Communication Systems

Lecture 12
**Probability of bit error for noncoherently detected binary orthogonal FSK**

- Consider the equally likely binary orthogonal FSK signal set \([S_i(t)]\), defined previously as follows;

\[
s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_t + \phi) \quad 0 \leq t \leq T \quad \cdots \cdots \cdots (12.1)
\]

The phase term \(\phi\) is unknown and assumed constant. The detector is characterized by \(M=2\) channels of bandpass filters and envelope detector, as shown in fig(12.1). The input to the detector consists of received signal.

\[
r(t) = S_i(t) + n(t)
\]

where \(n(t)\) is a white two sided power spectral density \(N_0/2\).

- Assume that \(S_1(t)\) and \(S_2(t)\) being equally likely, we start the bit error probability \(P_B\) computation, which is used in baseband signaling.

\[
P_B = \frac{1}{2} P(H_2|S_1) + \frac{1}{2} P(H_1|S_2) \quad \cdots \cdots (12.2)
\]

Where \(P(H_2|S_1)\) and \(P(H_1|S_2)\) are conditional probabilities

\[
P_B = \frac{1}{2} \int_{-\infty}^{0} P(Z|S_1)dz + \frac{1}{2} \int_{0}^{\infty} P(Z|S_2)dz \quad \cdots \cdots (12.3)
\]

Where \(P(Z|S_1)\) and \(P(Z|S_2)\) are conditional pdfs.
- For binary case, the test statistic $Z(T)$ is defined by $Z_1(T) - Z_2(T)$.

- Assume that the bandwidth of the filter $W_f$ is $1/T$, so that the envelope of the FSK signal is preserved at the filter output.

- If there was no noise at the receiver, the value of

$$Z(T) = \sqrt{2E/T} \text{ when } S_1(t) \text{ is sent}$$

and

$$Z(T) = -\sqrt{2E/T} \text{ when } S_2(t) \text{ is sent}$$

Because of symmetry, the optimum threshold is $\gamma_0 = 0$.

- The pdf $P(Z/S_1)$ is similar to $P(Z/S_2)$, that is

$$P(Z/S_1) = P(-Z/S_2) \quad \ldots \ldots (12.4a)$$

Therefore, we can write

$$P_B = \int_{-\infty}^{\infty} P(Z/S_2) dz \quad \ldots \ldots (12-4b)$$

or

$$P_B = (Z_1)Z_2/S_2 \quad \ldots \ldots (12-4c)$$

Where $Z_1$ and $Z_2$ denote the outputs $Z_1(T)$ and $Z_2(T)$ from the envelope detections shown in Fig(12.1).
For the case in which the tone $S_2(t) = \cos \omega_2 t$ is sent, such that $r(t) = S_2(t) + n(t)$, the output $Z_1(t)$ is a Gaussian noise random variable only, it has no signal component.

A Gaussian distribution into the nonlinear envelope detector yields a Rayleigh distribution at the output, so that

$$p(z_1|s_2) = \begin{cases} \frac{z_1}{\sigma_0^2} \exp \left( -\frac{z_1^2}{2\sigma_0^2} \right) & z_1 \geq 0 \\ 0 & z_1 < 0 \end{cases}$$

Where $\sigma_0^2$ is the noise at the filter output. On the other hand $Z_2(t)$ has a Rician distribution, since the input to the lower envelope detector is a sinusoid pulse noise. The pdf $P(Z_2/S_2)$ is written as

$$p(z_2|s_2) = \begin{cases} \frac{z_2}{\sigma_0^2} \exp \left( -\frac{(z_2^2 + A^2)}{2\sigma_0^2} \right) I_0 \left( \frac{z_2 A}{\sigma_0} \right) & z_2 \geq 0 \\ 0 & z_2 < 0 \end{cases}$$

Where $A = \sqrt{2E/T}$ and as before $\sigma_0^2$ is the noise at the filter output. The function $I_0(x)$ known as the modified Bessel function of the first kind, is defined as

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp (x \cos \theta) \, d\theta$$

