Fourier Transform

Definition:

$$f(t) \longleftrightarrow F(jw)$$

$$F(jw) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$f(t) = \int_{-\infty}^{\infty} F(jw)\, e^{jwt}\, df$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(jw)\, e^{jwt}\, dw$$

$F(jw)$ is Fourier transform of $f(t)$
Fourier transform is used for energy signal

Note:

① Fourier series is used for periodic signal, and V_n is discrete defined for discrete frequency $f_n = 0, 2f, 3f, \ldots$

② Fourier transform is used for aperiodic signal «Energy signal». The domain is continous and defined for all freqs.

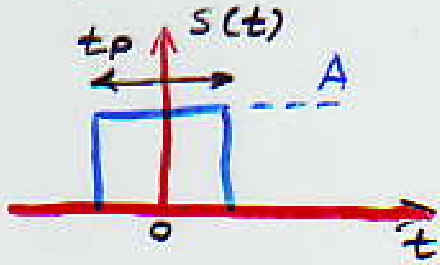
properties of Fourier transform are the same for Fourier series

properties of Fourier transform:

1. Uniqueness $\quad v(t) \longleftrightarrow V(jw)$
2. Superposition $\quad a v(t) + b w(t) \longleftrightarrow a V(jw) + b W(jw)$
3. differentiation $\quad \dfrac{d^n v(t)}{dt^n} \longleftrightarrow (jw)^n V(jw)$
4. Scaling $\quad v(t/a) \longleftrightarrow a V(ajw)$
5. Modulation $\quad v(t)\, e^{jw_0 t} \longleftrightarrow V(jw - jw_0)$
6. Delay $\quad v(t - t_0) \longleftrightarrow V(jw)\, e^{-jw t_0}$
7. Convolution $\quad \int_{-\infty}^{\infty} v(\lambda)\, w(t-\lambda)\, d\lambda \longleftrightarrow V(jw) \cdot W(jw)$
8. Multiplication $\quad v(t) w(t) \longleftrightarrow \int_{-\infty}^{\infty} V(\lambda)\, W(jw - \lambda)\, d\lambda$

- Examples on Fourier transform and its properties.

Ex: Find the Fourier transform of $s(t)$.



Solution: $S(jw) = \int_{-\infty}^{\infty} s(t)\, e^{-jwt}\, dt$

$$S(jw) = \int_{-tp/2}^{tp/2} A\, e^{-jwt}\, dt$$

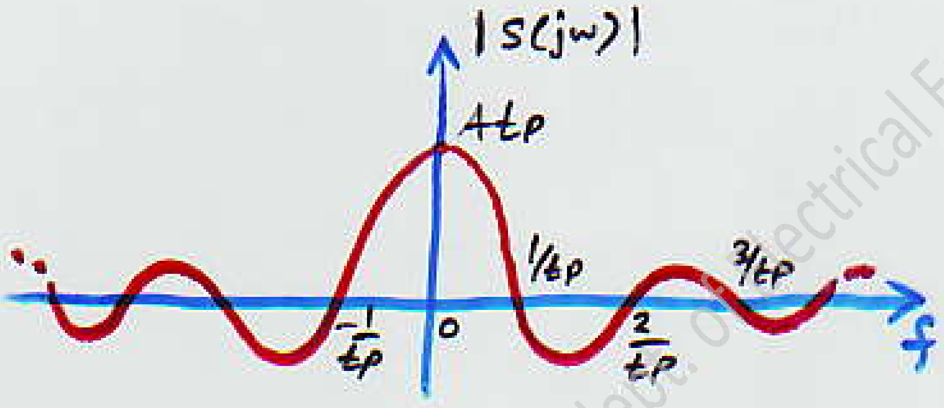
$$= A \left[\frac{1}{-jw} e^{-jwt} \right]_{-tp/2}^{tp/2} = \frac{A}{jw} \left[e^{jw\,tp/2} - e^{-jw\,tp/2} \right.$$

$$S(j\omega) = \frac{2A}{\omega}\sin\frac{\omega t_p}{2} = A t_p \frac{\sin \omega t_p/2}{\omega t_p/2}$$

$$= A t_p \left[\frac{\sin \pi f t_p}{\pi f t_p}\right]$$

$S(j\omega) = 0$ when $\sin(\pi f t_p) = 0$

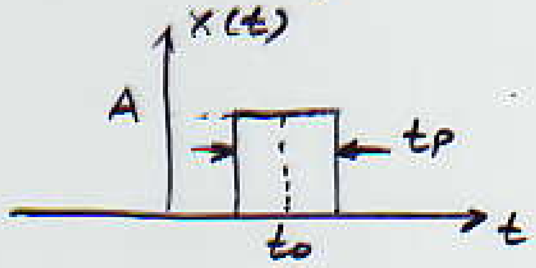
$\therefore \pi f t_p = n\pi \Rightarrow f = n/t_p \quad , n = \mp 1, \mp 2, \mp 3 \dots$



$$|S(j\omega)| = A \cdot t_p \frac{\sin \pi f t_p}{\pi f t_p} \quad = \text{Amplitude c/c}$$

$$\angle S(j\omega) = 0 \quad = \text{phase c/c}$$

Ex: Find the Fourier transform of the signal shown



Sol: This signal is the signal of the previous example shifted by t_o. using the delay property

$$x(t) = S(t - t_o) \Leftrightarrow S(j\omega)e^{-j\omega t_o}$$

$$= A \cdot t_p \left[\frac{\sin \pi f t_p}{\pi f t_p}\right] e^{-j\omega t_o}$$

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Ex: Find FT of delta funct. $\delta(t)$.

