CHAPTER SIX

Application of integrals

Definite integrals:

If $f(x)$ is continuous in the interval $a \leq x \leq b$ and it is integrable in the interval then the area under the curve: –

$$\int_{a}^{b} f(x) \, dx = F(x)|_{a}^{b} = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:

1) $\int_{a}^{b} c \, f(x) \, dx = c \int_{a}^{b} f(x) \, dx$, where $c$ is constant.
2) $\int_{a}^{b} f(x) + g(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx$
3) $\int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx$
4) Let $a < c < b$ then $\int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx$
5) $\int_{a}^{a} f(x) \, dx = 0$
6) If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_{a}^{b} f(x) \, dx \geq 0$
7) If $f(x) \leq g(x)$ for $a \leq x \leq b$ then $\int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} g(x) \, dx$

Ex 1: – Evaluate the following definite integrals:

1) $\int_{2}^{6} \frac{dx}{x+2}$
2) $\int_{\pi/2}^{3\pi/2} \cos x \, dx$
3) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$
4) $\int_{0}^{3/2} \frac{dx}{\sqrt{1-x^2}}$
5) $\int_{-2}^{4} e^{-x/2} \, dx$
6) $\int_{0}^{\pi} (\pi - x) \cdot \cos x \, dx$

Sol:

1) $\int_{2}^{6} \frac{dx}{x+2} = \ln(x+2)|_{2}^{6} = \ln(6+2) - \ln(2+2) = \ln 8 - \ln 4 = 3 \ln 2 - 2 \ln 2 = \ln 2$.
2) \[ \int_{\pi/2}^{3\pi/2} \cos x \, dx = \sin x \bigg|_{\pi/2}^{3\pi/2} = \sin \left(\frac{3\pi}{2}\right) - \sin \left(\frac{\pi}{2}\right) = -1 - 1 = -2. \]

3) \[ \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{1 + x^2} = \tan^{-1} x \bigg|_{-\sqrt{3}}^{\sqrt{3}} = \tan^{-1} \sqrt{3} - \tan^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2}{3} \pi \]

4) \[ \int_{0}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \bigg|_{0}^{\sqrt{3}/2} = \sin^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3} \]

5) \[ \int_{-2}^{4} e^{-x^2} \, dx = -2e^{-x^2} \bigg|_{-2}^{4} = -2(e^{-2} - e) = 2(e - e^{-2}) \]

6) Let \( u = \pi - x \Rightarrow du = -dx \) & \( dv = \cos x \, dx \Rightarrow v = \sin x \)

\[ \int_{0}^{\pi} (\pi - x) \cdot \cos x \, dx = (\pi - x) \sin x \bigg|_{0}^{\pi} + \int_{0}^{\pi} \sin x \, dx = (\pi - x) \sin x - \cos x \bigg|_{0}^{\pi} \]

\[ = (\pi - \pi) \sin \pi - \cos \pi - ((\pi - 0) \sin 0 - \cos 0) = 0 - (-1) - (0 - 1) = 2 \]

**Area between two curves:**

Suppose that \( y_1 = f_1(x) \) and \( y_2 = f_2(x) \) define two functions of \( x \) that are continuous for \( a \leq x \leq b \) then the area bounded above by the \( y_1 \) curve, below by \( y_2 \) curve and on the sides by the vertical lines \( x = a \) and \( x = b \) is:

\[ A = \int_{a}^{b} [f_1(x) - f_2(x)] \, dx \]

Ex2: – Find the area bounded by the \( x - axis \) and the curve: \( y = 2x - x^2 \)

Sol: –

\[ \begin{align*}
    y &= 0 \quad \cdots \cdots (1) \\
    y &= 2x - x^2 \quad \cdots (2)
\end{align*} \]

\[ \Rightarrow x(x - 2) = 0 \Rightarrow x = 0, 2 \]

The points of the intersection of the curve and the \( x - axis \) are (0,0) and (2,0).
(2,0) then the area bounded by \( x - \) axis and the curve is:

\[
\int_{0}^{2} (2x - x^2) \, dx = x^2 - \frac{x^3}{3} \bigg|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}
\]

**Ex 3:** Find the area bounded by the \( y - \) axis and the curve: \( x = y^2 - y^3 \)

**Sol:**

\[
x = 0 \quad \ldots \ldots \ldots (1)
\]
\[
x = y^2 - y^3 \quad \ldots \ldots \ldots (2)
\]

