \frac{5}{s + 1} \right] = \left[ \frac{1}{(s + 1)^2} \right]_{s = -2}
\]

\[
- \frac{1}{s + 1} \frac{1}{s + 2} + \int \frac{2}{(s + 2)^2} e^{\cdot -2t}
\]

\[
= e^{-t} + e^{-2t} + 2te^{-2t}
\]

\[
= e^{(1 + 2t)} e^{-t}
\]
(2) Quadratic Roots: $a_1 b_1 \frac{z}{z^2 + w^2}$

\[ f(s) = \frac{A_1}{s+a} + \frac{A_2}{s+b} + \frac{A_n}{s+m} = \frac{c_1 s + c_2}{s^2 + w^2} \]

by partial fractions method:

\[ f(s) = \frac{A_1}{s+a} + \frac{A_2}{s+b} + \frac{A_n}{s+m} = \frac{c_1 s + c_2}{s^2 + w^2} \]

$A_1$, $A_2$, $A_n$ obtained by previous method.

$c_1$, $c_2$, $c_3$:

\[ f(s) = \frac{A_1}{s+iw} + \frac{A_2}{s-iw} + \frac{A_n}{s+m} = \frac{c_1 s + c_2}{s^2 + w^2} \]

$s = iw$, where $i = \sqrt{-1}$

\[ (s^2 + w^2) x conjugate \text{ of } s \]

Example:

\[ f(s) = \frac{K}{(s^2 + w^2)(s^2 + w^3)} \]

\[ \frac{K}{(s^2 + w^2)(s^2 + w^3)} = \frac{A}{s + i} + \frac{c_1 s + c_2}{s^2 + w^2} \]

\[ A = \left[ \frac{K}{s + i} \right] s = -\frac{1}{2} = \frac{K + c_1 w}{s^2 + w^2} \]

\[ \frac{K}{s^2 + w^2} = \frac{A(s^2 + w^3) + c_1 s + c_2}{s^2 + w^2} \]
$$\frac{k}{(\tau^2+1)(\tau^2w^2+1)} = c_1 \tau w + c_2$$

Real part: $$c_2 = \frac{k}{1 + \tau^2 w^2}$$

Imaginary part: $$c_1 = -\frac{k \tau}{1 + \tau^2 w^2}$$

$$\int \frac{k}{(\tau^2+1)(\tau^2w^2+1)} \, d\tau = \int \frac{k \tau^2 / \tau w^2}{\tau^2+1} \, d\tau + \int \frac{k \tau s / \tau w}{s^2 w^2} \, ds + \int \frac{k / \tau w^2}{s \tau w} \, ds$$

$$= \frac{k \tau}{\tau w^2 + 1} e^{-\frac{k \tau}{\tau w^2}} - \frac{k \tau / \tau w}{\tau w^2 + 1} \cos \tau w + \frac{k / \tau w^2}{(\tau w^2 + 1)w} \sin \tau w$$

$$= \frac{k}{\tau w^2 + 1} \left[ e^{-\frac{k \tau}{\tau w^2}} + \frac{1}{w} \sin \tau w - \frac{k}{\tau w} \cos \tau w \right]$$
\[ f(x) = \frac{s + 1}{s^2 + 5s + 26.5} \]

\[ g(x) = \frac{s + 1}{s^2 + 5s + 26.5} \]

\[ \int f(x) \, dx = \frac{s + \frac{5}{2}}{(s + \frac{5}{2})^2 + (4.5)^2} - \frac{1.5}{4.5} \frac{4.5}{(s + \frac{5}{2})^2 + (4.5)^2} \]

\[ \int g(x) \, dx = \frac{s + \frac{5}{2}}{(s + \frac{5}{2})^2 + (4.5)^2} - \frac{1.5}{3} \frac{s + \frac{5}{2}}{s + \frac{5}{2}} \]
Initial & Final Value Theorems

(a) Initial Value Theorem:

\[ \lim_{t \to 0} \{ f(t) \} = \lim_{s \to \infty} \{ s f(s) \} \]

\[ f(t) = \frac{1}{s(s^2 + 25 + 25)} \]

\[ \lim_{t \to 0} \{ f(t) \} = \lim_{s \to \infty} \left[ s \times \frac{1}{s^3 + 25 + 25} \right] = 0 \]

(b) Final Value Theorem:

\[ \lim_{t \to \infty} \{ f(t) \} = \lim_{s \to 0} \{ s f(s) \} \]

\[ f(t) = \frac{5}{s(s+1)} \]

\[ \lim_{t \to \infty} \{ f(t) \} = \lim_{s \to 0} \left[ s \times \frac{5}{s(s+1)} \right] = 5 \]
Function derivatives

1. 1st order derivative:
   \[ \frac{dy}{dt} = \frac{y(t) - y(0)}{t - 0} \quad \text{where } y(t): \text{value of } y \text{ at } t \]

2. 2nd order derivative:
   \[ \frac{d^2y}{dt^2} = \frac{y(t) - 2y(0) + y(0)}{t - 0} \quad \text{where } y(t): \text{value of } y \text{ at } t \]

Example:

- \[ \frac{dy}{dt} = 2y + \cos t \]
  \[ t = 0 \quad y = 1 \quad \text{y(0)} = 1 \]
  \[ 1 = \frac{dy}{dt} = 2y + \cos t \]

- \[ y(0) = \frac{5}{5^2 + 1} + \frac{1}{5^2 + 1} \]

- Solution:
  \[ y(t) = \frac{5}{(5^2 + 1)} + \frac{1}{5^2 + 1} \]

- \[ \frac{5}{(5^2 + 1)} = A + \frac{B}{s^2 + 1} + \frac{C}{s + 1} \]

- \[ A(s^2 + 1) = 5 \] \[ 5 = A(5 + 1) \]

- \[ B = -1 \] \[ \quad A = \frac{2}{5} \]

- \[ \frac{5}{(5^2 + 1)} = A + \frac{B}{s^2 + 1} + \frac{C}{s + 1} \]
\[ S = i \omega \]

\[ \frac{B \cdot i \omega}{\omega + \xi} \quad \chi \bar{\chi} \]

\[ \frac{i \omega (\omega + \zeta)}{(\omega + \zeta)(\omega - \zeta)} = B \cdot \omega + C \]

\[ -\omega^2 - 2i \omega \]

\[ \frac{-\omega^2 - 2i \omega}{\omega + \chi} = B \cdot \omega + C \]

\[ \frac{\omega^2 + 2i \omega}{\omega + \zeta} = B \cdot \omega + C \]

\[ \beta = \frac{2}{\omega + \zeta}, \quad \chi = \frac{\omega^2}{\omega + \zeta} \]

\[ \text{Since } \omega = 1 \quad \text{(cos } \chi \text{)} \rightarrow \text{(cos } \omega \zeta \text{)} \]

\[ \beta = \frac{2}{5}, \quad \chi = \frac{1}{5} \]

\[ \therefore g(1) = -\frac{2}{5} + \frac{2}{5} \cdot \frac{5}{5} + \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{5 + 2} \]

\[ g(t) = \int \left( -\frac{2}{5} + \frac{2}{5} \cdot \frac{5}{5} + \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{5 + 2} \right) dt \]

\[ = -\frac{1}{5} e^{-\frac{t}{5}} + \frac{2}{5} \cos \frac{t}{5} + \frac{1}{5} \sin \frac{t}{5} + C \]

\[ = \frac{1}{5} \sin \frac{t}{5} + \frac{2}{5} \cos \frac{t}{5} + \frac{3}{5} e^{-\frac{t}{5}} \]
Ex. \( \frac{\text{d}y}{\text{d}t} + ay = \sin bt \)

\[ s^2 y(s) - sy(0) - y'(0) + a^2 y(s) = \frac{b}{s^2 + b^2} \]

\[ y(s) \left( s^2 + at \right) - s y'(0) - y(0) = \frac{b}{s^2 + b^2} \]

\[ y(s) = \frac{b}{(s^2 + b^2)(s^2 + at^2)} + \frac{sy(0)}{s^2 + at^2} + \frac{y'(0)}{s^2 + b^2} \]

\[ A \frac{b}{s^2 + at^2} \]

\[ \frac{b}{(s^2 + b^2)(s^2 + at^2)} \]

By partial fractions:

\[ A = \frac{b}{(s^2 + b^2)} \]

\[ \frac{b}{(s^2 + at^2)} \]

\[ y(s) = \frac{b}{s^2 + at^2} \left( \frac{1}{s^2 + at^2} - \frac{1}{s^2 + b^2} \right) + \frac{y(0)}{s^2 + at^2} + \frac{y'(0)}{s^2 + b^2} \]

\[ y(t) = \frac{b \sin at}{a (s^2 + at^2)} - \frac{b \sin bt}{s^2 + b^2} + \frac{y(0) \cos at}{s^2 + at^2} + \frac{y'(0) \sin at}{s^2 + b^2} \]
Ex.: Heating tank $A$ used to heat water by using steam coil, as shown in the figure. Heat losses to ambient by natural convection.