Sol:

$$s(t) = \delta(t)$$

$$S(jw) = \int_{-\infty}^{\infty} s(t)\, e^{-jwt}\, dt = e^{-jwo} = 1$$

$$\therefore\ \delta(t) \longleftrightarrow 1$$

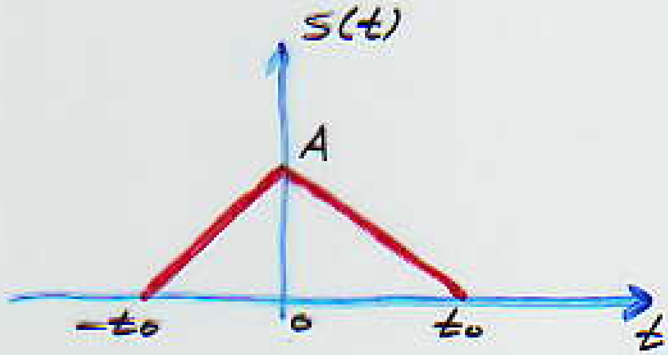
Note that $$\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1$$

and $$\int_{-\infty}^{\infty} \delta(t-t_0)\, f(t)\, dt = f(t_0)$$

for any function $f(t)$

Ex: Find FT of the signal shown in the figure

$$S(jw) = \int_{-t_0}^{t_0} s(t)\, e^{-jwt}\, dt$$



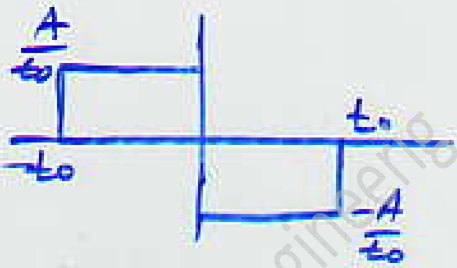
$$s(t) = \begin{cases} A + \dfrac{A}{t_0}\, t & -t_0 \leq t \leq 0 \\ A - \dfrac{A}{t_0}\, t & 0 \leq t \leq t_0 \end{cases}$$

$$S(jw) = \int_{-t_0}^{0} \left(A + \frac{A}{t_0} t\right) e^{-jwt}\, dt + \int_{0}^{t_0} \left(A - \frac{A}{t_0} t\right) e^{-jwt}$$

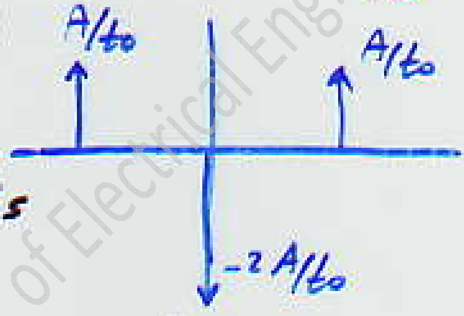
$$S(jw) = \frac{2A}{t_0 w^2} \left[1 - \cos w t_0\right]$$

Ex: Solve the preivous example using the properties of FT.

$$s(t) = \begin{cases} A + \frac{t}{t_o}A & -t_o \le t \le 0 \\ A - \frac{t}{t_o}A & 0 \le t \le t_o \end{cases}$$



$$\frac{ds(t)}{dt} = \begin{cases} \frac{A}{t_o} & -t_o \le t \le 0 \\ -\frac{A}{t_o} & 0 \le t \le t_o \end{cases}$$



$$\frac{d^2 s(t)}{dt^2} = \frac{A}{t_o}\left[\delta(t+t_o) - 2\delta(t) + \delta(t-t_o)\right]$$

FT of the 2^{nd} derivation is

$$\frac{A}{t_o}\int_{-\infty}^{\infty} \delta(t+t_o) - 2\delta(t) + \delta(t-t_o)\, e^{-j\omega t}\, dt$$

$$= \frac{A}{t_o}\left[e^{j\omega t_o} - 2 + e^{-j\omega t_o}\right] = \frac{A}{t_o}\left[2\cos\omega t_o - 2\right]$$

since $\frac{d^2 s(t)}{dt^2} \longleftrightarrow (j\omega)^2 S(j\omega)$

$$\mathcal{F}\left\{\frac{d^2 s(t)}{dt^2}\right\} = -\omega^2 S(j\omega)$$

$$S(j\omega) = -\frac{\mathcal{F}\left\{\frac{d^2 s(t)}{dt^2}\right\}}{\omega^2} = -\frac{\frac{A}{t_o}\left\{2\cos\omega t_o - 2\right\}}{\omega^2}$$