When $S_2(t)$ is transmitted, the receiver makes an error whenever the envelope sample $Z_1(T)$ obtained from the upper channel (due to noise alone) exceeds the envelope sample $Z_2(T)$ obtained from the lower channel (due to channel plus noise). Thus the probability of error is given by
\[ P_B = P(Z_1)Z_2 / S_2 \]
\[ = \int_0^\infty P(Z_2 / S_2) \left[ \int P(Z_1 / S_2) dz_1 \right] dz_2 \]

\[ P_B = \frac{1}{2} \exp \left( -\frac{A^2}{4\sigma_0^2} \right) \] \hspace{1cm} (12-7)

where \( A = \sqrt{2E/T} \)

The filter output noise = \( \sigma_0^2 = 2 \left( \frac{N_o}{2} \right) W_f \) \hspace{1cm} (12-8)

Where \( G(f)=N_o/2 \) and \( W_f \) is the filter bandwidth.

substitute eq(12.8) into eq(12.7)

\[ \therefore P_B = \frac{1}{2} \exp \left( -\frac{A^2}{4\sigma_0^2} \right) \] \hspace{1cm} (12-9)

Eq(12.9) indicates that the error performance depends on the bandpass filter bandwidth, and that \( P_B \) becomes smaller as \( W_f \) is decreased. The minimum \( W_f \) allowed is obtained by using raised-cosine filter with roll of \( r=0 \).

\[ W_f = (1+r)R_s \]

\[ \therefore W_f = R_s \text{ if } r = 0 \quad (\text{where } R_s = \text{symbol rate}) \]

\[ W_f = R \text{ bits/s} = \frac{1}{T} \quad (\text{for binary case}) \] \hspace{1cm} (12-10)

Substitute eq(12.10) into eq(12.9)
\[ P_B = \frac{1}{2} \exp \left( \frac{-A^2 T}{4N_0} \right) \ldots \ldots (12.11) \]

\[ P_B = \frac{1}{2} \exp \left( \frac{-E_b}{2N_0} \right) \ldots \ldots (12.12) \]

*where* \( E_b = \frac{1}{2} A^2 T \) *is the energy per bit*

-When comparing the error performance of noncoherent FSK with coherent FSK, it is seen that
a-For the same \( P_B \), noncoherent FSK requires approximately 1dB more than for coherent FSK.
b-The noncoherent receiver is easier to implement because coherent reference signals need not be generated.

**Probability of bit error for DPSK**

-Let us define a BPSK set as follows:-

\[ x_1(t) = \sqrt{\frac{E_b}{T_b}} \cos(w_b t + \phi) \quad 0 \leq t \leq T \]

\[ x_2(t) = \sqrt{\frac{E_b}{T_b}} \cos(w_b t + \phi + \pi) \quad 0 \leq t \leq T \] (12.3)

fig(12.2a)
Where $T_b$ is the bit duration and $E_b$ is the signal energy per bit.

A characteristic of DPSK is that there are no fixed decision regions in the signal space. The decision is based on the phase difference between successively received signals. Suppose the transmitted DPSK signal is $\chi_1(t)$ for $0 \leq t \leq T$. Then for DPSK signaling we are transmitting each bit with the binary signal pair as follow:

$S_1(t) = \chi_1(t)$, $\chi_1(t) \iff$ if we have binary symbol 1 at the transmitter input for the second part of the interval $T_b \leq t \leq 2T_b$

$S_2(t) = \chi_1(t)$, $\chi_2(t) \iff$ if we have binary symbol 0 at the transmitter input for the second part of the interval $T_b \leq t \leq 2T_b$

The interval $T_b \leq t \leq 2T_b$

<table>
<thead>
<tr>
<th>$b_k$</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
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<tbody>
<tr>
<td>Differential Encoded</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Sequenced $d_k$</td>
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<td>0</td>
<td>$\pi$</td>
<td>0</td>
<td>0</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Transmitted</td>
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<td>0</td>
<td>$\pi$</td>
<td>0</td>
<td>0</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>

Phase shift

where $\chi_i, \chi_j (i, j = 1, 2)$ denotes $\chi_i(t)$ followed by $\chi_j(t)$

also $\chi_1(t)$ and $\chi_2(t)$ are antipodal signal. Fig(12.3) shows the equivalent two channel detector for binary DPSK.