\( \Rightarrow \) intersection points \((0,0), (0,1)\)

The area = \( A = \int_{0}^{1} (y^2 - y^3) \, dy = \frac{y^3}{3} - \frac{y^4}{4} \bigg|_{0}^{1} = \frac{1}{3} - \frac{1}{4} - (0 - 0) = \frac{1}{12} \)

**Ex 4:** Find the area bounded by the curve \( y = x^2 \) and the line \( y = x \).

**Ex 5:** Find the area bounded by the curves \( y = x^4 - 2x^2 \) and \( y = 2x^2 \)

**Sol:**

\[
y = x^4 - 2x^2 \quad \ldots \ldots (1)
\]
\[
y = 2x^2 \quad \ldots \ldots (2)
\]

\( \Rightarrow \) intersection points are \((0,0), (2,8), (-2,8)\)

The area = \( A = \int_{-2}^{0} (2x^2 - (x^4 - 2x^2)) \, dx + \int_{0}^{2} (2x^2 - (x^4 - 2x^2)) \, dx = 2 \int_{0}^{2} (4x^2 - x^4) \, dx = 2\left[ \frac{4x^3}{3} - \frac{x^5}{5} \right]_0^2 = 2\left[ \frac{4}{3} \cdot 8 - \frac{32}{5} - 0 \right] = \frac{128}{15} \)

**Notice:** We can use the double integration to calculate the area between two curves which bounded above by the curve \( y = f_2(x) \) below by \( y = f_1(x) \) on the left by the line \( x = a \) and on the right by \( x = b \), then:
$$A = \int_{a}^{b} \int_{f_{1}(x)}^{f_{2}(x)} dy \, dx$$

To evaluate above integrals we follow:

a) Integrating ∫ dy with respect to y and evaluating the resulting integral the limits y = f_{1}(x) and y = f_{2}(x), then:

b) Integrating the result of (a) with respect to x between the limits x = a and x = b.

If the area is bounded on the left by the curve x = g_{1}(y), on the right by x = g_{2}(y), below by the line y = c, and above by the line y = d, then it is better to integrate first with respect to x and then with respect to y.

$$A = \int_{c}^{d} \int_{g_{1}(y)}^{g_{2}(y)} dx \, dy$$

Ex 6: – Find the area of the triangular region in the first quadrant bounded by the y – axis and the curve y = sin x , y = cos x.

Sol:

\[ y = \sin x \quad \ldots \ldots (1) \]
\[ y = \cos x \quad \ldots \ldots (2) \]

\[ \Rightarrow \sin x = \cos x \quad \therefore x = \frac{\pi}{4} \]

The area = \[ A = \int_{0}^{\pi/4} \int_{\sin x}^{\cos x} dy \, dx = \int_{0}^{\pi/4} y|_{\sin x}^{\cos x} dx = \int_{0}^{\pi/4} (\cos x - \sin x) dx \]

\[ = \sin x + \cos x|_{0}^{\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) = \sqrt{2} - 1 = 0.414 \]

Ex 7: Calculate: \[ \int_{0}^{1} \int_{y}^{1} \frac{\sin x}{x} dx \, dy \]

Sol: We cannot solve the integration

\[ \int_{0}^{1} \int_{y}^{1} \frac{\sin x}{x} dx \, dy \], hence we reverse the order of integration as follow: –
\[
x = 1 \text{ and } y = 1 \\
x = y \quad y = 0
\]

\[
A = \int_0^1 \int_0^x \frac{\sin x}{x} \, dy \, dx = \int_0^1 \frac{\sin x}{x} \, y \bigg|_0^x \, dx = \int_0^1 \frac{\sin x}{x} (x - 0) \, dx = \int_0^1 \sin x \, dx \\
- \cos x \bigg|_0^1 = -(\cos 1 - \cos 0) = 1 - \cos 1
\]

Ex 8: Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.