$$S(j\omega) = \frac{2A}{\omega^2 t_o}\left[1 - \cos\omega t_o\right]$$

Summary: $f(t) \longleftrightarrow F(j\omega) = R(\omega) + jX(\omega)$

$F(j\omega) = A(\omega)\underline{/\phi(\omega)}$

$A(\omega)$ is even function and $\phi(\omega)$ is odd if $f(t)$ is even function then

$$R(w) = 2\int_0^{\infty} f(t)\cos wt\,dt$$

$$X(w) = 0$$

if $f(t)$ is odd function then

$$R(w) = 0 \quad \& \quad X(w) = -2\int_0^{\infty} f(t)\sin wt\,dt$$

Q: Find FT of $s(t) = \begin{cases} A\cos W_c t & -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$

Q: Find FT of $s(t) = \begin{cases} e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$

Rayleigh theorm & energy Spectral

Rayleigh theorem relates the energy in the freq. domain to the energy in the time domain. it is like Parseval's theorem which relates the power.

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(jw)|^2 df$$

we can now define the energy spectral density $W(w)$

$$W(w) = |F(jw)|^2$$

Correlation & Convolution

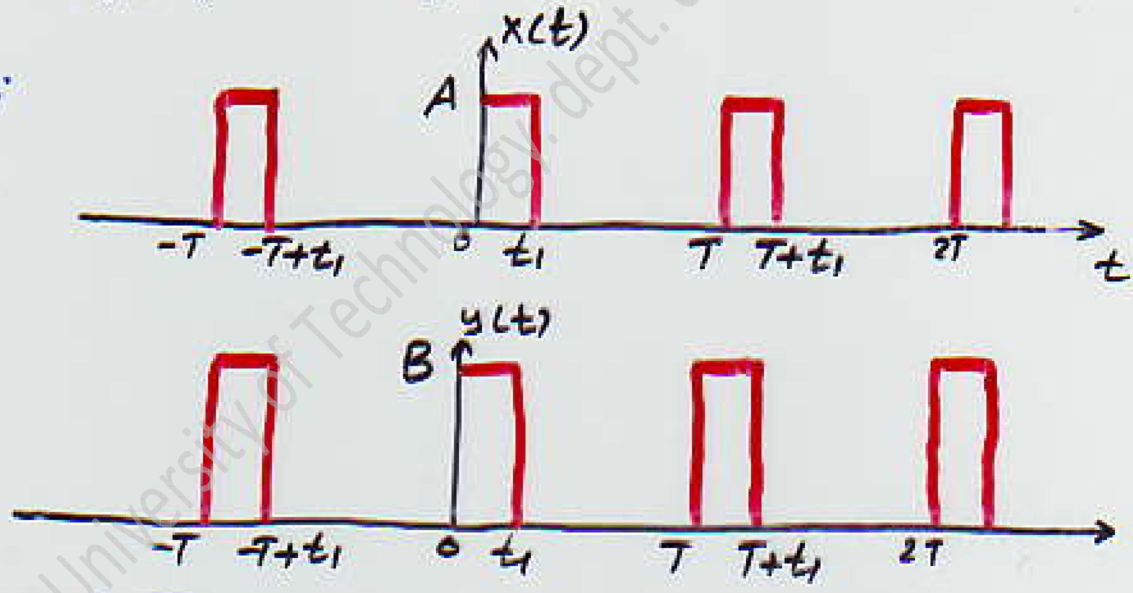
Cross-Correlation function: is a measure of similarities between two signals.

if $x(t)$ and $y(t)$ are two periodical signals with period T, then the cross correlation is:

$$R_{xy}(\tau) = \frac{1}{T}\int_{t'}^{t'+T} x(t)\,y(t+\tau)\,dt$$

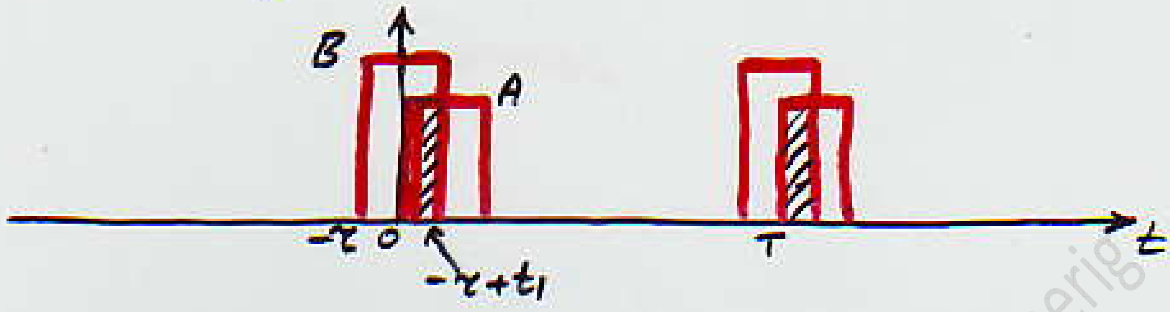
where τ is continuous time displacement in the range $(-\infty, \infty)$. $x(t)$, $y(t)$ are periodic so $R_{xy}(\tau)$ is also periodic.

