CHAPTER NINE

Vector Analysis

Vector Analysis:

- Scalar: Quantities that are completely known or determined from their magnitude only like length and weight.

- Vector: Quantities that have magnitude and direction like velocity and force.

Component form:

A vector is directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point $A$ and terminal point $B$; its length is denoted by $|\overrightarrow{AB}|$. Two vectors are equal if they have the same length and direction.

If $v$ is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1, v_2)$, then the component form of $v$ is $v = (v_1, v_2)$. $v = (x_2 - x_1, y_2 - y_1)$

If $v$ is a three-dimensional vector equal to the vector with initial point at the origin and terminal point $(v_1, v_2, v_3)$, then the component form of $v$
is \( \mathbf{v} = (v_1, v_2, v_3) \).

\[ \mathbf{v} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \]

* Two vectors are equal if and only if and only if \( u_1 = v_1 \), \( u_2 = v_2 \), and \( u_3 = v_3 \).

* The magnitude or length of the vector \( \mathbf{v} = \overrightarrow{PQ} \) is the non-negative number

\[ |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \]

* The only vector with length 0 is the zero vector \( \mathbf{0} = (0,0) \) or \( \mathbf{0} = (0,0,0) \). This vector is also the only vector with no specific direction.

**Ex1:** Find the (a) component form and (b) length of the vector with initial point \( P(-3,4,1) \) and terminal point \( Q(-5,2,2) \).

**Sol:** (a) \( v_1 = x_2 - x_1 = -5 - (-3) = -2 \)

\( v_2 = y_2 - y_1 = 2 - 4 = -2 \)

\( v_3 = z_2 - z_1 = 2 - 1 = 1 \)

The component form of \( \overrightarrow{PQ} \) is \( \mathbf{v} = (-2, -2, 1) \)

(b) The length or magnitude of \( \mathbf{v} = \overrightarrow{PQ} \) is

\[ |\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = 3. \]

**Vector Algebra operations:**

Let \( u = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) be vectors with \( k \) a scalar

Addition: \( u + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \)

Scalar multiplication: \( ku = (ku_1, ku_2, ku_3) \)

**Vector Analysis**
If \( k > 0 \) \( \Rightarrow \) \( ku \) has the same direction as \( u \).

If \( k < 0 \) \( \Rightarrow \) the direction of \( ku \) is opposite to that of \( u \).

\[
|ku| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)}
\]

\[
= \sqrt{k^2} \sqrt{u_1^2 + u_2^2 + u_3^2} = |k||u|
\]

\((-1)u = -u \) has the same length as \( u \) but points in the opposite direction.

\( u - v = u + (-v) \) difference of two vectors.

if \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \)

\( u - v = (u_1 - v_1, u_2 - v_2, u_3 - v_3) \)

**Ex.2:** Let \( u = (-1,3,1) \) and \( v = (4,7,0) \).

**Find:**

- a) \( 2u + 3v \)
- b) \( u - v \)
- c) \( \frac{1}{2}u \)

**Sol:**

- a) \( 2u + 3v = 2(-1,3,1) + 3(4,7,0) = (-2,6,2) + (12,21,0) = (10,27,2) \)
- b) \( u - v = (-1,3,1) - (4,7,0) = (-1 - 4,3 - 7,1 - 0) = (-5,-4,1) \)
Properties of Vector Operations

Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) be vectors and \( a, b \) be scalars.

1. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)
2. \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \)
3. \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)
4. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \)
5. \( a\mathbf{u} = 0 \)
6. \( 1\mathbf{u} = \mathbf{u} \)
7. \( a(b\mathbf{u}) = (ab)\mathbf{u} \)
8. \( a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \)
9. \( (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \)

Unit Vectors:

A vector \( \mathbf{v} \) of length 1 is called a unit vector. The standard unit vectors are: \( \mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \text{and } \mathbf{k} = (0,0,1) \)

Any vector \( \mathbf{v} = (v_1, v_2, v_3) \) can be written as a linear combination of the standard unit vectors as follows:

\[
\mathbf{v} = (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3)
\]

\[
= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}
\]

We call the scalar (or number) \( v_1 \) the \( i \) – component of the vector \( \mathbf{v} \), \( v_2 \) the \( j \) – component, and \( v_3 \) the \( k \) – component.