Find the temperature response equation of the tank $(T_2)$ as a function of time $t$ if $T_1$ varied as a ramp change as $(T_{1(t)} = \alpha t)$, where $\alpha$: constant.

\[ F \times \Delta \text{water} = h_1 A (T_s - T_1) \]

\[ A = \text{area} \quad h = \frac{J}{\text{water}} \quad J = \text{energy} \quad C_p = \frac{J}{\text{water} \cdot \text{temp}} \]

\[ F = \text{flow rate} \quad \text{V} = \text{volume} \text{ of water} \]

\[ h_2 A (T_2 - T_a) \]

\[ h = \frac{J}{\text{water} \cdot \text{temp}} \]

\[ -h_2 A (T_2 - T_a) \]

\[ Fc \Delta T_1 = h_1 A (T_s - T_1) \]

\[ Fc \Delta T_1 + h_1 A (T_s - T_2) - Fc \Delta T_2 = \frac{d(\rho v c A T_2)}{dt} \]

\[ Fc \Delta T_1 + h_1 A (T_s - T_2) - Fc \Delta T_2 + h_2 A (T_2 - T_a) = \frac{\rho A T_s}{dt} \]

\[ F = \rho v c A T_2 \]

\[ dT_2 \quad + \frac{h_1 A + Fc}{\rho v c} T_2 \quad + \frac{(h_1 A + Fc) T_2}{\rho v c} \quad + \frac{h_2 A (T_2 - T_a)}{\rho v c} \quad = \frac{Fc}{\rho v c} \alpha t \]

\[ \frac{dT_2}{dt} + \alpha T_2 + b + c = \beta t \]

By taking $L.T.$ to eq. $(2)$, note: for $t > 0 \quad a + b + c \quad (\text{constant})$

\[ a + b + c \quad = \text{neglected from equilibrium} \]
\[
S T_2(t) - T_2(0) + a T_2(t) \text{ where } \frac{\beta}{s} = \beta
\]

\[
T_2(t) = T_2(0) + \frac{\beta}{s^2} t + \frac{\beta}{s^3} (s+a)
\]

\[
T_2(t) = T_2(0) + \frac{\beta}{s(s+a)}
\]

\[
T_2(t) = \int_0^t T_2(0) dt + \int_0^t \frac{\beta}{s(s+a)} dt
\]

\[
= (T_2(0) \Delta t) e^{-at} + \frac{\beta}{s^2(s+a)}
\]

\[
\frac{\beta}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}
\]

\[
A(s+a) = 1
\]

\[
A = \frac{1}{a}
\]

\[
B s^2 = 1
\]

\[
B = \frac{1}{a}
\]

\[
T_2(t) = (T_2(0) \Delta t) e^{-at} + \frac{\beta}{a} t + \frac{\beta}{a^2} e^{-at}
\]

or

\[
T_2(t) = \frac{\beta}{a} t + (\frac{\beta}{a} \Delta t) e^{-at}
\]

\[\text{Original equation.}\]
well-mixed

Ex: \[ \text{CSTR} \text{ carried out the first order reaction } (A \rightarrow B) \text{ at isothermal condition. If the inlet flow rate varies suddenly as } \dot{b}, \text{ find the response equation of the outlet concentration.} \]

\[ \text{i.e. } C_2(t) = \dot{C}(t) \text{, outlet tank} \]

\[ A \xrightarrow{K} B \text{, } CA = C_2 \]

\[ \dot{V} = kCA = kC_2 \]

\[ V = \text{volume} = m^3, \text{ } C_1 \& C_2: \text{mole/m}^3, \text{ } k = \text{sec}^{-1}, \text{F: flow rate} \text{ m}^3/\text{sec}. \]

\[ \dot{V} = \frac{\text{mole}}{\text{m}^3 \cdot \text{sec}} \]

\[ \text{rate of mass in} = \text{rate of mass reacted} = \text{accumulated mass} \]

\[ FCI - FCE = \dot{V} \frac{dc_2}{dt} \]

\[ FCI - FCE - kVc_2 = \dot{V} \frac{dc_2}{dt} \hspace{1cm} (i) \]

\[ \dot{V} \frac{dc_1}{dt} + c_2 (F + kV) = FCI \]

\[ \frac{dc_2}{dt} + \frac{(F + kV)}{V} c_2 = \frac{F}{V} c_1 \hspace{1cm} (ii) \]

\[ \text{assume: } a = \frac{F + kV}{V}, \text{ } b = \frac{F}{V} \]

\[ \frac{dc_1}{dt} + a c_2 = b c_1 \hspace{1cm} (iii) \]

By taking L.T. to (ii) \( \beta \), where \( C_{10E} \frac{1}{s} \)

\[ s \text{ } CE(s) = C_2(s) + a \text{ } C_2(s) = \frac{b}{s} \text{ } \]

\[ \therefore C_2(s) (s + a) = \frac{b}{s} + C_2(s) \]
\[ C_2(s) = \frac{b}{s(s+a)} + \frac{C_2(0)}{s+a} \]

\[ C_2(t) = \int \frac{b}{s(s+a)} \, ds + \int \frac{C_2(0)}{s+a} \, ds \]

\[ = \frac{b}{s+a} \int \frac{1}{s+a} \, ds + \int \frac{C_2(0)}{s+a} \, ds \]

\[ = \frac{b}{s+a} \ln |s+a| + C_2(0) \ln |s+a| \]

\[ \text{Response equation.} \]

\[ C_2(t) = \frac{b}{a} (1 - e^{-at}) + C_2(0) e^{-at} \]

\[ H.W. = \text{Problems 1-3} \]

\[ C_1(t) = \sin 2t \]
A linear differential equations with variable coefficients can often be solved by assuming a solution in the form of a power series in \( x \). The series itself can be considered to be a solution of the differential equation. For practical use, it is necessary to consider the convergence of the series before differential equations can be solved by their use.

**Infinite Series**

A series of numbers: \( u_1 + u_2 + u_3 + \cdots + u_n = S_n \) is classified by determining the behaviour of \( S_n \) as \( n \) increases to infinity.

(a) If \( S_n \to c \) as \( n \to \infty \), the series is termed 'convergent'.

(b) If \( S_n \to \infty \) as \( n \to \infty \), the series is termed 'divergent'.

(c) In other cases, the series is termed 'oscillatory'; the values of \( S_n \) may oscillate between either finite or infinite limits.

\( u_n \) will be a function of \( x \), then \( S_n \) will also be a function of \( x \).

\( x \) is positive or negative values, or even complex values.

\[ |u_n| = \text{absolute value of } u_n. \]

The series (eq. 1) is said to be 'absolutely convergent' if the series

\[ |u_1| + |u_2| + |u_3| + \cdots |u_n| \]

is convergent.

**EX.**

\[ u_n = z^n \]

\[ S_n = z + z^2 + z^3 + \cdots + z^n \]
This series is a geometric progression with common ratio $z$:

$$S_n = \frac{z(1-z^n)}{1-z} = \frac{z}{1-z} \cdot \frac{z^n - 1}{z - 1}$$

(a) If $|z| < 1$, $\lim_{n \to \infty} \frac{z^n}{1-z} = 0$, as $n \to \infty$. Thus, $S_n = \frac{z}{1-z}$.

Hence the series is convergent.

(b) If $|z| > 1$, $\lim_{n \to \infty} \frac{z^n}{1-z} \to \infty$, as $n \to \infty$. The series diverges if $z$ is real and positive.

(c) If $z = 1$, $S_n = n$, which tends to infinity as $n$ increases indefinitely. Hence, the series is divergent.

(d) If $|z| = 1$ and $z \neq 1$, the series oscillates between finite limits.

**Comparison Test.**

The comparison test is the simplest of all and consists of two parts:

(i) If $|U_n| \leq V_n$ for all $n > N$ where $N$ is finite integer, and $\Sigma V_n$ is convergent, then $\Sigma U_n$ is absolutely convergent.

(ii) If $|U_n| \geq V_n$ for all $n > N$ and $\Sigma V_n$ is divergent to $\infty$, then $\Sigma U_n$ is also divergent.

**Example:**

$$U_n = n^p \quad (p \text{ real})$$

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} = \Sigma \frac{1}{n^p}$$

The above series has different properties for different values of $p$:

(a) $p > 1$ (convergent)

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} = \Sigma \frac{1}{n^p}$$

(b) $p = 1$ (convergent)

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \Sigma \frac{1}{n}$$

(c) $p < 1$ (divergent)
(C) $P < 1$ diversifiable (unbounded)

**Ratio test.**

(i) If $\left| \frac{u_n}{u_{n+1}} \right| > 1$ for all $n > N$, then the series is absolutely convergent.

(ii) If $\left| \frac{u_n}{u_{n+1}} \right| < 1$ for all $n > N$, then the series is not convergent.