Ex:



Find $R_{xy}(\tau)$?

$$R_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)\,y(t+\tau)\,dt$$

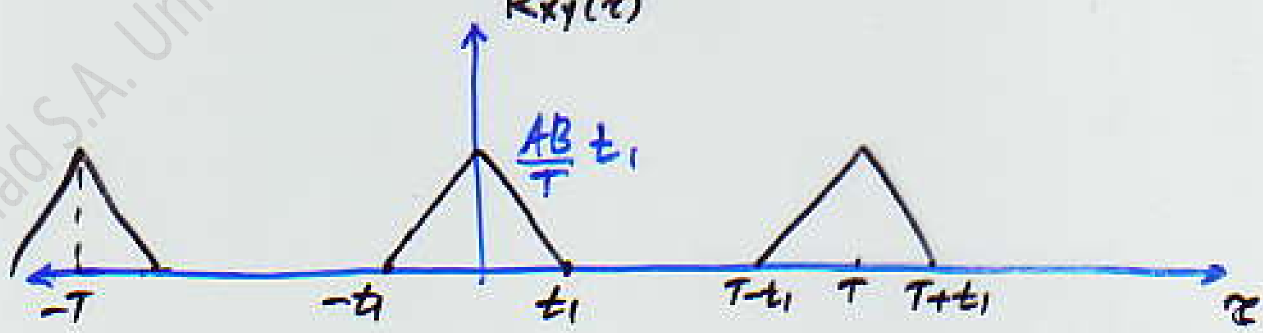


for $0 \leq \tau < t_1 \longrightarrow 0 \geq -\tau > -t_1$

$$R_{xy}(\tau) = \frac{1}{T}\int_0^{-\tau+t_1} AB\,dt = \frac{AB}{T}(t_1-\tau)$$

for $-t_1 \leq \tau < 0 \longrightarrow t_1 \geq -\tau > 0$

$$R_{xy}(\tau) = \frac{1}{T}\int_{-\tau}^{t_1} A\cdot B\,dt = \frac{AB}{T}(t_1+\tau)$$



it is also periodic function

Correlation theorm

if $X_n \longleftrightarrow x(t)$

$Y_n \longleftrightarrow y(t)$

Then $$R_{xy}(\tau) = \frac{1}{T}\int_{t'}^{t'+T} x(t)\,y(t+\tau)\,dt$$

$$(R_{xy})_n = X_n \cdot Y_n^*$$

Auto-correlation:

The auto-correlation is a special case from the cross-correlation. Auto-correlation represents the cross-correlation of the function with its self.

$$R_{xx}(\tau) = \frac{1}{T}\int_{t'}^{t'+T} x(t)\,x(t+\tau)\,dt$$

$$(R_{xx})_n = X_n \cdot X_n^*$$

$$(R_{xx})_n = |X_n|^2$$

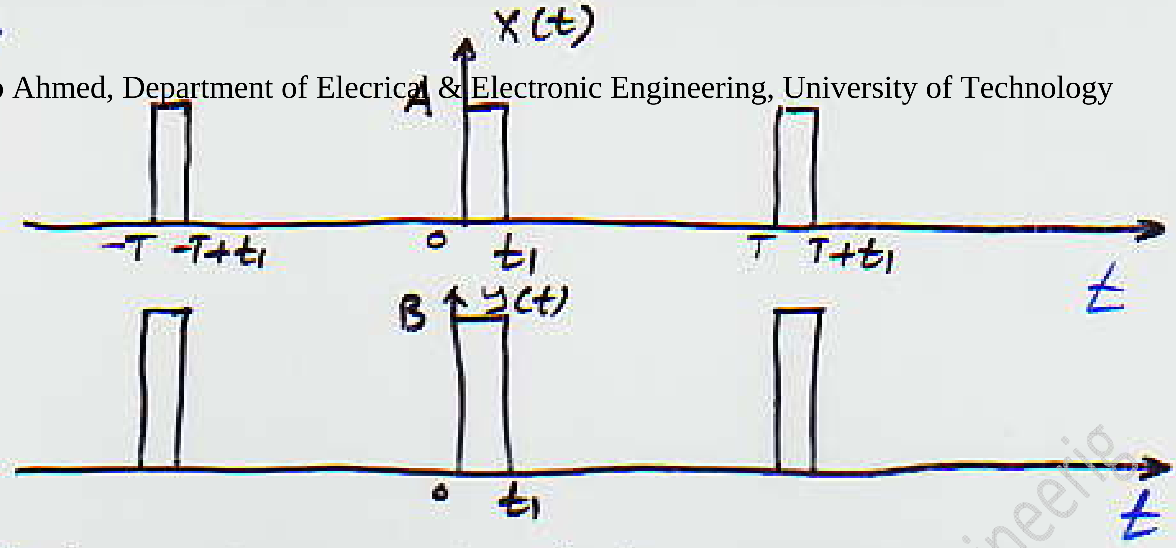
Convolution:

if $x(t)$ is periodic function and $y(t)$ is also periodic of the same period T, Then the convolution between $x(t)$ & $y(t)$ is $z(t)$

$$z(\tau) = \frac{1}{T}\int_{t'}^{t'+T} x(t)\,y(\tau - t)\,dt$$

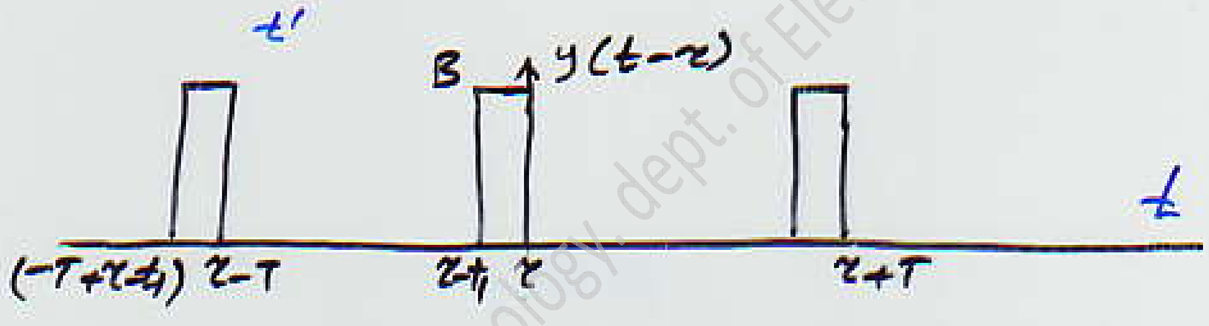
$z(\tau)$ is also periodic function of period T.

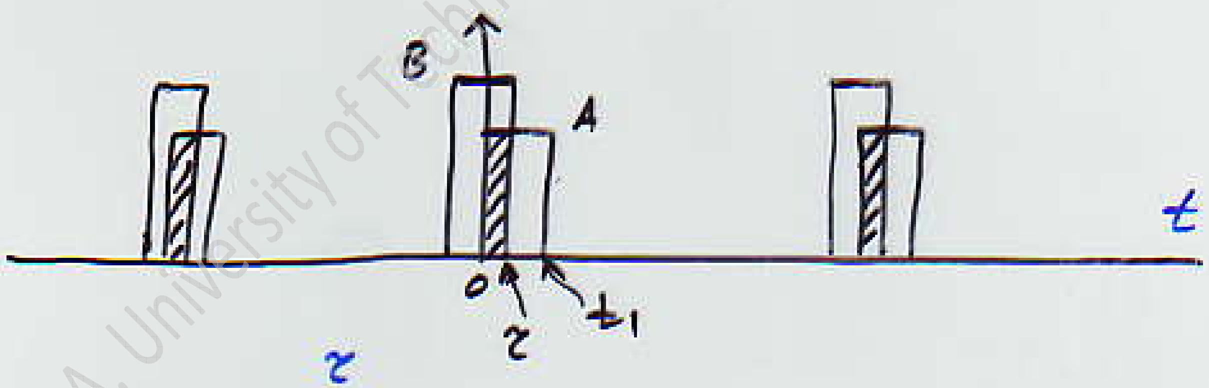
Ex:



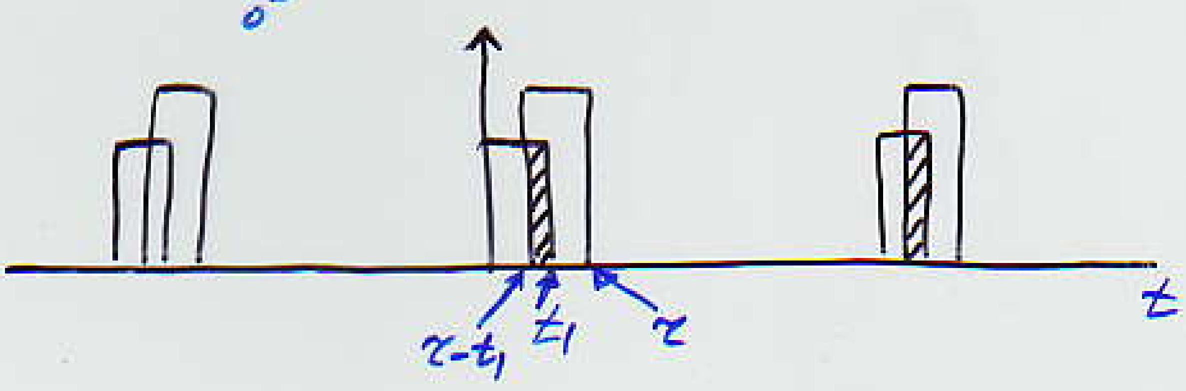
Find the Convolution between x(t) & y(t).