\[
\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k
\]

\[
\left| \frac{1}{|v|} \mathbf{v} \right| = \frac{1}{|v|} |v| = 1
\]

\[
\frac{\mathbf{v}}{|v|}
\]

is a unit vector in the direction of \( \mathbf{v} \), called the direction of the nonzero vector \( \mathbf{v} \).
Ex₃:

1. Find a unit vector \( \mathbf{u} \) in the direction of the vector from \( P_1(1,0,1) \) to \( P_2(3,2,0) \)

Sol: \( \overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k \)
\[ \overrightarrow{P_1P_2} = (3 - 1)i + (2 - 0)j + (0 - 1)k = 2i + 2j - k \]
\[ |\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3 \]
\[ \mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2i + 2j - k}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k \]

The unit vector \( \mathbf{u} \) is the direction of \( \overrightarrow{P_1P_2} \).

2. If \( \mathbf{v} = 3i - 4j \) is a velocity vector, express \( \mathbf{v} \) as a product of its speed times a unit vector in the direction of motion.

Sol: Speed is the magnitude (length) of \( \mathbf{v} \)
\[ |\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5. \]

The unit vector \( \left( \mathbf{v}/|\mathbf{v}| \right) \) has the same direction as \( \mathbf{v} \):
\[ \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j \]
\[ \mathbf{v} = 3i - 4j = 5 \left( \frac{3}{5}i - \frac{4}{5}j \right) \quad \mathbf{v} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \]

3. A force of 6 newtons is applied in the direction of the vector \( \mathbf{v} = 2i + 2j - k \). Express the force \( \mathbf{F} \) as a product of its magnitude and direction.

Sol:

The force vector has magnitude 6 and direction \( \frac{\mathbf{v}}{|\mathbf{v}|} \)
\[ \mathbf{F} = 6 \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2i + 2j - k}{\sqrt{(2)^2 + (2)^2 + (-1)^2}} = \frac{2i + 2j - k}{3} = 6 \left( \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k \right) = 6 \left( \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k \right) \]
Mid point of a line Segment :

The midpoint \( M \) of the line segment joining points

\[ P_1(x_1, y_1, z_1) \text{ and } P_2(x_2, y_2, z_2) \]

is the point

\[ \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \]

\[ \overrightarrow{OM} = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{P_1P_2}) = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \]

\[ = \frac{x_1 + x_2}{2} i + \frac{y_1 + y_2}{2} j + \frac{z_1 + z_2}{2} k. \]

Ex.4: Find midpoint of the segment joining \( P_1(3, -2, 0) \) and \( P_2(7, 4, 4) \).

Sol: \( M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \)

\[ = \left( \frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2} \right) = (5, 1, 2) \]

Vector tangent and normal to the curve:

Two vectors are said to be parallel if are scalar multiples of each other.

\[ \overrightarrow{v_2} = c \overrightarrow{v_1} \Rightarrow \overrightarrow{v_2} \text{ parallel to } \overrightarrow{v_1}. \]

Steps to find that vector:

1. Find the slope of the curve at that point = \( y' \) which is equal to slope of the vector.

for any vector like \( \overrightarrow{v} = c_1 i + c_2 j \), slope of the vector = \( \frac{c_2}{c_1} = y' \)

2. Find the unit vector \( \overrightarrow{u} = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} i + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} j \)

\[ \overrightarrow{u} = \frac{\overrightarrow{v}}{|\overrightarrow{v}|}, \text{ The vector } \overrightarrow{u} \text{ is tangent to the curve at that point} \]

because it has the same direction as \( \overrightarrow{v} \), and \( \theta = \tan^{-1} \frac{c_2}{c_1} \)

\[ -\overrightarrow{u} = -\frac{c_1}{\sqrt{c_1^2 + c_2^2}} i - \frac{c_2}{\sqrt{c_1^2 + c_2^2}} j \text{ is the vector which points in the direction opposite to } \overrightarrow{u}, \text{ is also tangent to the curve at that point.} \]
3. To find unit vector normal to the curve at that point, we look for unit vectors whose slopes are the negative reciprocal of the slope of \( u \)

\[
  n = c_2 i - c_1 j, \quad \text{slope} = -\frac{c_1}{c_2} = -\frac{1}{\frac{c_1}{c_2}}, \quad \theta = \tan^{-1} -\frac{c_1}{c_2}.
\]

**Ex_5:** Find the unit vector tangent to the curve \( y = x^2 \) at a point \((2, 4)\), and it's perpendicular to the concave up side (normal).