**Example:**

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \rightarrow \frac{1}{n}$

$\left| \frac{u_n}{u_{n+1}} \right| = \frac{n+1}{n} = 1 + \frac{1}{n} > 1$ (divergent)

---

**Type of series**

a) **Alternative series:**

$u_1 - u_2 + u_3 - u_4 + \cdots$ \quad If $u_n \to 0$ as $n \to \infty$ then the series is convergent.

b) **Power series.**

$a_0 + a_1 z + a_2 z^2 + \cdots \Rightarrow \sum a_n z^n$

If $\left| \frac{a_n}{a_{n+1}} \right| \to R$ as $n \to \infty$, the series convergent.
c) Binomial series:

The series obtained by expanding \((1 + Z)^p\) is:

\[
1 + pZ + \frac{p(p-1)}{1 \times 2} Z^2 + \frac{p(p-1)(p-2)}{1 \times 2 \times 3} Z^3 + \cdots = \sum_{n=0}^{\infty} \binom{p}{n} Z^n
\]

where \(\binom{p}{n}\) signifies the binomial coefficient; \(Z\), may be complex, but \(p\) is real.

\[
a_n = \frac{p(p-1)(p-2)\cdots(p-n+1)}{1 \times 2 \times 3 \cdots n} = \frac{p(!)}{(p-n)!!}
\]

\[
\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+1}{p-n} \rightarrow 1 \text{ as } n \rightarrow \infty
\]

Therefore, the binomial series is convergent for \(|Z| < 1\).

d) Exponential series:

\[
e^Z = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{Z^n}{n!}
\]

Here, \(a_n = \frac{1}{n!}\) and \(\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+1}{n} \rightarrow 1 \text{ as } n \rightarrow \infty\)

Therefore, the exponential series is always convergent.

e) Logarithmic series:

\[
\ln(1+Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \frac{Z^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} Z^n}{n}
\]

Here, \(a_n = (-1)^n (n + 1)\) and \(\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+1}{n} \rightarrow 1 \text{ as } n \rightarrow \infty\)

Therefore, the series for \(\ln(1+Z)\) is convergent for \(|Z| < 1\).
f) Trigonometric Series:
\[
\sin Z = Z - \frac{Z^3}{3!} + \frac{Z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n Z^{2n+1}}{(2n+1)!}
\]
\[
\cos Z = 1 - \frac{Z^2}{2!} + \frac{Z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n Z^{2n}}{(2n)!}
\]

These two series are convergent for all values of Z.

g) Hyperbolic Series:
\[
\sinh Z = Z + \frac{Z^3}{3!} + \frac{Z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{Z^{2n+1}}{(2n+1)!}
\]
\[
\cosh Z = 1 + \frac{Z^2}{2!} + \frac{Z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{Z^{2n}}{(2n)!}
\]

The two series are convergent for all values of Z.

Method of Frobenius

The linear differential equation with constant coefficients was solved and shown to have solutions for the complementary function of the types:
\[
y_1 = A_1 e^{wx} + A_2 e^{wx}
\]
\[
y_2 = (A + Bx) e^{wx}
\]
\[
y_3 = (A \cos wx + B \sin wx) e^{wx}
\]

Regarding the previous section (type of series), all of their solutions can be expanded in ascending power series of x. Thus, for the first case,
\[
y = A(1 + mx + \frac{1}{2} m^2 x^2 + \cdots) + B(1 + mx + \frac{1}{2} m^2 x^2 + \cdots)
\]

This accepted solution if series is convergent.
The series always used for differential equation with variable coefficients (expected solution by series).

If the general second order diff. equs cannot be expressed in the form

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + F(x) y = 0 \quad (1) \]

where:

\[ F(x) = F_0 + F_1 x + F_2 x^2 + \cdots \quad (2) \]

and

\[ G(x) = G_0 + G_1 x + G_2 x^2 + \cdots \quad (3) \]

with eqns (2) and (3) having a radius of convergence \( R \); then the equation can be solved completely by the method of Frobenius in the form of a power series which is also convergent for \( |x| \leq R \).

In order to solve eqn (1), put

\[ y = \sum_{n=0}^{\infty} a_n x^n \quad \text{(eqn)} \quad (4) \]

where \( c \) is constant (usually fractional).

\[ \frac{dy}{dx} = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (5) \]

\[ \frac{d^2 y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad (6) \]

Substituting these equations into eqn (1) gives:

\[ \sum_{n=0}^{\infty} n(n+1) a_n x^n + (F_0 + F_1 x + F_2 x^2 + \cdots) \sum_{n=0}^{\infty} a_n x^n = 0 \quad (7) \]

The power of \( x \):

\[ \sum_{n=0}^{\infty} \frac{n(n+1) a_n}{c(n+1)} x^n = 0 \quad (7) \]

Power \( x \):

\[ c_0 + c F_0 \equiv c_0 + c_0 = 0 \]

\[ c^2 + (F_0 - 1) c + G_0 = 0 \quad (8) \]
This quadratic equation \( \Rightarrow \) (indicial equation).

The coefficients of next power \( x \)

\[ a_{1} (1+c) x + \sum_{i=1}^{\infty} (x+c)^{i} + C x^{2} + \sum_{i=1}^{\infty} C x^{i} \]

\[ a_{0} (x+c)^{2} + C (x+c)^{2} + \sum_{i=1}^{\infty} (x+c)^{i} \]

\[ = \frac{C (x+c)^{2}}{\sum_{i=1}^{\infty} (x+c)^{i}} \]

\[ = \frac{C (x+c)^{2}}{\sum_{i=1}^{\infty} (x+c)^{i}} \]

Then determines \( a_{0} \) in terms of \( a_{0} \) and \( C \).

In general, the coefficient of \( a_{j} \) in the equation obtained by equating coefficients of \( x \) can be obtained from equation (8) by replacing \( C \) by \( (c+j) \).

If the two roots of equation (8) are denoted by \( c_{1} \) and \( c_{2} \) and they differ by a positive integer \( j \) so that \( c_{2} = c_{1} + j \) when the root \( c_{2} = c_{1} \) is used, the coefficient of \( a_{j} \) on the first occasion that it appears will be zero by the above explanation since \( C c_{1} + j \) satisfies eq(8).

**Case I.** Roots of Indicial Equation Different but not by an Integer.

\[ S_{x} \]

\[ u \times \frac{dy}{dx} + 6 \frac{dy}{dx} + y = 0 \]

\[ y = C_{o} x^{c} + a_{1} x^{c+1} + \ldots = \sum_{n=0}^{\infty} a_{n} x^{n+c} \]

\[ \frac{dy}{dx} = a_{0} c x^{c+1} + a_{1} (c+1) x^{c+2} = \sum_{n=0}^{\infty} a_{n} (n+c) x^{n+c+1} \]

\[ \frac{d^{2}y}{dx^{2}} = a_{0} c (c-1) x^{c-1} + a_{1} (c+1) x^{c+1} = \sum_{n=0}^{\infty} a_{n} (n+c) (n+c+1) x^{n+c+2} \]

These relations may be substituted into eq (1) to give:

\[ 4 \left[ a_{0} c (c-1) x^{c-1} + a_{1} (c+1) x^{c} + \ldots \right] + 6 \left[ a_{0} c x^{c} + a_{1} (c+1) x^{c+1} + \ldots \right] \]

\[ + a_{0} x^{c} + a_{1} x^{c+1} + \ldots = 0 \]
Equating coefficients of like powers of \( x \), starting with the smallest \( x \):

\[
4a_0 c (c-1) + 6a_0 c = 0
\]

\( \Rightarrow \) Indicial eqn.

\[
4a_1 (c+1) c + 6a_1 (c+1)c + a_0 = 0
\]  \((2)\)

\[
4a_2 (c+2) (c+1) + 6a_2 (c+2) + a_1 = 0
\]  \((5)\)

and for the coefficient of \( x^r \),

\[
4a_{r+1} (c+r+1) (c+r) + 6a_{r+1} (c+r+1) + a_r = 0
\]  \((6)\)

The indicial equation \((2)\) has the two roots:

\[ c = 0 \quad \text{or} \quad c = -\frac{1}{2} \]

At \( c = 0 \), the equation \((5)\) becomes:

\[ 6a_1 + a_0 = 0 \quad \Rightarrow \quad a_1 = -a_0/6 \]

\[ 8a_2 + 12a_2 + a_1 = 0 \quad \Rightarrow \quad a_2 = -a_1/10 \]

and

\[ 4(r+1)a_{r+1} + 6(r+1)a_r + a_{r-1} = 0 \]

\[ \Rightarrow \quad \frac{a_{r+1}}{a_r} = \frac{-1}{(2r+2)(2r+1)} \]  \((7)\)

Equation \((7)\) true for \( r \geq 3 \), the previous assumed answer are confirmed by using:

and \( r = 1 \). Replacing \( r \) with \((r-1)\) gives:

\[ \frac{a_r}{a_{r-1}} = \frac{-1}{2r(2r+1)} \]

Repeating this process, by reducing the value of \( r \) by unity each time:

\[ \frac{a_{r-1}}{a_{r-2}} = \frac{-1}{(2r-2)(2r-1)} \]

Until:

\[ \frac{a_1}{a_0} = \frac{-1}{2(3)} \]

\[ \frac{a_r}{a_0} = \frac{(-1)^r}{(2r+1)!} \]  \((8)\)

\[ a_2 = \frac{a_1}{2!} \]

\[ a_3 = \frac{a_2}{3!} \]

\[ a_4 = \frac{a_3}{4!} \]

\[ \vdots \]
The solution therefore takes the form:

\[ y_j = a_0 \sum_{0}^{\infty} \left( \frac{-1}{(2n+1)!} \right)^n x^n \]

\[ \Rightarrow \sin x = a_0 \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

\[ = a_0 \left( 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right) \quad -(9) \]

\[ \therefore \text{for } c = 0 \text{ the solutions:} \]

\[ y_j = a_0 \frac{\sin \sqrt{x}}{\sqrt{x}} \quad -(10) \]

\[ \text{for } c = -\frac{1}{2} \text{ equations (4 & 6) become:} \]

\[ -a_1 + 3a_1 + a_4 = 0 \quad \text{and} \quad (2r+1)(2r-1) + 3a_{2r} + 7a_{2r-1} + a_{2r-2} = 0 \]