![Diagram](12.3)
-Since pairs of DPSK signals are orthogonal, such noncoherent detection operates with the bit error probability given by

\[
P_B = \frac{1}{2} \exp\left( -\frac{E_b}{2N_0} \right) \quad \text{(12.13) [previously derived]}
\]

However since the DPSK signals have a bit interval of 2T, thus, the Si(t) signals have twice the energy defined over a single symbol duration. Thus we may write the eq(12.13) as:-

\[
P_B = \frac{1}{2} \exp\left( -\frac{E_b}{N_0} \right) \quad \text{(12.14)}
\]

**Comparison of Bit error performance for various modulation methods**

<table>
<thead>
<tr>
<th>Modulation</th>
<th>PB</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSK (coherent)</td>
<td>( Q\left( \sqrt{\frac{2E_b}{N_0}} \right) )</td>
</tr>
<tr>
<td>DPSK</td>
<td>( \frac{1}{2} \exp\left( -\frac{E_b}{N_0} \right) )</td>
</tr>
<tr>
<td>Orthogonal FSK (coherent)</td>
<td>( Q\left( \sqrt{\frac{E_b}{N_0}} \right) )</td>
</tr>
<tr>
<td>Orthogonal FSK (noncoherent)</td>
<td>( \frac{1}{2} \exp\left( -\frac{E_b}{2N_0} \right) )</td>
</tr>
</tbody>
</table>
M-ary signaling and performance

In the case of M-ary, the modulator produce one of $M=2^k$ waveforms, for each k-bits, binary signaling is the special case where $k=1$.

Fig(12.5) shows the probability of bit error $P_B(M)$ versus $E_b/N_0$ for coherently detected MFSK signaling over Gaussian channel

Fig(12.6) shows $P_B(M)$ versus $E_b/N_0$ for coherently detected MPSK signaling over Gaussian channel
Fig(12.5) $P_B$ for coherently detected MFSK

Fig(12.6) $P_B$ for coherently detected MPSK

Fig(12-5) and Fig(12.6) show that
1-M-ary signaling produces improved error performance with MFSK and degraded error performance with MPSK.

2-For the curves characterizing MFSK, as k increases the required bandwidth also increases.

3-For the MPSK, as k increases a larger bit rate can be transmitted within the same bandwidth. In other words for fixed data rate, the required bandwidth is decreased.

**BPSK and QPSK have the same bit error Probability**

In equation (6.10) we stated the general relationship between $E_b/N_0$ and $S/N$ which is rewritten:

$$\frac{E_b}{N_0} = S \left( \frac{W}{R} \right) \text{.......(6.10)}$$

Where $S$ is the average signal power and $R$ is the bit rate QPSK (quadriphase shift keying). This is four phase PSK with $M=4$. QPSK can be consider of as two binary PSK systems in parallel in which the carrier are in phase quadrature. Thus QPSK bit stream is usually partitioned into an even and odd (I and Q) stream, each new stream modulate an orthogonal component of the carrier at half the bit rate of the original stream. The I stream modulate the $\cos w_0 t$ and the Q stream modulate the $\sin w_0 t$.

-If the magnitude of the original QPSK vector has the value $A$, the magnitude of the I and Q component vectors each has a value of $A/\sqrt{2}$ as shown in fig(12.7). Hence, the original QPSK waveform
has a bit rate $R$ bit/s and average power of $S$ watts, the quadrature partitioning results in each of the BPSK waveforms having a bit rate $R/2$ and average power of $S/2$ watts.

$$\therefore \frac{E_b}{N_0} = \frac{S/2}{N_0} \left( \frac{W}{R/2} \right)$$

$$\frac{E_b}{N_0} = \frac{S}{N_0} \frac{W}{R} \quad (12.15)$$

Thus each of the orthogonal BPSK (components of QPSK) has the same $E_b/N_0$ and hence the same $P_B$.