**Sol:** \( y = x^2 \Rightarrow y' = 2x |_{(2, 4)} \),

\[
y' = 4 \quad \text{slope of the curve equal to slope of the vector}.
\]

\[
\text{slope of the vector} = \frac{c_2}{c_1} = 4 = \frac{4}{1}
\]

\[
\vec{v} = i + 4j \Rightarrow u = \frac{\vec{v}}{|v|} = \frac{1}{\sqrt{17}} i + \frac{4}{\sqrt{17}} j
\]

and in the opposite direction is \(- \vec{v} = -i - 4j\).

\[
-u = -\frac{1}{\sqrt{17}} i - \frac{4}{\sqrt{17}} j \quad \& \quad \theta = \tan^{-1} \frac{c_2}{c_1} = \tan^{-1} 4.
\]

**perpendicular (normal) vector slope** \(- \frac{1}{4} \Rightarrow \frac{1}{4} \), \quad \( n = 4i - j \)

\[
u_n = \frac{4}{\sqrt{17}} i - \frac{1}{\sqrt{17}} j \quad \& \quad \theta = \tan^{-1} -\frac{c_1}{c_2} = \tan^{-1} -\frac{1}{4},
\]

and opposite normal \(- n = -4i + j\)

**Ex_6:** Express the vectors in form of \( ai + bj \) for the following:

1. \( \overrightarrow{P_1 P_2} \) where \( P_1 = (1, 3) \) and \( P_2(2, -1) \)

\[
\overrightarrow{P_1 P_2} = (x_2 - x_1)i + (y_2 - y_1)j = (2 - 1)i + (-1 - 3)j = i - 4j
\]

or \( \overrightarrow{P_1 P_2} = \overrightarrow{P_1 O} + \overrightarrow{OP_2} = (0 - 1)i + (0 - 3)j + (2 - 0)i + (-1 - 0)j
\]

\[
= -i - 3j + 2i - j = i - 4j
\]

or \( \overrightarrow{P_1 P_2} = \overrightarrow{OP_2} + \overrightarrow{OP_1} = (2 - 0)i + (-1 - 0)j - [(1 - 0)i + (3 - 0)j] \)

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**Vector Analysis**
\[ 2i - j - i - 3j = i - 4j \]

2. \( \overline{OP_3} \) if \( O \) is the origin and \( P_3 \) is the midpoint of vector \( \overline{P_1P_2} \) (\( P_1(2,-1) \) & \( P_2(-4,3) \))

Midpoint = \( \left( \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right) \)

\[ P_3 = \left( \frac{2+(-4)}{2}, \frac{-1+3}{2} \right) \Rightarrow P_3(-1,1) \]

\( \overline{OP_3} = (x_3 - x_0)i + (y_3 - y_0)j = (-1 - 0)i + (1 - 0)j \)

\( \overline{OP_3} = -i + j \)

3. From \( A(2,3) \) to the origin

\( \overline{AO} = (x_0 - x_A)i + (y_0 - y_A)j = (0 - 2)i + (0 - 3)j \)

\( \overline{AO} = -2i - 3j \).

4. The unit vector has the same direction of \( 3i - 4j \).

\( \vec{v} = 3i - 4j \), \( |\vec{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5 \)

\( u = \frac{v}{|v|} = \frac{3}{5}i - \frac{4}{5}j \).

5. Find the length and direction of the vector \( \vec{v} = -2i + 3j \).

\( |v| = \sqrt{(-2)^2 + 3^2} = \sqrt{13} \)

\( u = -\frac{2}{\sqrt{13}}i + \frac{3}{\sqrt{13}}j \), \( \vartheta = \tan^{-1} \frac{3}{-2} = 56.3 \text{ in quarter 2} \)

\( \therefore \vartheta = 180 - 56.3 = 123.7 \)

\( \vartheta = \tan^{-1} \frac{\frac{c_2}{c_1}}{\frac{x}{y}} \quad \{ x = -ve \} \Rightarrow \text{quarter 2} \)
The Dot Product:

Angle Between Vectors:

The angle between two nonzero vectors \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) is given by

\[
\theta = \cos^{-1}\left( \frac{u_1v_1 + u_2v_2 + u_3v_3}{|u||v|} \right).
\]