Rearranging:

\[ \frac{a_{2r+1}}{a_{2r+1-1}} = \frac{-1}{(2r+1)(2r-1)} \]

Similarly:

\[ \frac{a_{2r}}{a_{2r-2}} = \frac{-1}{(2r-1)2r} \]

\[ \begin{align*}
\text{when:} & \quad \frac{a_{1}}{a_{0}} = \frac{-1}{1 \times 2} \\
\text{Multiply these equations together gives:} & \quad \frac{a_{2}}{a_{2}} = \frac{-1}{2 \times 1} \times \frac{(-1)^2}{4 \times 3} \\
\text{Substitution into the general series gives the solution:} & \quad a_x = \frac{(-1)^n a_n}{(2n)!} \quad -(11) \\
\cos x: & \quad y_2 = a_0 x^\frac{1}{2} \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) = \sum_{0}^{\infty} \frac{(-1)^n a_n x^{n+\frac{1}{2}}}{(2n)!} \\
\therefore & \quad y_2 = a_0 \left( \cos \sqrt{x} \right) \sqrt{x} \quad -(12) \\
\text{The general solution of equation (1) is:} & \quad y = y_1 + y_2 = (A \sin \sqrt{x} + B \cos \sqrt{x}) \sqrt{x} \quad -(13) \]

Case II  Roots of Indicial equation equal:

\[ c_1 = c_2 \]

The complete solution:

\[ y = A u(x; c_1) + B u(x; c_2) \]

\[ A \neq B \text{ constant} \]

\[ y = A u(x; c_1) + B \left[ \frac{u(x; c_2) - u(x; c_1)}{c_2 - c_1} \right] \]

\[ y = A u(x; c_1) + B \left( \frac{du}{dc} \right)_{c = c_1} \]

\[ y = c_1 \cdot \frac{du}{dc} \]

\[ \text{Ex: } x \frac{dy}{dx} + (1-x) \frac{dy}{dx} - y = 0 \]

Put \( y = \frac{a}{x} \cdot e^{\alpha x} \)

By substituting the 1st & 2nd differentiation into eqn. and equate coefficients of \( x^0, x, x^2 \) and \( \frac{1}{x} \) to obtain:

\[ a_0 \cdot c (c-1) + a_0 c = 0 \]

\[ a_1 (c+1)^2 = a_0 (c+1) \]

and \( a_2 (c+1) = a_0 (c+1) \)

The indicial equation \( a_1 = 0 \) has double root \( c = 0 \).

\[ a_2 (c+2)^2 = a_0 (c+1) \]

\[ \frac{a_1}{a_0} = \frac{1}{c+1} \]

\[ \frac{a_2}{a_0} = \frac{1}{(c+1)(c+2)} \]

\[ \frac{a_{n+1}}{a_n} = \frac{1}{(c+n)(c+n-1)} \]
\[ u(x) = \sum_{n=1}^\infty \frac{a_n}{x^n} \ln x \]
Case III: Roots of Indicial Equations differing by an integer

The case is very similar to case I, in that the 2nd solution involves a partial differentiation with respect to \( C \); thus generating a term \( \ln x \).

\[ y = A u(x/C) + B \frac{d}{dC} \left[ (C-C_0) u(x/C) \right] \]

\[ x(1-x) \frac{d^2y}{dx^2} + (2-5x) \frac{dy}{dx} - 4y = 0 \]

with the usual substitution, so that the indicial equation is:

\[ a_0 C [C - 1] + 2 a_0 C = 0 \]

(2)

The 1st recurrence relation is:

\[ a_1 (C + 1) = a_0 (C + 2) \]

and the general recurrence relation is:

\[ a_{k+1} = \frac{Y + C + 2}{Y + C + 1} a_k \]

(4)

The roots of the indicial equation are:

\[ C_1 = -1 \text{ or } C_2 = 0 \]

The difference is integer (between roots):

when \( C = -1 \) equation (3):

\[ a_{-1} = a_0 \]

\[ a_0 = a_0 \]

\[ a_1 = a_0 \]

\[ a_2 = a_0 \frac{a_1}{a_0} \]

\[ a_3 = a_0 \frac{a_2}{a_1} \]

\[ a_4 = a_0 \frac{a_3}{a_2} \]

\[ a_n = a_0 \frac{a_{n-1}}{a_{n-2}} \]

(11)

Thus, the series for \( C = -1 \) (no values).

The series for \( C = 0 \) is still valid.
The equation (1.1) becomes
\[ a_{n+1} = \left( \frac{x+2}{n+1} \right) a_n \]
and
\[ a_{n+1} = \frac{n+2}{n+1} a_n \]

When \( n = 0 \),
\[ y_1 = a_0 \left( \frac{1}{1-x} \right) \]

By differentiation and substitution in Eq. (1),
\[ x(1-x) \frac{dy}{dx} + (2+5x) \frac{dy}{dx} - 4y = a_0 e^{(c+1)x} \]

For \( c = 0 \), the term \( a_0 e^{(c+1)x} x^0 = 0 \) in the right-hand side of Eq. (7), also \( \lim_{x \to -1} y(x-1) = \infty \).

An artificial way of generating second solution by multiplying
Eq. (7) by the factor \((c+1)\) to cancel out the denominator of \(U(x)c\).

\[ y_2 = \frac{d}{dx} \left[ (c+1) U(x)c \right]_{c=1} \]

But from Eq. (6):
\[ (c+1) U(x)c = \sum_{n=0}^{\infty} \left( \frac{n+1}{n+2} \right) a_0 x^n \]

\[ \frac{d}{dx} \left( c+1 \right) U(x)c = \sum_{n=0}^{\infty} a_0 x^n \left[ (n+1)(\ln x + 1) \right]_{c=1} \]

\[ y_2 = a_0 \left( \ln x \right)(1 + 2x + 3x^2 + \cdots) + a_0 \left( \frac{1}{1-x} \right) \]
Once again, the term multiplying \( Bx \) is the first solution. The complete solution of equation (1) is thus:

\[
y = \frac{A}{(1-x)^2} + \frac{B\ln x}{(1-x)^2} + \frac{B}{x(1-x)}
\]

Case 1b. Roots of Indicial Equation Differing by an Integer.

There is just one further variation in the type of solution which can arise in the method of Frobenius. In the last example, equation (3) gave:

\[
a_1 (c+1) = a_0 (c+2)
\]

and when \( c = -1 \), \( a_1 \) became infinite. Had the right-hand side also contained a factor \( (c+1) \) this particular difficulty would not have arisen; equation (3) would have been automatically satisfied, and both parts of the solution would have been normal power series as in case 1.

**Ex.**

\[
x \frac{dy}{dx} + (x-1) \frac{dy}{dx} - y = 0
\]

With the usual series substitution, as follow the same method as before, the indicial equation is:

\[
a_0 c (c-1) - a_0 c = 0
\]

and the general recurrence relation is:

\[
x a_{n+1} (c+1)(c+n+1) + a_n (c+n+1) = 0
\]

Since the roots of equation (1) differ by 2, consider the first of equation (3), in which \( a_1 \) occurs:

\[
a_2 (c+2) e + a_1 c = 0
\]
\[ \text{when } C = 0 \text{, the coefficient of } x^3 \text{ in equation (3) is zero, but both terms in equation (4) are zero when } C = 0 \text{ and } a_3 \text{ remains indeterminate instead of infinite.} \]

\[ \therefore \text{If } C = 0, \text{ eqns. can be simplified by cancellation, but if } C = 0, \text{ the term } (x^3 + 1) \text{ can only be cancelled if } x \neq 1 \]

\[ = a_4 + (x + 1) = -a_4 \text{ (unless } C = 0, \ y = 1) \]

The 1st solution is given by \( C = 2, \) thus:

\[ a_4 + (x + 3) = -a_4 \]

and by successive substitution:

\[ a_{n+1} = \frac{(-1)^n 2a_0}{(n+3)!} \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n 2a_0}{(n+3)!} x^{n+3} = a_0 (e^{-x} - 1 + x) \]

The 2nd solution when \( C = 0, \) becomes:

\[ y_2 = a_0 - a_0 x + a_2 x^2 - \frac{a_3 x^3 + a_2 x^4}{3} \]

\[ = a_0 (1 - x) + 2a_2 \left( x - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots \right) \]

\[ = a_0 (1 - x) + 2a_2 \left( e^{-x} - 1 + x \right) \]

The complete solution of eqn. (1):

\[ y = A (1 - x) + B e^x \]

where \( A = a_0 - 2a_2 \) and \( B = 2a_2. \)
Example 1: A supply of hot air is to be obtained by drawing cool air through a heated cylindrical pipe (insulated).

D = 0.1 m, L = 1.2 m, maintained at Tw = 300°C,

Flux rate (W) = 0.009 W/m²

Outlet lim: 20°C, overall h = 50 W/m²K

where x is the distance measured from the pipe inlet.

\[ \text{Solution:} \]

Heat balance:

Input heat rate - output heat rate = accumulated heat.