$$Z(\tau) = \frac{1}{T}\int_{t'}^{t'+T} x(t)\, y(\tau - t)\, dt$$



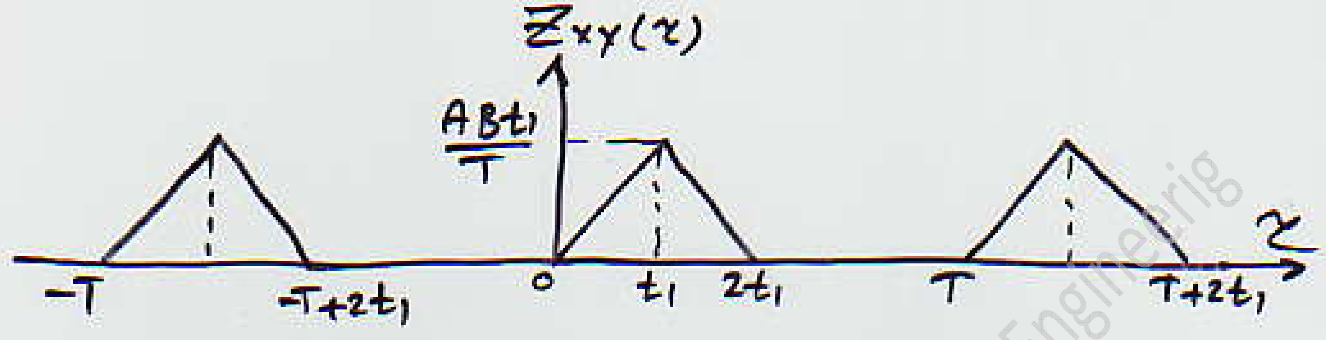


$$Z(\tau) = \frac{1}{T}\int_{0}^{\tau} A.B\, dt = \frac{AB}{T}\tau \qquad 0<\tau\leq t_1$$



$$Z(\tau) = \frac{1}{T}\int_{\tau-t_1}^{t_1} A\cdot B\,dt = \frac{AB}{T}(2t_1-\tau)$$

$$t_1 < \tau < 2t_1$$



Convolution theorm

$$x(t) \longleftrightarrow X_n$$
$$y(t) \longleftrightarrow Y_n$$

$$Z(\tau) = \frac{1}{T}\int_{t'}^{t'+T} x(t)\,y(\tau-t)\,dt$$

$$Z_n = X_n \cdot Y_n$$

Correlation and convolution of energy Signal

Auto correlation function

$$R(\tau) = \int_{-\infty}^{\infty} f(t)\cdot f(t+\tau)\,d \qquad -\infty < \tau < \infty$$

$$R(\tau) = R(-\tau) \qquad \text{(even function)}$$

$$R(0) = \int_{-\infty}^{\infty} |f(t)|^2\,dt = \text{energy of the signal}$$

$$R(0) = \int_{-\infty}^{\infty} |f(t)|^2 = \int_{-\infty}^{\infty} |F(jw)|^2\,df$$

Theorm :

if $s(t)$ is energy signal then

$$s(t) \longleftrightarrow S(j\omega)$$

$$R(\tau) \longleftrightarrow |S(j\omega)|^2 = W(\omega)$$

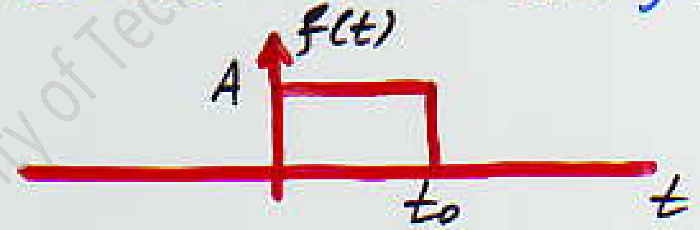
it means $R(\tau) = F^{-1}\{|S(j\omega)|^2\}$

$$|S(j\omega)|^2 = F\{R(\tau)\}$$

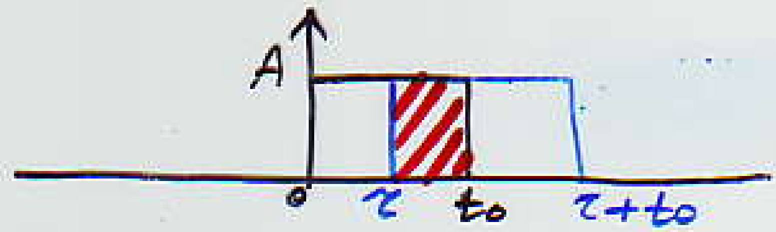
$$|S(j\omega)|^2 = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R(\tau) = \int_{-\infty}^{\infty} |S(j\omega)|^2\, e^{-j\omega\tau}\, df$$