The dot product \( u \cdot v \) (u dot v) of vectors \( u(u_1, u_2, u_3) \) and \( v(v_1, v_2, v_3) \) is

\[
u \cdot v = u_1v_1 + u_2v_2 + u_3v_3
\]

\[
\therefore \cos^{-1}\left( \frac{u \cdot v}{|u||v|} \right)
\]

Ex.7:

1. \((1, -2, -1) \cdot (-6, 2, 3) = (1)(-6) + (-2)(2) + (-1)(-3) = -6 - 4 + 3 = -7
\]

2. \(\left( \frac{1}{2}i + 3j + k \right) \cdot (4i - j + 2k) = \left( \frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1 \)

Note: The dot product of a pair of two - dimensional vectors is defined in a similar fashion:

\((u_1, u_2) \cdot (v_1, v_2) = u_1v_1 + u_2v_2\)

Ex.8:

1. Find the angle between \( u = i - 2j - 2k \) and \( v = 6i + 3j + 2k \)

Sol: \( u \cdot v = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4 \)

\[|u| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3\]

\[|v| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7\]

\[\theta = \cos^{-1}\left( \frac{u \cdot v}{|u||v|} \right), \quad \theta = \cos^{-1}\left( \frac{-4}{3 \cdot 7} \right) \approx 1.76 \text{ radians}.
\]

2. Find the angle \( \theta \) in the triangle \( ABC \) determined by the vertices \( A = (0,0), \quad B = (3,5), \) and \( C = (5,2) \).

Vector Analysis
Sol: The angle $\theta$ is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

The component forms of these two vectors are

$\overrightarrow{CA} = (-5, -2)$ and $\overrightarrow{CB} = (-2, 3)$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$

$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$

$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$

$\theta = \cos^{-1} \left( \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} \right) = \cos^{-1} \left( \frac{4}{(\sqrt{29})(\sqrt{13})} \right) \approx 78.1^\circ$ or $1.36$ radians.

**Perpendicular (Orthogonal) Vectors:**

$\theta = \cos^{-1} \left( \frac{u \cdot v}{|u||v|} \right) \Rightarrow u \cdot v = |u||v| \cos \theta$

Two nonzero vectors $u$ & $v$ are perpendicular or orthogonal if the angle between them is $\frac{\pi}{2}$.

$\cos(\pi/2) = 0 \Rightarrow u \cdot v = 0 \Rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$

**Ex 9:**

1. $u = (3, -2)$ and $v = (4, 6)$

   $u \cdot v = (3)(4) + (-2)(6) = 0$ (u & v are orthogonal).

2. $u = 3i - 2j + k$ and $v = 2j + 4k$

   $u \cdot v = (3)(0) + (-2)(2) + (1)(4) = 0$ (u & v are orthogonal).

**Dot Product Properties and Vector Projections:**

Properties of the Dot Product

If $u$, $v$, and $w$ are any vectors and $c$ is a scalar, then

1. $u \cdot v = v \cdot u$
2. $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
3. $u \cdot (v + w) = u \cdot v + u \cdot w$
4. $u \cdot u = |u|^2$
5. $0 \cdot u = 0$. 

**Vector Analysis**
**Vector projection of \( u \) onto \( v \):**

\[
\text{proj}_v u = \left( \frac{u \cdot v}{|v|^2} \right) v = \left( \frac{u \cdot v}{v \cdot v} \right) v
\]

**Scalar component of \( u \) in the direction of \( v \):**

\[
|u| \cos \theta = \frac{u \cdot v}{|v|} = \frac{u}{|v|} v
\]

*Note: Both the vector projection of \( u \) onto \( v \) and the scalar component of \( u \) onto \( v \) depend only on the direction of the vector \( v \) and not its length (because we dot \( u \) with \( v/|v| \), which is the direction of \( v \)).*

**Ex10:**

1. **Find the vector projection of \( u = 6i + 3j + 2k \) onto \( v = i - 2j - 2k \) and the scalar component of \( u \) in the direction of \( v \).**

**Sol:**

\[
\text{proj}_v u = \left( \frac{u \cdot v}{|v|^2} \right) v = \left( \frac{u \cdot v}{v \cdot v} \right) v = \frac{6 - 6 - 4}{1 + 4 + 4} (i - 2j - 2k)
\]

\[
= -\frac{4}{9} (i - 2j - 2k) = -\frac{4}{9} i + \frac{8}{9} j + \frac{8}{9} k.
\]

\[
|u| \cos \theta = u \cdot \frac{v}{|v|} = (6i + 3j + 2k) \cdot \left( \frac{1}{3} i - \frac{2}{3} j - \frac{2}{3} k \right) = 2 - 2 - \frac{4}{3} = -\frac{4}{3}
\]

*\( |v| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \)*

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**Vector Analysis**
2. Find the vector projection of a force \( F = 5i + 2j \) onto \( v = i - 3j \) and the scalar component of \( F \) in the direction of \( v \).