(by convection & by conduction)

\[ -kA \frac{dT}{dx} + \rho C_p T \left( \frac{dT}{dx} \right) + \rho \left( \frac{dH_T}{dx} \right) = 0 \]

\[ \text{or} \quad \frac{dT}{dx} = -\frac{\rho C_p T \frac{dT}{dx} + \rho \left( \frac{dH_T}{dx} \right)}{kA} \]

\[ \text{or} \quad \frac{dH_T}{dx} = -\frac{kA}{\rho C_p} \frac{dT}{dx} \quad \text{(1)} \]

\[ \text{or} \quad \frac{dH_T}{dx} = -\frac{kA}{\rho C_p} \frac{dT}{dx} \quad \text{(2)} \]

\[ \text{or} \quad \frac{dH_T}{dx} = -\frac{kA}{\rho C_p} \frac{dT}{dx} \quad \text{(3)} \]

\[ \text{or} \quad \frac{dH_T}{dx} = -\frac{kA}{\rho C_p} \frac{dT}{dx} \quad \text{(4)} \]
Put \( x = Z^2 \)
\[
2 \frac{dk}{dz} - (1 + 52400 Z^2) \frac{dk}{dz} = 45720 Z^2 k \to 0
\]

By the method of Frobenius,

\[ C = 0 \text{ or } 2 \text{ from the indicial equation.} \]

By Taking \( C = 2 \to a_1 = 0 \to a_3 = 994.8 a_0 \)
\[ a_1 = 180 a_0 \to a_{10} = 2.1 \times 10^8 a_0 \]

The tail of the series must be converted but it need to have at least 105 terms for the 5th solution.

The trouble arises because two of the coefficients in \( k \) are much larger than the coefficient of \( (dk/dx^2) \).

The 2nd derivative arises from the gas conduction term is neglected. Therefore, equate between

\[
\frac{dk}{dx} + 0.136 Z^2 k = 0
\]

By separation method,

\[ t = C \exp(-0.0872 Z^2) \]

B.C. \( x = 0.1 \to t = 300 - 280 = 280 \to C = 280 \)
\[ t = 280 \exp(-0.0872 Z^2) \]
\[ T = 300 - 280 \exp(-0.0872 Z^2) \]

for \( x = 1.2 \to T = 192 e^0 \text{ (exit gas temperature).} \)
Bessel's Equation

The equation:

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0 \]  

where \( \nu \) is a real constant, is called Bessel's equation and its solutions are the Bessel functions of order \( \nu \). Applying the Frobenius method and assuming a series solution of (1) of the type:

\[ y_j = \sum_{m=0}^{\infty} a_m x^{\nu+m} \]  

we obtain the recurrence relation:

\[ a_{\nu+2} = -\frac{a_\nu}{(\nu+2)(\nu+1)} \]  

where \( a_0 = 0 \) and \( \nu \neq 0 \).

From the indicial equation:

Since \( a_0 = 0 \), eq. (3) gives \( a_2 = a_4 = \cdots = 0 \) and hence, with \( m = \nu \),

we have the series solution:

\[ y_j = a_0 x^\nu \left[ 1 - \frac{x^2}{2(2\nu+2)} + \frac{x^4}{2.4(2\nu+2)(2\nu+4)} - \cdots \right] \]  

provided \( \nu \) is not a negative integer.

Similarly with \( m = -\nu \), we obtain series:

\[ y_j = a_0 x^{-\nu} \left[ 1 + \frac{x^2}{2(2\nu-2)} + \frac{x^4}{2.4(2\nu-2)(2\nu-4)} + \cdots \right] \]  

provided \( \nu \) is not a positive integer.
The general solution of Bessel's equation for non-integral \( \nu \) is therefore,

\[
Y = A J_{\nu}(x) + B J_{-\nu}(x)
\]  \hspace{2cm} (6)

where \( A \) and \( B \) are arbitrary constants.

If \( \nu = n \), where \( n \) is a positive integer,

\[
J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! (n+r)!} \left( \frac{x}{2} \right)^{n+r}  \hspace{2cm} (7)
\]

Likewise,

\[
J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! (-n+r)!} \left( \frac{x}{2} \right)^{-n+r}  \hspace{2cm} (8)
\]

Now the first \( n \) terms of this series are zero, hence putting \( \nu = n+p \) in eq(8):

\[
J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! (-n+r)!} \left( \frac{x}{2} \right)^{-n+r}  \hspace{2cm} (9)
\]

\[
= \sum_{p=0}^{\infty} \frac{(-1)^{n+p}}{p! (n+p)!} \left( \frac{x}{2} \right)^{n+p}  \hspace{2cm} (10)
\]

\[
= (-1)^n J_n(x)
\]

we see that \( J_n(x) \) and \( J_{-n}(x) \) are linearly dependent.

Finally, we note that from eq (7):

\[
J_0(x) = 1 - \frac{x^2}{(1!)^2} + \frac{x^4}{(2!)^2} - \frac{x^6}{(3!)^2} + \cdots \hspace{2cm} (11)
\]

and \( J_1(x) = \frac{x}{2} - \frac{x^3}{2 \cdot 3!} + \frac{x^5}{2^2 \cdot 5!} - \frac{x^7}{2^3 \cdot 7!} + \cdots \hspace{2cm} (11') \)

From which:

\[
\frac{d J_0(x)}{dx} = J_1(x) \hspace{2cm} (12)
\]
Any equation containing partial differential coefficients is called a partial differential equation, the order of the equation being equal to the order of the highest partial differential coefficient occurring in it. For example, the equations:

\[ 3y^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2u \quad \text{--- Eq. 1} \]

\[ \frac{\partial^2 u}{\partial x^2} + f(x,y) \frac{\partial u}{\partial y} = 0 \quad \text{--- Eq. 2} \]

where \( f(x,y) \) is an arbitrary function, are typical partial differential equations of the first and second orders respectively, \( x \) and \( y \) being independent variables and \( u \) the function to be found. Both these equations are linear in the sense that both \( u \) and its derivatives occur only to the 1st power, and that products of \( u \) and its derivatives are absent. A typical non-linear equation in two independent variables is:

\[ u \frac{\partial^2 u}{\partial x^2} + \left( \frac{\partial u}{\partial y} \right)^2 = u^2 \quad \text{--- Eq. 3} \]

However, we shall not consider equations of this type here. In general, the solution of partial differential equations presents a much more difficult problem than the solution of ordinary differential equations, and except for certain special types of linear partial differential equations no general method of solution is available.
Several methods are used to solve p.d. equations which are:
- methods of unknown I & B.C.s.
- known I & B.C.s.
- 1st method: method of combination of variables (or the method of similarity solutions). I.C & B.C may be combined into a single new B.C.
- 2nd method: method of separation of variables, in which P-diff equation is split up into two or more ordinary diff. equs. The solution is then an infinite sum of products of the solutions of ordinary diff. equs.
- 3rd method: method of sinusoidal response, which is useful into describing the way a system responds to external periodic disturbances.

Laplace method:
A. General solutions of p.d equations (unknown I & B.C.s).

Second order constant coefficient equations

Any equation of the type:

\[ a \frac{\partial^2 u}{\partial x^2} + 2h \frac{\partial^2 u}{\partial x \partial y} + b \frac{\partial^2 u}{\partial y^2} + 2f \frac{\partial u}{\partial x} + 2g \frac{\partial u}{\partial y} + cu = 0 - 14 \]

(where a, h, b, f, g, and c are constants) is a linear 2nd order constant coefficient partial diff. equation in two variables (x and y).

By comparison with the equation of the general conic:
we say that (eq. 1) is of:

elliptic type when $ab - h^2 > 0$
parabolic type when $ab - h^2 = 0$
hyperbolic type when $ab - h^2 < 0$

For example, Laplace's equation in two variables (harmonic function)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

this equation may be obtained from (eq. 1) by putting $a = 1, h = 0, b = 1, f = g = c = 0$, and hence, since $ab - h^2 > 0$, is of elliptic type. Similarly, the equation:

$$\frac{\partial^2 u}{\partial x^2} - k^2 \frac{\partial^2 u}{\partial y^2} = 0$$

(where $k$ is a real constant) may be obtained from (eq. 1) by putting $a = 1, h = 0, b = -k^2, f = g = c = 0$. Hence, since $ab - h^2 = -k^2 < 0$, the equation is of hyperbolic type. However, the equation:

$$\frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial y} = 0$$

is of parabolic type, since $a = 1, h = 0, b = 0, g = -\frac{1}{2} k, f = c = 0$ and $ab - h^2 = 0$.

The general solution of a given 2nd order constant coefficient p. diff. equation depends very much on whether the equation is of elliptic, parabolic, or hyperbolic type.
Euler's equation:

The equation: \( a \frac{\partial^2 u}{\partial x^2} + 2h \frac{\partial^2 u}{\partial x \partial y} + b \frac{\partial^2 u}{\partial y^2} = 0 \) — (7)

where \( a, h \) and \( b \) are constants, is a special case of (1.40) (obtained by putting \( f = c = 0 \)), and is usually known as Euler's equation.

Assume two roots, \( X_1 \) and \( X_2 \).