Fig (12.7)

**Probability of symbol error for MPSK**

- For large energy –to noise ratios, the symbol error Performance $P_E(M)$ for equally likely, coherently detected $M$-ary PSK signaling, can be expressed as

$$P_E(M) \approx 2Q \left( \sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{M} \right) \quad (12.16)$$

12-12
Where $P_E(M)$ is the probability of symbol error.

$E_s = E_b \log_2 M$ is the energy per symbol.

$M = 2^k$ is the size of the symbol set.

Fig (12.8) shows the performance curves for coherently detected MPSK signaling versus $E_b / N_o$.

The symbol error performance for differentially coherent detection of M-ary DPSK, can be expressed as

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{\sqrt{2M}}\right)$$

...............(12.17)
Probability of symbol error for MFSK

The symbol error performance $P_E(M)$ for equally likely, coherently detected M-ary orthogonal signaling can be upper bounded as.

$$P_E(M) \leq (M - 1) Q \left( \sqrt{\frac{E_s}{N_0}} \right)$$

Where $E_s = E_b \log(M)$ is the energy per symbol

$M = 2^k$ is the size of the symbol set.

Fig(12.9) shows the $P_E(M)$ performance curves for coherently detected M-ary orthogonal signaling are plotted versus $\frac{E_b}{N_0}$.

Fig (12.9)

-An upper bound for noncoherent reception of MFSK is given by

$$P_E(M) < \frac{M - 1}{2} \exp \left( -\frac{E_s}{2N_0} \right)$$

...............(12.19)
**Bit error Probability versus symbol error probability**

1-For MFSK

The relationship between probability of bit error \( P_B \) and probability of symbol error \( P_E \) for an MFSK is

\[
P_B = P_E \frac{2^{k-1}}{2^k - 1} = P_E \frac{M/2}{M - 1} \quad (12.20)
\]

Fig(12.10) shows an octal message set. The example in fig(12.10) indicates that the symbol comprising bits 011 was transmitted.

1-Notice that just because a symbol error is made does not mean that all bits with symbol will be in error.

2-If the receiver decides that the transmitted symbol is 111, thus comprising two of the three transmitted symbol bits will be correct, only one bit will be in error.

<table>
<thead>
<tr>
<th>Transmitted symbol</th>
<th>Bit position</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 1</td>
<td>0 0 0</td>
</tr>
<tr>
<td>1 0 0</td>
<td>0 0 1</td>
</tr>
<tr>
<td>1 0 1</td>
<td>0 1 0</td>
</tr>
<tr>
<td>1 1 0</td>
<td>1 0 0</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1 0 1</td>
</tr>
<tr>
<td></td>
<td>1 1 0</td>
</tr>
<tr>
<td></td>
<td>1 1 1</td>
</tr>
</tbody>
</table>

Fig (12.10)
3-For nonbinary signaling $P_B \ll P_E$

4-For each bit position the digit occupancy consists of 50% ones and 50% zero. There are $2^{k-1} = 4$ ways (four places where zeros appear in the column) that a bit error can be made.

5-Thus, the final relationship $P_B / P_E$ is obtained for forming the following ratio, the number of ways a bit error can be made ($2^{k-1}$) divided by the number of ways that a symbol error can be made ($2^k - 1$).

2. For MPSK (with Gray code)

\[ P_B \approx \frac{P_E}{\log_2 M} \quad \text{(for } P_E \ll 1) \quad ........(12.21) \]

**Goals of the communication system designer**

- The goals of the designer may include any of the following:

1-To maximize transmission bit rate $R$

2-To minimize probability of bit error $P_B$

3-To minimize required power or equivalently to minimize required bit energy to noise power spectral density $\frac{E_b}{N_0}$

4-To minimize required system bandwidth $W$

5-To maximize system utilization, that is, to provide reliable service for a maximum number of users with minimum delay and with maximum resistance to interference.