Sol: \( \text{proj}_v F = \left( \frac{F \cdot v}{v \cdot v} \right) v = \frac{5 - 6}{1 + 9} (i - 3j) = -\frac{1}{10} i + \frac{4}{10} j. \)

\[ |F| \cos \theta = \frac{F \cdot v}{|v|} = \frac{5 - 6}{\sqrt{1 + 9}} = -\frac{1}{\sqrt{10}}. \]

**The Cross Product:**

Two nonzero vectors \( u \) and \( v \) in space are not parallel. We select a unit vector \( n \) perpendicular to the plane by the right-hand rule. \( u \times v = (|u||v| \sin \theta) n \)

Two nonzero vectors \( u \) and \( v \) in space are parallel if and only if \( u \times v = 0 \).

**Properties of the Cross Product:**

<table>
<thead>
<tr>
<th>Properties of the Cross Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( u, v, ) and ( w ) are any vectors and ( r, s ) are scalars, then</td>
</tr>
<tr>
<td>1. ( (ru) \times (sv) = (rs)(u \times v) )</td>
</tr>
<tr>
<td>2. ( u \times (v + w) = u \times v + u \times w )</td>
</tr>
<tr>
<td>3. ( (v + w) \times u = v \times u + w \times u )</td>
</tr>
<tr>
<td>4. ( v \times u = -(u \times v) )</td>
</tr>
<tr>
<td>5. ( 0 \times u = 0 )</td>
</tr>
</tbody>
</table>

\( |u \times v| \) is the area of a parallelogram.

Because \( n \) is a unit vector, the magnitude of \( u \times v \) is

**Vector Analysis**
\[ |u \times v| = |u||v||\sin \theta||n| = |u||v| \sin \theta \]

**Determinant Formula for \( u \times v \):**

If \( u = u_1i + u_2j + u_3k \) and \( v = v_1i + v_2j + v_3k \), then

\[
\begin{vmatrix}
i & j & k \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 \\
\end{vmatrix}
\]

**Ex_{11}:**

1. Find \( u \times v \) and \( v \times u \) if \( u = 2i + j + k \) and \( v = -4i + 3j + k \).

**Sol:**

\[
\begin{vmatrix}
i & j & k \\2 & 1 & 1 \\-4 & 3 & 1 \\
\end{vmatrix} = \begin{vmatrix}1 & 2 & 1 \\1 & -4 & 1 \\2 & 3 & 1 \\
\end{vmatrix} = k
\]

\[ u \times v = -2i - 6j + 10k \]
\[ \therefore v \times u = -(u \times v) = 2i + 6j - 10k \]

2. Find a) vector perpendicular to the plane of \( P(1,-1,0), Q(2,1,-1), \) and \( R(-1,1,2) \). b) area of the triangle. c) unit vector perpendicular to the plane.

**Sol: a)** The vector \( \overrightarrow{PQ} \times \overrightarrow{PR} \) is perpendicular to the plane because it is perpendicular to both vectors.

\[ \overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)k = i + 2j - k. \]

\[ \overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)k = -2i + 2j + 2k. \]
\[ \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & -4 & 1 \\ -4 & 1 & 3 \end{vmatrix} k \\
= 6i + 6k \]

b) \( |\overrightarrow{PQ} \times \overrightarrow{PR}| = |6i + 6k| = \sqrt{6^2 + 6^2} = \sqrt{2 \times 36} = 6\sqrt{2} \) is the area of the parallelogram.

\[ \therefore \text{The triangle's area is half of the parallelogram area} \quad \frac{6\sqrt{2}}{2} = 3\sqrt{2} \]

c) Since \( \overrightarrow{PQ} \times \overrightarrow{PR} \) is perpendicular to the plane, its direction \( n \) is a unit vector perpendicular to the plane

\[ n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6i + 6k}{6\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k. \]