Ex: Find the auto-correlation of the signal



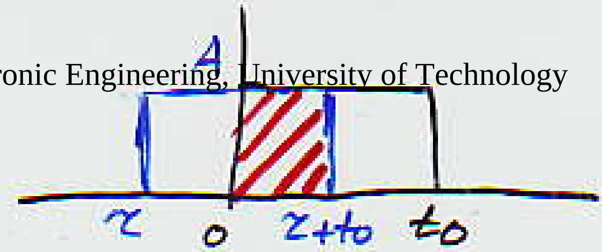
Sol:



for $0 \leq \tau \leq t_o$

$$R(\tau) = \int_{\tau}^{t_o} A.A\, dt = A^2(t_o - \tau)$$

For $-t_0 \le \tau \le 0$



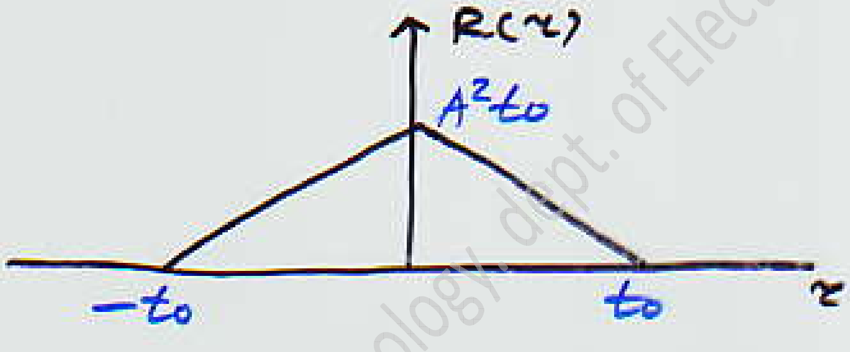
$$R(\tau) = \int_0^{t_0+\tau} A \cdot A \, dt$$

$$R(\tau) = A^2 (t_0 + \tau)$$

for $\tau > t_0$ $\quad R(\tau) = 0$

hence

$$R(\tau) = \begin{cases} A^2 (t_0 + \tau) & -t_0 \le \tau \le 0 \\ A^2 (t_0 - \tau) & 0 \le \tau \le t_0 \\ 0 & \text{otherwise} \end{cases}$$



Note that :

1. $R(\tau)$ is even function
2. $R(0) = A^2 t_0$ = energy of the signal
3. $R(0) \geq R(\tau)$

   or $|R(\tau)| \le R(0)$

Convolution :

Convolution integral represents the relation between the input and the output of Linear system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) \circledast h(t) = h(t) \circledast x(t)$$

$\circledast$ denote convolution

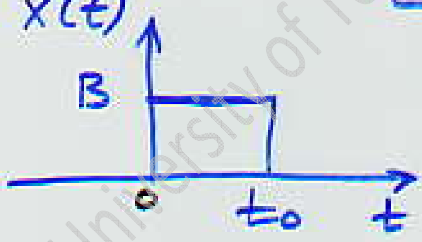
$$y(\tau) = \int_{-\infty}^{\infty} x(t)\, h(\tau - t)\, dt$$

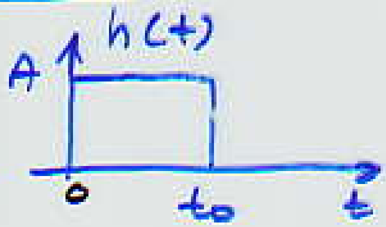
$$= \int_{-\infty}^{\infty} h(t)\, x(\tau - t)\, dt$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

Ex: Find the output signal of the system shown

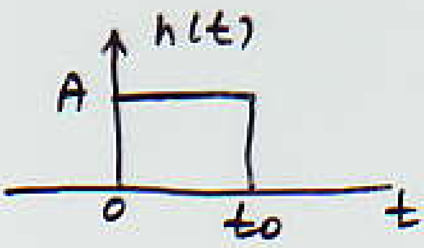
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

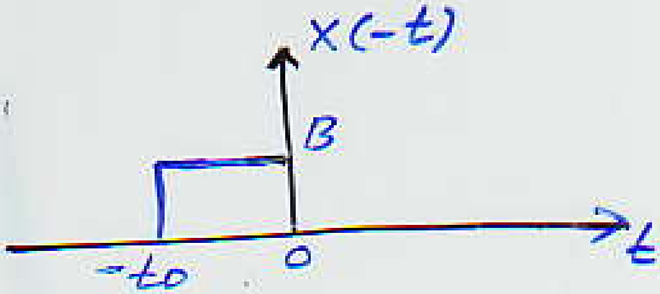




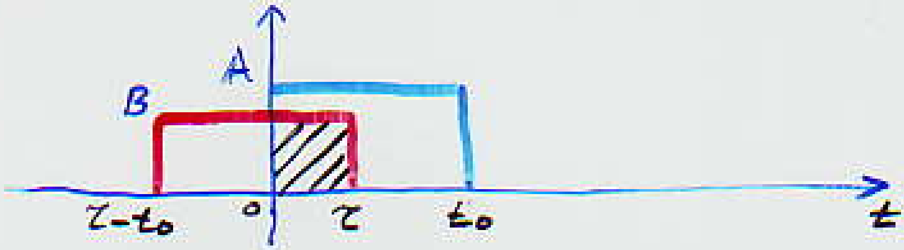
Sol:

$$y(\tau) = \int_{-\infty}^{\infty} h(t)\, x(\tau - t)\, dt$$



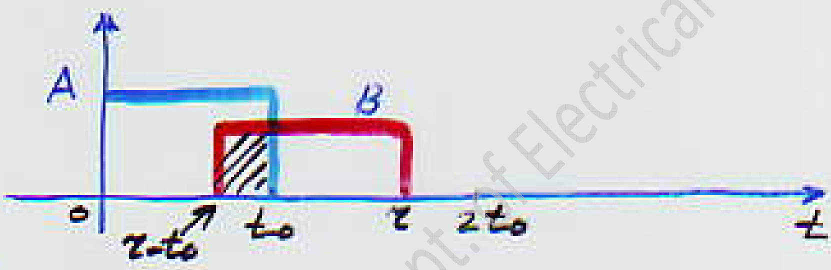


For $0 \le \tau \le t_0$, $x(\tau - t)$ is shown



$$y(\tau) = \int_0^{\tau} A \cdot B \, dt = AB\tau$$

For $t_0 \le \tau \le 2t_0$



$$y(\tau) = \int_{\tau - t_0}^{t_0} A \cdot B \, dt = AB\left[t_0 - (\tau - t_0)\right]$$

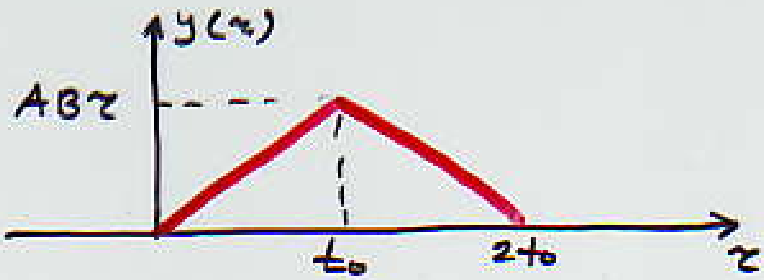
$$= A \cdot B\,[2t_0 - \tau]$$

$y(\tau) = 0$ for $\tau > 2t_0$

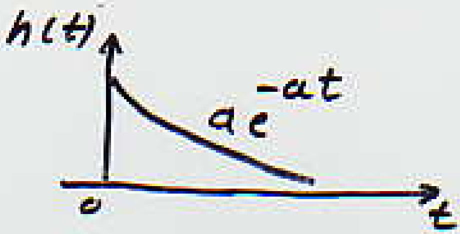
$y(\tau) = 0$ for $\tau < 0$

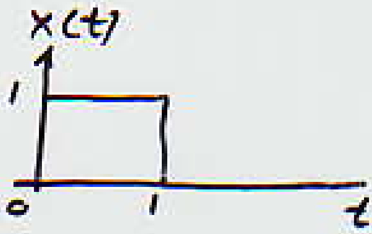
hence

$$y(\tau) = \begin{cases} AB\tau & 0 \le \tau < t_0 \\ AB[2t_0 - \tau] & t_0 \le \tau \le 2t_0 \\ 0 & \text{otherwise} \end{cases}$$



Ex: Find $y(t)$ for the following input $x(t)$ and system $h(t)$.



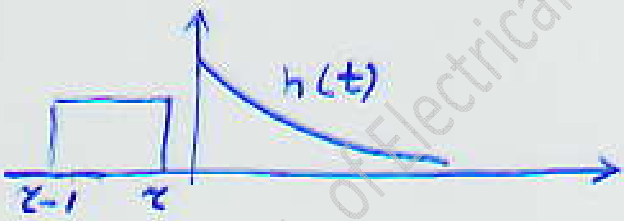


Solution:

$$y(\tau) = \int_{-\infty}^{\infty} h(t)\, x(\tau - t)\, dt$$

for $\tau < 0$

$y(\tau) = 0$



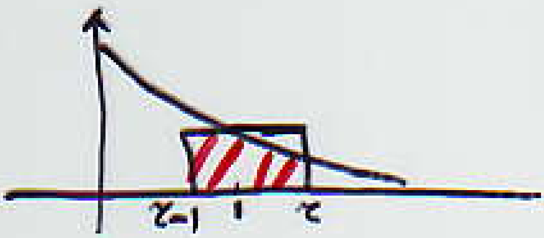
For $0 < \tau < 1$

$$y(\tau) = \int_{0}^{\tau} 1 \cdot a e^{-at}\, dt$$

$$= -e^{-at}\Big]_{0}^{\tau} = 1 - e^{-a\tau}$$

for $\tau > 1$

$$y(\tau) = \int_{\tau-1}^{\tau} a e^{-at}\, dt$$



$$= -e^{-at}\Big]_{\tau-1}^{\tau} = e^{-a(\tau-1)} - e^{-a\tau} = e^{-a\tau}\left[e^{a} - 1\right]$$