6-To minimize system complexity and system cost
-A system designer cannot achieve all these goals simultaneously because of several constraints and theoretical limitations that necessitate the trading off of any one system requirement with each of the others;-

a-The Nyquist theoretical minimum bandwidth requirements
b-The Shannon Hartly capacity theorem (and the Shannon limit)
c-Government regulations (e.g. frequency allocations)
d-Technological limitations (e.g. state of the art components )

-There are two performance planes used to study the different types of modulation and coding. These planes will be referred to as probability plane and the bandwidth efficiency plane.

**Error probability plane**

- Fig(12.11) shows the error probability performance curves, and to the plane on which they are plotted an error probability plane. Such a plane describes the locus of operating points available for a particular type of modulation and coding.

- Movement of the operating point along line 1, between points a and b can be viewed as trading off between $P_B$ and $E_b/N_0$ performance with fixed (W). Similarly movement along line 2 between point c and d, is seen trading $P_B$ versus W(with fixed $E_b/N_0$). Finally movement along line 3, between points e and f illustrates trading W versus $(E_b/N_0)$ (with $P_B$ fixed).

- Movement along line 1 is affected by increasing or decreasing the available $E_b/N_0$. This can be achieved by increasing or decreasing
the power. Also movement along lines (2) and (3) involve some
changes in the system modulation or coding.

![Bit error probability versus $E_b/N_0$](image1)

![Bit error probability versus $E_b/N_0$ for MPSK](image2)

**Nyquist minimum bandwidth**

- Nyquist showed that the theoretical minimum bandwidth (Nyquist bandwidth) needed for the baseband transmission of $R_s$ symbol per second without intersymbol interference (ISI) is $R_s/2$ Hertz. In practice, the Nyquist minimum bandwidth is expanded by about 10% to 40%, because of the constants of real filters.

- If the number of bits per symbol can be expressed as $k = \log_2 M$ and the data rate or bit rate $R$ must be $K$ times faster than the symbol rate $R_s$, as expressed by this equation
\[ R = K R_s \quad \text{as} \quad R_s = \frac{R}{K} = \frac{R}{\log_2 M} \quad \text{(12.22)} \]

- For signaling at a fixed symbol rate, eq(12.22) shows that, as \( K \) is increased, the data rate \( R \) is increased.

In this case of MPSK, increasing \( K \), results in an increased bandwidth efficiency \( R/W \) measured in bits/s/Hz. In other words with the same system bandwidth, one can transmit MPSK signals at an increased data rate and hence increased \( R/W \).

**Ex12.1**

a- Does error performance improve or degrade with increasing \( M \) for MPSK and MFSK.

b- The choice available in digital communication almost always involves a trade off. If error performance improves, what price must be paid.

c- If error performance degrades, what benefit is exhibited?

**Solution**

a- When examining the error probability plane for MFSK and MPSK, we see that error performance improvement or degradation depends upon the class of singling (orthogonal e.g MFSK or nonorthogonal e.g MPSK) as shown in fig (12.11)

1- Consider the orthogonal MFSK, where error performance improves with increased \( K \) or \( M \). There are only two ways to compare \( P_B(M) \). A vertical line can be drawn through some
fixed value of $\frac{E_b}{N}$, and as $K$ increased, it is seen that $P_B(M)$ is reduced. Or, horizontal line can be drawn through some fixed $P_B(M)$, as $K$ is increased, it is seen that $E_b/N_0$ required is reduced.

2-Similarly, it can be seen the curves for nonorthogonal MPSK, behave in the opposite performance. Error performance degrades as $K$ of $M$ is increased.

b-In the case of orthogonal signaling (MFSK), where error performance improves with increasing $K$ or $M$. If $k$ or $M$ is increased, it should be the cost of improved error performance is an expansion (increased) of required bandwidth.

c-In the case of nonorthogonal signaling, such as MPSK or QAM, where error performance degrades as $K$ or $M$ is increased, the bandwidth is reduced because

$$R_s(symbol/S) = \frac{R(bit/s)}{\log_2 M} = \frac{R}{\log_2 2^k} = \frac{R}{K}$$

When slower signaling ($R_s$ small) the bandwidth can be reduced

**Shannon – Hartley capacity theorem.**

- Shannon showed that the system capacity $C$ of a channel changed by additive white Gaussian noise (AWGN) is a function of the function of the average received signal power $S$, the average noise power $N$ and the bandwidth $W$. The capacity relationship (Shannon-Hartley) can be expressed as
The capacity \( C \) is given in bits/S. It is theoretically possible to transmit information over such a channel at any rate \( R \), where \( R \leq C \).