1) \[ \int_{-2}^{1} \int_{x^2+4x}^{3x+2} dy \, dx \]
2) \[ \int_{-1}^{0} \int_{-2}^{1-x} dy \, dx + \int_{0}^{2} \int_{-x/2}^{1-x} dy \, dx \]

**Volume by slicing and Rotation about an Axis:**

volume = area \times \text{height}, \ v = A \cdot h

**Volume of a solid of known integrable cross-section:**

A(x) from x = a to x = b is the integral of A from a to b,

\[
v = \int_{a}^{b} A(x) \, dx
\]

**Calculating the volume of a solid:**

1) Sketch the solid and a typical cross-section.
2) Find a formula for A(x), the area of a typical cross-section.
3) Find the limits of integration.
4) Integrate A(x) using the fundamental theorem.
Ex: A pyramid 3m high has a square base that is 3m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid?

Sol:

1) A sketch.
2) A formula for \( A(x) \).
   \[ A(x) = x^2 \]
3) The limits of integration
   The square lies on the planes from \( x = 0 \) to \( x = 3 \)
4) Integrate to find the volume.
   \[ v = \int_0^3 A(x) \, dx = \int_0^3 x^2 \, dx = \frac{x^3}{3} \bigg|_0^3 = 9 \]

**Solids of revolution: The Disk Method:**

\[ A(x) = \pi (\text{radius})^2 = \pi [R(x)]^2 \]

\[ v = \int_{-b}^{b} A(x) \, dx = \int_{-b}^{b} \pi \left( R(x) \right)^2 \, dx \]  
   Revolution about x-axis

\[ A(y) = \pi (\text{radius}) = \pi [R(y)]^2 \]

\[ v = \int_{c}^{d} A(y) \, dy = \int_{c}^{d} \pi \left( R(y) \right)^2 \, dy \]  
   Revolution about y-axis

**Application of integrals**
Examples:

1. The region between the curve $y = \sqrt{x}, 0 \leq x \leq 4$, and the $x$-axis is revolved about the $x$-axis to generate a solid. Find its volume?

   Sol: $v = \int_{a}^{b} \pi [R(x)]^2 \, dx$, \quad $R(x) = \sqrt{x}$.

   $v = \int_{0}^{4} \pi [\sqrt{x}]^2 \, dx = \pi \int_{0}^{4} x \, dx = \pi \left[ \frac{x^2}{2} \right]_{0}^{4} = 8\pi$.

2. The circle $x^2 + y^2 = a^2$ is rotated about the $x$-axis to generate a sphere. Find its volume?

   Sol: We imagine the sphere cut into thin slices by planes perpendicular to the $x$-axis. The cross-sectional area at a typical $x$ between
\(-a\) and \(a\) is: 
\[ A(x) = \pi y^2 = \pi(a^2 - x^2) \]

\[ v = \int_{-a}^{a} A(x) \, dx = \int_{-a}^{a} \pi (a^2 - x^2) \, dx = \pi \left[ a^2 x - \frac{x^3}{3} \right]_{-a}^{a} = \frac{4}{3} \pi a^3. \]

3. Find the volume of the solid generated by revolving the region between the \(y\)-axis and the curve \(x = \frac{2}{y}, 1 \leq y \leq 4\), about the \(y\)-axis?

Sol:

\[ v = \int_{1}^{4} \pi [R(y)]^2 \, dy \]

\[ v = \int_{1}^{4} \pi \left( \frac{2}{y} \right)^2 \, dy \]

\[ v = 4\pi \int_{1}^{4} y^{-2} \, dy \]

\[ v = 4\pi \left[ -\frac{1}{y} \right]_{1}^{4} = 4\pi \left[ \frac{3}{4} \right] = 3\pi \]

Application of integrals
4. Find the volume of the solid generated by revolving the region between the parabola \( x = y^2 + 1 \) and the line \( x = 3 \) about the line \( x = 3 \)?

**Sol:** \( R(y) = 3 - (y^2 + 1) \)

\[
R(y) = 3 - y^2 - 1 = 2 - y^2
\]

\[
v = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 \, dy
\]

\[
v = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 \, dy
\]

\[
v = \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] \, dy
\]

\[
v = \pi [4y - \frac{4}{3}y^3 + \frac{1}{5}y^5]_{-\sqrt{2}}^{\sqrt{2}}
\]

\[
v = \frac{64\pi\sqrt{2}}{15}
\]
**Solids of Revolution: The Washer Method:**

\( R(x) \): Outer radius.

