Solution of 2nd order D.E. by Frobenius Method (power series method)

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Frobenius Method

Let b(x) and c(x) be any functions that are analytic at x = 0. Then the ODE

(1) $$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

has at least one solution that can be represented in the form

(2) $$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0)$$

where the exponent r may be any (*real or complex*) *number* (*and r is chosen so that* $a_0 \neq 0$).

The ODE (1) *also has a second solution* (*such that these two solutions are linearly independent*) *that may be similar to* (2) (*with a different r and different coefficients*) *or may contain a logarithmic term.*

Indicial Equation. Indicating the Form of Solutions

We shall now explain the Frobenius method for solving (1). Multiplication of (1) by x^2 gives the more convenient form

(1′) $$\boldsymbol{x^2y'' + xb(x)y' + c(x)y = 0.}$$

We first expand $b(x)$ and $c(x)$ in power series,

$b(x) = b_0 + b_1x + b_2\,x^2 + \dots$, $c(x) = c_0 + c_1x + c_2\,x^2 + \dots$ Then we differentiate (2) term by term, finding

(2*)
$$y'(x) = \sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + \cdots]$$

$$y''(x) = \sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-2}$$

$$= x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + \cdots].$$

By inserting all these series into (1′) we obtain

(3) $$x^r[r(r-1)a_0+\cdots]+(b_0+b_1x+\cdots)x^r(ra_0+\cdots)$$
$$+(c_0+c_1x+\cdots)x^r(a_0+a_1x+\cdots)=0.$$

We now equate the sum of the coefficients of each power x', x'^{r+1}, x''^{r+2}, … to zero. This yields a system of equations involving the unknown coefficients a_m. The smallest power is x^r and the corresponding equation is

$$[r(r-1)+b_0r+c_0]a_0=0.$$

Since by assumption $a_0 \neq 0$, the expression in the brackets […] must be zero. This gives

(4) $$r(r-1)+b_0r+c_0=0.$$

This important quadratic equation is called the **indicial equation** of the ODE (1).

Frobenius Method. Basis of Solutions. Three Cases

Let r_1 and r_2 be the roots of the indicial equation (4). *Then we have the following three cases.*

Case* 1. *Distinct Roots Not Differing by an Integer.
A basis is

(5) $$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdorts)$$

and

(6) $$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots)$$

with coefficients obtained successively from (3) *with $r = r_1$ and $r = r_2$, respectively.*

Case 2. Double Root $r_1 = r_2 = r$

A basis is

(7) $y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \dots)$ $\qquad [r = \frac{1}{2}(1 - b_0)]$

(*of the same general form as before*) *and*

(8) $y_2(x) = y_1(x) \ln x + x^r(A_1 x + A_2 x^2 + \dots)$ $\qquad (x > 0)$

Case 3. Roots Differing by an Integer.

A basis is

(9) $y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$

(*of the same general form as before*) *and*

(10) $y_2(x) = k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots),$

where the roots are so denoted that $r_1 - r_2 > 0$ *and k may turn out to be zero.*

For all cases $\quad y = y_1(x) + y_2(x)$