Shannon’s work showed that the values of \( S \), \( N \) and \( W \) set a limit on transmission rate, not on error probability.

Fig (12.12) shows the normalized channel capacity \( C/W \) in bit/s/s as a function of the channel signal to noise ratio (SNR).

\[
C = W \log_2 \left( 1 + \frac{S}{N} \right) \quad \text{......(12.23)}
\]

- The detected noise power is proportional to bandwidth

\[
N = N_0 W \quad \text{...........(12.24)}
\]

where \( N_0 \) is the noise spectral density, substituting eq(12.24) into eq(12.23) and rearranging terms yields
\[
\frac{C}{W} = \log_2 \left[ 1 + \frac{S}{N_0 W} \right] \hspace{1cm} \text{...........(12.25)}
\]

for the case where transmission bit rate is equal to channel capacity \( R = C \)

But \[ \frac{E_b}{N_0} = \frac{S}{N} \frac{W}{R} \hspace{1cm} \text{.................(6.10)} \]

Substituting \( R = C \), \( N = N_0 W \) in eq (6.10)

\[ \therefore \frac{E_b}{N_0} = \frac{S}{N_0 W} \frac{W}{C} \]

\[ \frac{E_b}{N_0} = \frac{S}{N_0 C} \hspace{1cm} \text{..................(12.26)} \]

Hence, We can modify eq(12.25) as follows

\[ \frac{c}{W} = \log_2 \left[ 1 + \frac{E_b}{N_0} \left( \frac{c}{W} \right) \right] \hspace{1cm} \text{...........(12.27 \(a\))} \]

\[ \frac{c}{w} = \log_2 \left[ 1 + \frac{C}{N_0 W} \right] \hspace{1cm} \text{...........(12.27 \(b\))} \]

\[ \frac{c}{w} = 1 + \frac{E_b}{\frac{C}{W}} \hspace{1cm} \text{...........(12.27 \(c\))} \]

\[ \frac{E_b}{N_0} = W \frac{2^{\frac{c}{w}} - 1}{C} \hspace{1cm} \text{...........(12.27 \(c\))} \]

Fig (12.13) shows the variation of \( W/C \) versus \( E_b/\sqrt{N_0} \) according eq(12.27c)
Shannon limit

There is a limiting value of $\frac{E_b}{N_0}$ below which there can be no error free communication at any information rate from eq(12.27a)

$$\frac{c}{W} = \log_2 \left[ 1 + \frac{E_b}{N_0} \left( \frac{c}{W} \right) \right] \ldots \ldots (12.27 a)$$

Let $\chi = \frac{E_b}{N_0} \left( \frac{C}{W} \right)$

Substituting for $\chi$ in eq(12.27a)
\[
\frac{C}{W} = \chi \log_2 (1+\times)^\frac{1}{\times}
\]

\[
as 1 = \frac{W}{C} \chi \log_2 (1+\times)^\frac{1}{\times}
\]

\[
since \quad \times = \frac{E_b}{N_0} \left( \frac{C}{W} \right)
\]

\[
1 = \frac{E_b}{N_0} \log_2 (1+\times)^\frac{1}{\times}
\]

\[
\lim_{x \to 0} (1+\times)^\frac{1}{\times} = e
\]

In the limit \(C/W \to 0\) we get
\[
\frac{E_b}{N_0} = \frac{1}{\log_2 e} = 0.693
\]

or in dB
\[
\frac{E_b}{N_0} = -1.6\, dB
\]

Equation (12.28) shows the Shannon limit and the value of \(E_b/N_0 = -1.6\, dB\) is called Shannon limit.