\( r(x) \): Inner radius.

\[ A(x) = \pi [R(x)]^2 - \pi [r(x)]^2 = \pi ([R(x)]^2 - [r(x)]^2). \]

\[
\nu = \int_a^b A(x) \, dx = \int_a^b \pi ([R(x)]^2 - [r(x)]^2) \, dx
\]

**Examples:**

1. The region bounded by the curve \( y = x^2 + 1 \) and the line \( y = x + 3 \) is revolved about the \( x \)-axis to generate a solid. Find the volume of the solid?

**Sol:**

*Outer radius*: \( R(x) = -x + 3 \)

*Inner radius*: \( r(x) = x^2 + 1 \)

\[ x^2 + 1 = -x + 3 \]

\[ x^2 + x - 2 = 0 \]
\((x + 2)(x - 1) = 0\)
\[x = -2 \text{ or } x = 1\]
\[v = \int_{-2}^{1} \pi([x + 3]^2 - [x^2 + 1]^2) \, dx\]
\[v = \pi \int_{-2}^{1} (8 - 6x - x^4 - x^2) \, dx\]
\[v = \pi \left(8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right)_{-2}^{1}\]
\[v = \frac{117\pi}{5}\]

2. The region bounded by the parabola \(y = x^2\) and the line \(y = 2x\) in the first quadrant is revolved about the \(y\) - axis to generate a solid. Find the volume of the solid?

Sol:
\(R(y) = \sqrt{y}, \ y = x^2 \Rightarrow x = \sqrt{y}\)
\[r(y) = \frac{y}{2}, \ y = 2x \Rightarrow x = \frac{y}{2}\]
\[x^2 = 2x \Rightarrow x^2 - 2x = 0\]
\[x(x - 2) = 0, \ x = 0 \ \text{or} \ x = 2\]
\[x = 0 \Rightarrow y = 2x \Rightarrow y = 0\]
\[x = 2 \Rightarrow y = 2x \Rightarrow y = 4\]
\[v = \int_{c}^{d} \pi([R(y)]^2 - [r(y)]^2) \, dy\]
\[
v = \int_{0}^{4} \pi \left( \left[ \sqrt{y} \right]^2 - \left[ \frac{y}{2} \right]^2 \right) dy
\]

\[
v = \pi \int_{0}^{4} \left( y - \frac{y^2}{4} \right) dy
\]

\[
v = \pi \left( \frac{y^2}{2} - \frac{y^3}{12} \right)_{0}^{4} = \frac{8}{3} \pi
\]

**Volume by Cylindrical Shells:**

**The Shell Method:**

The volume of the solid generated by revolving the region between the $x$ – axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

\[
v = \int_{a}^{b} 2\pi \left( \frac{\text{shell}}{\text{radius}} \right) \left( \frac{\text{shell}}{\text{height}} \right) dx
\]

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region.

Examples: 1. The region bounded by the curve $y = \sqrt{x}$ , the $x$ – axis , and the line $x = 4$ is revolved about the $y$ – axis to generate a solid.

Find the volume of the solid?

Sol:

\[
v = \int_{a}^{b} 2\pi \left( \frac{\text{shell}}{\text{radius}} \right) \left( \frac{\text{shell}}{\text{height}} \right) dx
\]

\[
v = \int_{0}^{4} 2\pi (x) (\sqrt{x}) dx = 2\pi \int_{0}^{4} x^{3/2} dx
\]

Application of integrals
\[ v = 2\pi \left[ \frac{2}{5} x^\frac{5}{2} \right]_0^4 = \frac{128}{5} \pi \]

2. The region bounded by the curve \( y = \sqrt{x} \), the \( x \)-axis, and the line \( x = 4 \) is revolved about the \( x \)-axis to generate a solid. Find the volume of the solid?

Sol:

\[ v = \int_a^b 2\pi \left( \frac{\text{shell}}{\text{radius}} \right) \left( \frac{\text{shell}}{\text{height}} \right) dy \]

\[ v = \int_0^2 2\pi (y)(4 - y^2)dy = 2\pi \int_0^2 (4y - y^3) dy \]
\[ v = 2\pi \left[ 2y^2 - \frac{y^4}{4} \right]_0 = 8\pi \]

**Length of Plane Curves:**

**Length of a Parametric Curve:**

If a curve \( C \) is defined parametrically by \( x = f(t) \) and \( y = g(t) \), \( a \leq t \leq b \), where \( f' \) and \( g' \) are continuous and not simultaneously zero on \( [a, b] \), and \( C \) is traversed exactly once as \( t \) increases from \( t = a \) to \( t = b \), then the length of \( C \) is the definite integral:

\[
L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt
\]

if \( x = f(t) \) & \( y = g(t) \)