Equation (7) becomes:

\[
\alpha + 2h X + b X^2 = 0 \quad (8)
\]

\[
\frac{\partial^2 u}{\partial x \partial y} = X_1 \frac{\partial u}{\partial y} + X_2
\]

Note:
Relations between roots and coefficients:

\[
a_0 x^2 + a_1 x + a_2 = 0
\]

if \( X_1 \) and \( X_2 \) are the roots:

\[
a_0 x^2 + a_1 x + a_2 = a_0 (X - X_1) (X - X_2)
\]

By equating coefficients of power \( x \):

\[
X_1 X_2 = -\frac{a_2}{a_0}
\]

\[
X_1 + X_2 = \frac{a_1}{a_0}
\]

\[
a_0 x^2 + a_1 x^2 + a_2 x + a_3 = 0
\]

\[
X_1 + X_2 + X_3 = -\frac{a_1}{a_0}
\]

\[
X_1 X_2 + X_2 X_3 + X_1 X_3 = \frac{a_2}{a_0}
\]

\[
X_1 X_2 X_3 = -\frac{a_3}{a_0}
\]
Regarding to Eq. 8:

\[ x_1 + x_2 = -\frac{2h}{b} \]

\[ x_1 x_2 = \frac{a}{b} \]

The general solution of (eq. 7) is:

\[ U = F(x + x_1 y) + G(x + x_2 y) \]  \( \text{(8)} \)

Finally, we see from eq.(8) that the nature of the roots \( x_1 \) and \( x_2 \) which appear in this solution depends on whether the equation is of hyperbolic or elliptic type:

- When \( ab - h^2 < 0 \) (hyperbolic): \( x_1 \) and \( x_2 \) are real. \( x_1 \neq x_2 \) \( \text{(8a)} \).
- When \( ab - h^2 > 0 \) (elliptic): \( x_1 \) and \( x_2 \) are necessarily complex. \( \text{(8b)} \).
- When \( ab - h^2 = 0 \) (parabolic): \( x_i = x_i \) \( \text{(8c)} \).

The general solution becomes:

\[ U = F(x + xy) + (y + sx) G(x + xy) \]  \( \text{(8d)} \)

where \( r \) and \( s \) are constant (but not zero).

**Ex.** Laplace's equation in two variables:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]  \( \text{(9)} \)

(\( a = b = 1 \) and \( h = 0 \)).

\[ 1 + x^2 = 0 \]

\[ x_1 = i, \quad x_2 = -i \]

\( i \): imaginary roots; \( i = \sqrt{-1} \).
The equation:

\[ 2 \frac{\partial^2 u}{\partial x^2} + 3 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \]

is of hyperbolic type since \( a = 2, h = \frac{3}{2}, b = 2 \) and hence \( ab - h^2 < 0 \):

\[ 2 + 3 \lambda + \lambda^2 = 0 \]

\[ \lambda_1 = -1, \lambda_2 = -2 \]

The general solution:

\[ u = F(x-y) + G(x-2y) \]

The equation:

\[ \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = 0 \]

is of parabolic type since \( a = 1, h = 2, b = 4 \) and \( ab - h^2 = 0 \):

\[ 1 + 4 \lambda + 4 \lambda^2 = 0 \]

\[ \lambda_1 = \lambda_2 = -\frac{1}{2} \]

The general solution:

\[ u = F(x-\frac{1}{2}y) + (\lambda x + \lambda y) G(x-\frac{1}{2}y) \]

where \( \lambda \) and \( \delta \) are arbitrary constants.

For the inhomogeneous Euler equation \( \phi \)

\[ a \frac{\partial^2 u}{\partial x^2} + 2h \frac{\partial^2 u}{\partial x \partial y} + b \frac{\partial^2 u}{\partial y^2} = f(x,y) \]

the D-operator technique is used similar to that used for ordinary diff. equs.
(a) Laplace's equation in two dimensions:
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

In these equations C

B) General solutions of p.d equations (known TRBCs)
several methods are used such as:

1) Method of combination of variables (or the method of similarity solutions). I.e. B.c may be combined into a single new B.c.

2) Method of separation of variables, in which p.diff equs. is split up into two or more ordinary diff equs. The solution is then an infinite sum of products of the solutions of ordinary diff equs.

3) Method of sinusoidal response, which is useful in describing the way a system responds to external periodic disturbances.

4) Laplace - transform method.
It is not always possible to derive a general solution of a partial differential equation given boundary conditions, and it is usually a somewhat difficult matter to choose the functions $F$ & $G$ of the solution:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Such that the equation is satisfied inside a square region defined by the lines $x = 0$, $x = a$, $y = 0$, $y = b$ and such that $u$ takes prescribed values on the boundary of this region. To overcome this difficulty, it is best to obtain a less general type of solution which is governed by the type of boundary conditions to be imposed. One method of doing this depends on assuming the solution to be a product of functions each of which contains only one of the independent variables. This is the method of separation of variables.

We shall now illustrate the use of this method by considering the solutions of three equations of physical importance. These three equations are:

(a) The one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

(b) The one-dimensional heat conduction (and diffusion) equation:

$$\frac{\partial u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}$$
(C) Laplace's equation in two-dimensions:

In these equations C & k are physical constants, x & y are the space variables of a Cartesian coordinate system, and t is the time variable. The physical interpretation of the dependent variable (u) is different in each equation.

\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \]  \hspace{1cm} (1)

which is periodic in x and t, and which satisfies the B.C.
\[ u(0,t) = u(l,t) = 0 \quad t \geq 0 \]  \hspace{1cm} (2)
\[ u(x,0) = f(x) \quad 0 \leq x \leq l \]  \hspace{1cm} (3)
\[ \left[ \frac{\partial u(x,t)}{\partial t} \right]_{t=0} = g(x) \quad 0 \leq x \leq l \]  \hspace{1cm} (4)

where f & g are given functions and l is a given constant.

We now assume a solution of (eq.1) of the form:

\[ \frac{\partial u}{\partial x} = X, \quad \frac{\partial u}{\partial t} = T \]

\[ u(x,t) = X(x)T(t) \]  \hspace{1cm} (5)

where X is a function of x only and T is a function of t only, then (eq.1) becomes:

\[ \frac{1}{c^2} \frac{d^2 X}{dt^2} = \frac{1}{X} \frac{d^2 T}{dx^2} \]  \hspace{1cm} (6)

This equation is satisfied if we now write:

\[ \frac{1}{c^2} \frac{d^2 X}{dt^2} = -\omega^2 \]  \hspace{1cm} (7)

\[ \frac{1}{X} \frac{d^2 T}{dx^2} = -\omega^2 \]  \hspace{1cm} (8)

and,

\[ \frac{d}{dx} + \frac{m^2}{\tan \gamma} = 0 \]

\[ \frac{1}{T} \frac{dt}{dx} = -\omega^2 \]  \hspace{1cm} (9)
where \( \omega \) is any real number. The solutions of these equations are periodic, as required, with negative signs introduced in equ.(7)-(8), and have the well-known forms:

\[
X(x) = A \cos \frac{wx}{c} + B \sin \frac{wx}{c} \quad -(9)
\]

\[
T(t) = C \cos \omega t + D \sin \omega t \quad -(10)
\]

where \( A/B, C \) and \( D \) are arbitrary constants. Hence (eq.5) becomes:

\[
W(x,t) = (A \cos \frac{wx}{c} + B \sin \frac{wx}{c})(C \cos \omega t + D \sin \omega t)
\]

which clearly satisfies (eq.1) for all values of the constants \( A/B, C, D \) and \( \omega \). By using B.C. into eq.(11):

\[
W(0,t) = 0 \quad t \geq 0 \quad \text{gives:}
\]

\[
0 = A(C \cos \omega t + D \sin \omega t) \quad -(12)
\]

for all \( t \), which implies:

\[
A = 0
\]

Secondly, putting \( x = c \) in eq.(11), B.C. \( W(c,t) = 0 \) to.

and by using \( \text{eq.(17)} \):

\[
0 = B \sin \frac{wc}{c} (C \cos \omega t + D \sin \omega t) \quad -(14)
\]

However, since \( B \) cannot be equal to zero without making \( W(x,t) \) identically zero, (eq.(14)) can only be satisfied for all \( t \) provided:

\[
\sin \frac{wc}{c} = 0 \quad \Rightarrow \quad w = \frac{yc}{x} \quad \text{where} \quad y = 1, 2, 3, \cdots
\]

(see the case \( \gamma = 0 \), which gives \( w = 0 \), hence excluded since this again makes \( W(x,t) \) identically zero).
Finally, since eqn (13) is linear, the final general solution is:

$$u(x,t) = \sum_{r=1}^{\infty} \left( Cr \sin \frac{r\pi x}{L} + Dr \cos \frac{r\pi x}{L} \right) \sin \frac{r\pi x}{L}$$

$Cr$ and $Dr$ are constants.

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

By putting $t=0$ in eqn (13):

$$f(x) = \sum_{r=1}^{\infty} Dr \sin \frac{r\pi x}{L} \quad - (16)$$

Similarly,

$$\left[ \frac{\partial u(x,t)}{\partial t} \right]_{t=0} = g(x), \quad 0 \leq x \leq L$$

is satisfied by differentiating eqn (13) with respect to $t$ and then putting $t=0$ in this way:

$$g(x) = \sum_{r=1}^{\infty} Cr \sin \frac{r\pi x}{L} \quad - (17)$$

The coefficients $Cr$ and $Dr$ may now be determined from eqns (16 & 17) by a Fourier series technique, consequently:

$$Dr = \frac{2}{\pi L} \int_{0}^{L} f(x) \sin \frac{r\pi x}{L} \, dx \quad - (18)$$

and

$$Cr = \frac{2}{r \pi L} \int_{0}^{L} g(x) \sin \frac{r\pi x}{L} \, dx \quad - (19)$$

where $r=1,2,3,...$

Hence finally, substituting eqns (18 & 19) into eqn (13), we have the solution:

$$u(x,t) = \sum_{r=1}^{\infty} \left[ \left( \frac{2}{r \pi L} \int_{0}^{L} g(x) \sin \frac{r\pi x}{L} \, dx \right) \sin \frac{r\pi x}{L} \cos \frac{r\pi ct}{L} \sin \frac{r\pi x}{L} \right] + \sum_{r=1}^{\infty} \left[ \left( \frac{2}{r \pi L} \int_{0}^{L} f(x) \sin \frac{r\pi x}{L} \, dx \right) \cos \frac{r\pi ct}{L} \sin \frac{r\pi x}{L} \right] \quad - (20)$$

where we have written $\chi$ for the variable of integration to distinguish it from the independent variable $x$. 
The method of separation of variables may be readily extended to the solution of the two- and three-dimensional wave equations:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]

and

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]

However, it is not always convenient or desirable to solve these equations using a cartesian coordinate system.

**EX:** To obtain the solution of the heat conduction equation:

\[
\frac{\partial u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}
\]

which decreases exponentially with time and which satisfies the boundary conditions,

\[u(0,t) = u(L,t) = 0, \quad t > 0\]

\[u(x,0) = f(x), \quad 0 < x < L\]

where \(f\) is a given function and \(L\) is a constant. Assuming a solution of the form:

\[u(x,t) = X(x)T(t)\]

and substituting in eqn., we find:

\[\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{kT} \frac{dT}{dt}\]

Hence putting:

\[\frac{1}{X} \frac{d^2 X}{dx^2} = -\omega^2\]

\[\frac{1}{kT} \frac{dT}{dt} = -\kappa\omega^2\]

where \(\omega\) is any real number (the negative sign in \(7\) ensuring an exponentially decreasing solution in \(t\)), we have

\[X = A \cos \omega x + B \sin \omega x\]

and

\[T = C e^{-\kappa \omega t}\]

where \(A, B, C\) are arbitrary constants.
The solution becomes:

\[ U(x,t) = (A \cos \omega x + B \sin \omega x) e^{-\omega^2 kt} \]  \hspace{1cm} (10)

where, \( A \) and \( B \) are new arbitrary constants. In order to satisfy the boundary conditions (eq.2) we first put \( x = 0 \) in (eq.10):

\[ 0 = A e^{-\omega^2 kt} \]  \hspace{1cm} (11)

for all \( t \), which implies:

\[ A = 0 \]  \hspace{1cm} (12)

Secondly putting \( x = \ell \) in (eq.10) we find (using eq.12):

\[ 0 = (B \sin \omega \ell) e^{-\omega^2 kt} \]  \hspace{1cm} (13)

which leads to non-trivial solutions provided:

\[ \sin \omega \ell = 0 \]  \hspace{1cm} (14)

Hence, \( \omega \ell = \frac{n \pi}{L} \) where \( n = 1, 2, 3, \ldots \) (15).

( the case \( \omega = 0 \) again being excluded to avoid making \( U(x,t) \) identically zero). Putting eqns (11) \& (15) into (eq.10) we now obtain the infinity of eigenvalues and corresponding eigenfunctions.

Therefore, the solution:

\[ U(x,t) = \sum_{n=1}^{\infty} B_n e^{-\omega_n^2 kt} \frac{\sin \frac{n \pi x}{L}}{\omega_n} \]  \hspace{1cm} (16)

\( B_n \) is constant must now be chosen to satisfy the remaining B.C. (eq.3).

Hence, putting \( t = 0 \) in eq.(16):

\[ f(x) = \sum_{n=1}^{\infty} B_n \frac{\sin \frac{n \pi x}{L}}{\omega_n} \]  \hspace{1cm} (17)

from which it follows, using the Fourier series technique, that

\[ B_n = \frac{2}{L} \int_{0}^{\ell} f(x) \sin \frac{n \pi x}{L} \, dx \]  \hspace{1cm} (18)

The solution of (eq.1) now:

\[ U(x,t) = \sum_{n=1}^{\infty} \left( \frac{2}{L} \int_{0}^{\ell} f(x) \sin \frac{n \pi x}{L} \, dx \right) e^{-\omega_n^2 kt} \]  \hspace{1cm} (19)

where \( x \) has been written as the variable of integration to
avoid confusion with $x$.

Similar remarks to those made at the end of previous example on the solution of the two- and three-dimensional wave equations apply equally to the solution of the two- and three-dimensional heat conduction equations:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k} \frac{\partial u}{\partial t} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k} \frac{\partial u}{\partial t}
\]

**Ex.** To obtain the solution of Laplace’s equation:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

which is periodic in $x$ inside the rectangular region defined by $0 \leq x \leq a$, $0 \leq y \leq b$, and which satisfies the B.C’s:

- $U(x, y) = 0$ when $x = 0$, $0 \leq y \leq b$
- $U(x, y) = 0$ when $x = a$, $0 \leq y \leq b$
- $U(x, y) = 0$ when $y = b$, $0 \leq x \leq a$
- $U(x, y) = f(x)$ when $y = 0$, $0 < x < a$

therefore, $U(x, y) = X(x) Y(y)$

So that, equi becomes:

\[
\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0 - (5)
\]

Hence, putting:

\[
\frac{1}{X} \frac{d^2 X}{dx^2} = -w^2 \quad (\text{for periodic solution})
\]

and,

\[
\frac{1}{Y} \frac{d^2 Y}{dy^2} = w^2 \quad - (6)
\]

where $w$ is any real number, we have:

- $X = A \cos wx + B \sin wx$
- $Y = C \cosh wy + D \sinh wy$

\[
X = A \cos wx + B \sin wx - (7)
\]

\[
Y = C \cosh wy + D \sinh wy - (8)
\]
A, B, C, and D being arbitrary constants.

Then eqn. becomes:

\[ u(x,y) = (A \cos \omega x + B \sin \omega x)(C \cosh \omega y + D \sinh \omega y) \quad (9) \]

Put \( x = 0 \)

\[ a = A(C \cosh \omega y + D \sinh \omega y), \quad 0 \leq y \leq b \quad \text{(10)} \]

or:

\[ A = 0 \quad \text{(11)} \]

Similarly, the 2nd B.C. requires (for non-trivial solution):

\[ \sin \omega x a = 0 \quad \text{(12)} \]

or:

\[ w = \frac{\pi n}{a} \quad \text{where} \quad n = 1, 2, 3, \ldots \quad \text{(13)} \]

Likewise, the 3rd B.C. of (eqn. 2) gives (putting \( y = b \) in (11))

\[ a = (A \cos \omega x + B \sin \omega x)(C \cosh \omega b + D \sinh \omega b) \quad \text{(14)} \]

or:

\[ \frac{C}{D} = -\tanh \omega b \quad \text{(15)} \]

Then substitute eqns. (11, 13 and 15) into eq. 9, we find the infinite set of eigenvalues and corresponding eigenfunctions.

Finally, as before, we now take a linear combination of these particular solutions:

\[ u(x,y) = \sum_{r=1}^{\infty} E_r \sin \frac{\pi r x}{a} \sinh \frac{\pi r (b-y)}{a} \quad \text{(16)} \]

\( E_r \), constant.

For fourth and last B.C. in eqn. 2:

Putting \( y = 0 \) into eqn. 16:

\[ f(x) = \sum_{r=1}^{\infty} E_r \sinh \frac{\pi r b}{a} \sin \frac{\pi r x}{a} \quad \text{(17)} \]

From which we find:

\[ E_r \sinh \frac{\pi r b}{a} = \frac{2}{a} \int_{0}^{a} f(x) \sin \frac{\pi r x}{a} \, dx \quad \text{(18)} \]
Substitute eqn(8) into eqn(6):

The final solution of eqn (8):

\[ u(x, y) = \sum_{r=1}^{\infty} \left( \frac{2}{a} \int_{0}^{a} f(x') \sin \frac{r \pi x'}{a} \, dx' \right) \frac{\sin \frac{r \pi y}{a} \sinh \frac{yA}{a} \left( \frac{y}{a} \right)^{m-1}}{\sinh \frac{yA}{a}} + \text{(4).} \]

where, as before, \( x' \) is the variable of integration.
The Laplace transform method.

L.T. would remove the derivatives from an ordinary diff.eq.
It is thus fairly obvious that the same technique can be used to
remove all derivatives with respect to one independent variable from
a partial diff.eqn. If the partial diff.eqn. has two independent
variables the removal of one of them yields an ordinary diff.eqn
which can be solved by obvious methods (ordinary eqn. L.T metod).