**Examples:**

**Application of integrals**
1. Find the length of the circle of radius \( r \) defined parametrically by \( x = r \cos t \) and \( y = r \sin t \), \( 0 \leq t \leq 2\pi \).

\[ L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

\[ L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

\[ x = r \cos t \Rightarrow \frac{dx}{dt} = -r \sin t \]

\[ y = r \sin t \Rightarrow \frac{dy}{dt} = r \cos t \]

\[ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-r \sin t)^2 + (r \cos t)^2 = r^2 (\sin^2 t + \cos^2 t) = r^2 \]

\[ L = \int_0^{2\pi} \sqrt{r^2} \, dt = [rt]_0^{2\pi} = 2\pi r \]

2. Find the length of the astroid?

\( x = \cos^3 t \), \( y = \sin^3 t \), \( 0 \leq t \leq 2\pi \)

\[ x = \cos^3 t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 3 \cos^2 t (-\sin t)^2 = 9 \cos^4 t \sin^2 t \]

\[ y = \sin^3 t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 3 \sin^2 t (\cos t)^2 = 9 \sin^4 t \cos^2 t \]

\[ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t} = \sqrt{9 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} \]

\[ = \sqrt{9 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} \] \( \cos^2 t + \sin^2 t = 1 \)

\[ = 3 \cos t \sin t \] \( \cos t \sin t \geq 0 \)

\[ = 3 \cos t \sin t \] \( 0 \leq t \leq \pi/2 \)

Application of integrals
Length of first - quadrant portion = \( \int_0^{\pi/2} 3 \cos t \sin t \, dt \)

\[
sin 2t = 2 \cos t \sin t \Rightarrow \cos t \sin t = \frac{\sin 2t}{2}
\]

\[
\frac{3}{2} \int_0^{\pi/2} \sin 2t \, dt = -\frac{3}{2} \left[ \frac{\cos 2t}{2} \right]_0^{\pi/2} = -\frac{3}{4} [\cos 2t]_0^{\pi/2} = 3/2
\]

The length of the astroid is four times this: \( 4 \left( \frac{3}{2} \right) = 6 \).

**Length of Curve \( y = f(x) \):**

If \( f \) is continuously differentiable on the closed interval \([a, b]\), the length of the curve (graph) \( y = f(x) \) from \( x = a \) to \( x = b \) is:

\[
L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx
\]

\[
L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

Example:

Find the length of the curve?

\[
y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 , \quad 0 \leq x \leq 1
\]

Sol:

\[
y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \Rightarrow \frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2} x^{1/2}
\]

\[
\left( \frac{dy}{dx} \right)^2 = \left( 2\sqrt{2} x^{1/2} \right)^2 = 8x
\]

\[
L = \int_0^1 \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_0^1 \sqrt{1 + 8x} \, dx
\]

\[
L = \int_0^1 (1 + 8x)^{1/2} \, dx = \frac{1}{8} \int_0^1 (1 + 8x)^{1/2} \, dx = \frac{1}{8} \left[ (1 + 8x)^{3/2} \right]_0^1 = \frac{13}{6}
\]

**Length of a curve \( x = g(y) \): Dealing with Discontinuities in \( dy/dx \)**

If \( g \) is continuously differentiable on \([c, d]\), the length of the curve
$x = g(y)$ from $y = c$ to $y = d$ is:

$$L = \int_c^d \sqrt{1 + (g'(y))^2} \, dy$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

**Example:**

Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x = 0$ to $x = 2$?

**Sol:**

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-\frac{1}{3}} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{\frac{1}{3}}$$

is not defined at $x = 0$, so we cannot find the curve’s length with $y = f(x)$

$$\frac{dy}{dx}$$ fails to exist.

$$y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y^{3/2} = \frac{x}{2} \Rightarrow x = 2y^{3/2}$$

$$\frac{dx}{dy} = 2 \cdot \frac{3}{2} y^{1/2} = 3 \cdot y^{1/2}$$

$x = 0 \Rightarrow y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y = \left(\frac{0}{2}\right)^{2/3} \Rightarrow y = 0$