Example: Linear heat conduction equation in a semi-infinite medium

\[ \lambda \frac{\partial^2 \mathcal{T}}{\partial x^2} = \frac{\partial \mathcal{T}}{\partial t} \]

J.C.: \( x = 0 \), \( T = T_0 \)

By L.T.

\[ \mathcal{L}\left( \frac{\partial T}{\partial t} \right) = s \mathcal{T}(s) - T(0) \]

Because \( x \) and \( t \) are independent variables, then:

\[ \mathcal{L}\left( \frac{\partial^2 T}{\partial x^2} \right) \]

\[ \frac{\partial^2 \mathcal{T}}{\partial x^2} = \frac{\partial T}{\partial x} \]

\( \mathcal{T}(s) = \mathcal{F}(s) \)

The eqn. becomes:

\[ \lambda \frac{\partial^2 \mathcal{F}}{\partial x^2} = s \mathcal{T} - T_0 \]

\[ \mathcal{F}(s) = A e^{-qx} + B e^{qx} + T_0 s \]

where \( q^2 = \frac{s}{\lambda} \)

\[ \mathcal{T}(s) = \frac{L}{s} = A + B + \frac{T_0}{s} \]

If \( x = 0 \), \( T = T_0 \) (changed

\( \mathcal{T}(s) = \mathcal{F}(s) \)

Transforming equation: \( x = 0 \)

\( \mathcal{T} = T_0 s \)

At \( x \rightarrow \infty \), \( T = T_f \), \( \mathcal{T} \rightarrow \mathcal{F} = \mathcal{F}(s) \)

\[ \mathcal{F}(s) = A \]

\[ B = 0 \]

\[ \mathcal{F}(s) = \mathcal{F}(s) \]

where \( \lambda = \frac{s}{q} \).
\[
\frac{\Delta T}{\Delta t} = A + \frac{T_0}{\Delta t}
\]

\[
T = \left( T_1 - T_0 \right) \left( e^{-\frac{1}{2} \frac{kS}{p}} \times \frac{kS}{p} \right) + T_0 \quad (5)
\]

By using Laplace inverse:

\[
\frac{1}{s} e^{-\frac{kS}{p}} (k \geq 0) \rightarrow \text{erfc} \left( \frac{k}{\sqrt{4p}} \right)
\]

\[
T = (T_1 - T_0) \text{erfc} \left[ \frac{1}{2} \sqrt{\frac{k}{4p}} \right] + T_0 \quad (6)
\]
Consider a fluid flowing with linear velocity $v$ along a pipe of length $L$ and diameter $D$. Find the temperature distribution of the fluid as a function of time and distance $T(x,t)$.

Where:

- $T$: is the mass of fluid in the element $dV$.
- $w$: is the mass rate of flow.
- $U$: is the overall heat transfer coe.
- $A$: surface area of pipe in the element $dx$.
- $T$: Temperature of fluid.
- $T_0$: ambient temperature.

Heat balance over element $dV$:

Input - output + heat losses = accumulated heat.

\[ \dot{Q} = \dot{W} c_p (T + \frac{dT}{dt} dx) - UA (T - T_0) = \frac{d}{dr} \frac{r c_p T}{r} \]

\[ T = \frac{M}{\rho V}, \quad W = \rho AV \]

\[ V = A dx, \quad A = \frac{D^2}{4} \]

\[ \frac{(\pi D^2 dx)}{4} \frac{d}{dr} \frac{r c_p (T + \frac{dT}{dr} dx)}{r} = -[(AV) D^2 \cdot (r c_p \frac{dT}{dr} dx) - UA (T_0 T)] \]

\[ \frac{dT}{dt} = -V (\frac{dT}{dr} dx) - (4 U I P c_p D) (T - T_0) \]
\[
\begin{align*}
\text{assume: } & \frac{y}{x} = \frac{1}{2} \\
\text{By L.T.:} & \\
ST(s) = & -v \frac{dT}{dx} - \alpha \left( T - T_a \right) \quad \text{(ordinary diff. eqn. in X)} \\
\text{Rearranging:} & \\
-v \frac{dT}{dx} = & \alpha \left( x + 1 \right) T - T_a \quad \text{(5)} \\
\frac{dT}{(x+1)T - T_a} = & -\frac{dx}{\alpha v} \quad \text{(6)} \\
\text{Integrating from } T = T_1 \text{ ab } x = 0 \text{ to } T = T_2 \text{ ab } x = X \\
\frac{(x+1)T_2 - T_a}{(x+1)T_1 - T_a} = & \exp \left[ -\frac{Xv(x+1)}{\alpha} \right] \quad \text{(7)} \\
\text{From eq (7), since } T_a = \text{constant; } \Rightarrow T_a = 0 \text{, eq (7) becomes:} & \\
\frac{T_2}{T_1} = & \exp \left[ -\frac{Xv(x+1)}{\alpha} \right] \quad \text{(8)} \\
& = \exp \left[ -\frac{XvS - X}{v \alpha} \right] \quad \text{(9)}
\end{align*}
\]
Numerical methods

There are many problems in mathematics for which no analytical solution is known. There are also others, for which the analytical solution is tedious and the answer may be in the form of an infinite series that can only be interpreted after much computational effort. A numerical method is the only one which yields a solution to the first kind of problem, and may be the most efficient method of solving the second, by using the high-speed digital computer. Numerical methods are used for algebraic, ordinary and partial differential equations.

First order ordinary differential equations

If the 1st order diff. eqn. is not linear and the variables will not separate, none of the methods given can be applied. The problem is thus to solve:

\[
\frac{dy}{dx} = f(x, y) \quad (1)
\]

Two kinds of methods are commonly used; Runge and Kutta and predictor corrector methods.

The Runge-Kutta method

The general principles can be described as follows:

at initial condition for eqn(1), \( y = y_0 \) at \( x = x_0 \), it is desired to find the value of \( y \) when \( x = x_0 + h \) where \( h \) is some given constant.
1. \( y_0 \rightarrow x_0 \)

2. Assume \( h: DX = x_2 - x_1 \)

3. Determine \( k_1 = f(x_0, y_0) = (\Delta y)_1 \)

4. Determine \( k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \)

5. Determine \( k_3 = f(x_0 + h, y_0 + 2k_2 - k_1) \)

\[ y = y_0 + \frac{1}{6} (k_1 + 4k_2 + k_3) \]

Fourth order \( y = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \) (to increase the accuracy)

**Example:** If \( y_3 = 1 \) when \( x = 1 \) and the equation

\[ \frac{dy}{dx} = \frac{x - y}{x} = f(x, y) \]

Find the value of \( y_3 \) when \( x = 2 \).

**Solution:**

Use \( h = 1 \)

\[ k_1 = f(1, 1) = \frac{1 - 1}{1} = 0 \]

\[ k_2 = f(1, \frac{3}{2}, 1) = \frac{\frac{3}{2} - 1}{\frac{3}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3} \]

\[ k_3 = f(2, 2, \frac{2}{3}) = \frac{2 - 2 \frac{2}{3}}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3} \]

\[ y = y_0 + \frac{1}{6} (k_1 + 4k_2 + k_3) \]

\[ = 1 + \frac{1}{6} (0 + \frac{2}{3} + \frac{2}{3}) = 1 + \frac{1}{3} \]

by analytical solution:

\[ y_3 = \frac{1}{3} x^2 \frac{2}{3} x = \frac{1}{3} x^2 \]

at \( x = 2 \)

\[ y_3 = \frac{1}{3} \left( \frac{2}{3} \right)^2 \]

(Identical)
This method uses one formula to predict an approximate value of \( y \) at the end of the interval of integration. The simplest and obvious method of this type is to predict the terminal value by using the initial gradient, thus:

\[
YP = y_0 + hf(x_0, y_0) \quad - (1)
\]

and

\[
y^{n+1}_c = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_0 + \frac{h}{2}, y^{n+1}_c) \right] \quad - (2)
\]

where \( y^{n}_c \) is the \( n \)th approximation to the solution, \( y_P \) having been used for the first.

**Ex.** If \( y = 1 \) when \( x = 1 \), and the equation:

\[
\frac{dy}{dx} = \frac{x^2 - y}{x}
\]

find the value of \( y \) when \( x = 2 \)

at \( x = 1 \), \( y = 1 \), and \( \frac{dy}{dx} = f(1,1) = 0 \)

Equation (1) thus predicts:

\[
y'' = 1 + 0 = 1
\]

The gradient at the end of the interval where \( x = 2 \) is

\[
f(2,1) = \frac{4-1}{2} = 1.5
\]

Substitute into \( y^{n+1}_c \) gives:

\[
y'' = 1 + \frac{h}{2} (0 + 1.5) = 1.75
\]

Repeated substitution gives:

\[
f(2,1.75) = \frac{4-1.75}{2} = 1.125
\]

\[
y'' = 1 + \frac{h}{2} (0 + 1.125) = 1.5625
\]
\[ y = 1 + \frac{1}{2} \left( 0 + \frac{4 - 1.562}{2} \right) = 1.6075 \]

\[ y = 1 + \frac{1}{2} \left( 0 + \frac{4 - 1.6075}{2} \right) = 1.5976 \]

\[ y = 1 + \frac{1}{2} \left( 0 + \frac{4 - 1.5976}{2} \right) = 1.6066 \]

\[ y = 1 + \frac{1}{2} \left( 0 + \frac{4 - 1.6066}{2} \right) = 1.6000 \]

The solution obtained is not very accurate. A more accurate solution can be obtained by covering the interval in two steps instead of one by using \( h = 0.5 \).