$x = 2 \Rightarrow y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y = \left(\frac{2}{2}\right)^{2/3} \Rightarrow y = 1 \quad 0 \leq y \leq 1$

$$L = \int_c^d \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dy = \int_0^1 \sqrt{1 + \left(3y^{\frac{1}{2}}\right)^2} \, dy = \int_0^1 \sqrt{1 + 9y} \, dy$$

$$L = \frac{1}{9} \int_0^1 (1 + 9y)^{1/2} \, 9 \, dy = \frac{1}{9} \cdot \left[\frac{2}{3} (1 + 9y)^{3/2}\right]_0^1 = \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27$$

**Areas of Surfaces of Revolution:**

**Surface Area for Revolution About the $x$ – axis:**

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the area of the surface generated by revolving the curve $y = f(x)$ about the $x$ – axis is:
\[ s = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx \]

\[ s = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

**Example:** Find the area of the surface generated by revolving the curve 

\[ y = 2\sqrt{x} \, , \, 1 \leq x \leq 2 \] about the \( x \)–axis?

**Sol:**

\[ s = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

\[ a = 1 \, , \, b = 2 \, , \, \quad y = 2\sqrt{x} \, , \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}} \]

\[ \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} = \sqrt{1 + \frac{1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}} \]

\[ s = \int_1^2 2\pi 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} \, dx = 4\pi \int_1^2 \sqrt{x+1} \, dx = 4\pi \frac{2}{3} [x + 1^{3/2}]_1^3 \]

\[ = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}) \]

**Surface Area for Revolution About the \( y \)–axis:**

If the function \( g(y) \geq 0 \) is continuously differentiable on \([c,d]\), the area of the surface generated by revolving the curve \( x = g(y) \) about the \( y \)–axis is:

\[ s = \int_c^d 2\pi g(y) \sqrt{1 + \left(g'(y)\right)^2} \, dy \]

\[ s = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \]

**Example:** The line segment \( x = 1 - y \, , \, 0 \leq y \leq 1 \) , is revolved about the \( y \)–axis to generate the cone. Find its surface area?

**Sol:** \( c = 0 \, , \, d = 1 \, , \, \quad x = 1 - y \, , \quad \frac{dx}{dy} = -1 \)
\[ \sqrt{1 + \left( \frac{dx}{dy} \right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2} \]

\[ s = \int_c^d 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_0^1 2\pi (1 - y) \sqrt{2} \, dy \]

\[ s = 2\pi \sqrt{2} \left[ y - \frac{y^2}{2} \right]_0 = 2\pi \sqrt{2} \left( 1 - \frac{1}{2} \right) = \pi \sqrt{2} \, . \]

**Surface Area of Revolution for Parametrized curves:**

If a smooth curve \( x = f(t), y = g(t), \ a \leq t \leq b, \) is traversed exactly once as \( t \) increases from \( a \) to \( b \), then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the \( x \)–axis \((y \geq 0)\):
   \[
   s = \int_a^b 2\pi y \sqrt{\left( \frac{dx}{dy} \right)^2 + \left( \frac{dy}{dx} \right)^2} \, dt
   \]

2. Revolution about the \( y \)–axis \((x \geq 0)\):
   \[
   s = \int_a^b 2\pi x \sqrt{\left( \frac{dx}{dy} \right)^2 + \left( \frac{dy}{dx} \right)^2} \, dt
   \]

**Example:**

The standard parametrization of the circle of radius 1 centered at the point (0,1) in the \( xy \)–plane is \( x = \cos t, \ y = 1 + \sin t, \ 0 \leq t \leq 2\pi \) use this parametrization to find the area of the surface swept out by revolving the circle?

**Sol:**

\[
 s = \int_a^b 2\pi y \sqrt{\left( \frac{dx}{dy} \right)^2 + \left( \frac{dy}{dx} \right)^2} \, dt
\]

**Application of integrals**
\[ a = 0 \, , \, b = 2\pi \]

\[ y = 1 + \sin t \, \Rightarrow \frac{dy}{dt} = \cos t \]

\[ x = \cos t \, \Rightarrow \frac{dx}{dt} = -\sin t \]

\[ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{1} = 1 \]

\[ s = \int_a^b 2\pi (1 + \sin t) \cdot dt = 2\pi \int_0^{2\pi} (1 + \sin t) \, dt = 2\pi [t - \cos t]_0^{2\pi} = 4\pi^2 \]