-Shanon`s work provided a theoretical proof for the existence of codes that could improve the \(P_B\) performance or reduce the \(E_b/N_0\) required, from the levels of the uncoded binary modulation schemes to levels approaching the limiting curve. For example, a bit error probability of \(10^{-5}\), binary phase shift keying (BPSK) modulation requires an \(E_b/N_0 = 9.6\, dB\) (the optimum uncoded binary modulation as shown in fig(12.11a)). For this example, Shannon works shows the existence of a theoretical performance improvement of 11.2
db(9.6+1.6) over the performance of optimum uncoded binary modulation, through the use of coding techniques.

-The selection of modulation and coding techniques to make the best use of transmitter power and channel bandwidth to reduce the cost of generating high power and to reduce the bandwidth.

**Bandwidth –Efficiency plane**

-Using eq(12.27c) to plot C/W versus $E_b/N_0$. This relationship is shown plotted on the R/W versus $E_b/N_0$ plane in fig (12.14). This plane is called the [bandwidth efficiency plane]

$$\frac{E_b}{N_0} = \frac{W}{C} (2^{C/W} - 1) \quad \text{.........(12.27c)}$$

-Thus R/W is a measure of how much data can be communicated in a specified bandwidth within a given time, therefore reflects how efficiently the bandwidth resource is utilized.

-For the case R=C in fig(12.14), the curve represents a boundary that separates a region characterizing practical communication systems from a region where such communication systems are not theoretically possible. Fig(12.14) is more useful for comparing digital communication modulation.
M-ary signaling

- For signaling schemes that process $k$ bits at a time, the signaling is called M-ary. Each symbol can be related to a unique sequence of $k$ bits, where

$$M = 2^k \quad \text{or} \quad k = \log_2 M \quad \text{..........}(12.28a)$$

where $M$ is the size of the alphabet. The term symbol refers to a member of the M-ary alphabet that is transmitted during each symbol duration $T_s$. 

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Fig (12.14)
In order to transmit the symbol, it must be mapped onto an electrical voltage or current waveform. Because the waveform represents the symbol, the term symbol and waveform are sometimes used interchangeably. Since one of \( M \) symbols or waveforms is transmitted during each symbol duration \( T_s \), the data rate \( R \) is given by:

\[
R = \frac{K}{T_s} = \frac{\log_2 M}{T_s} \quad \text{..........(12.28b)}
\]

The effective duration \( T_b \) of each bit in terms of the symbol duration \( T_s \) or the symbol rate \( R_s \) is

\[
T_b = \frac{1}{R} = \frac{T_s}{k} = \frac{1}{kR_s} \quad \text{........(12.28c)}
\]

\[
R_s = \frac{R}{k} = \frac{R}{\log_2 M} \quad \text{........(12.28d)}
\]

Using eq(12.28b) and(12.28c), it is seen that any digital scheme that transmits \( k - \log_2 M \) bit in \( T_s \), using a bandwidth \( W \) Hz, operates at a bandwidth efficiency of

\[
\frac{R}{W} = \frac{\log_2 M}{W T_s} = \frac{1}{W T_b} \text{bit} / S / H \quad \text{......(12.29)}
\]

**Bandwidth limited systems and power limited systems**

1-From eq(12.29), it can be seen that any digital communication system will become more bandwidth efficient as its \( W T_b \) product is decreased. Thus, signals will small \( W T_b \) product are often used
with bandwidth limited systems. For example the Global System for Mobile(GSM) communication uses Gaussian minimum shift keying (GMSK) modulation having a $W T_b$ product equal to 0.3Hz/bit/S where $W$ is the 3dB bandwidth of a Gaussian filter.

The required bandwidth, at an intermediate frequency, for MPSK or MQAM is related to symbol rate by

$$W = \frac{1}{T_s} = R_s$$

.........(12.30)

From eq(12.29) and (12.30), the bandwidth efficiency of MPSK or MQA using Nyquist filtering can be express as

$$\frac{R}{W} \log_2 M$$

.........(12.31)

2-For the case power limited systems in which power is low value but system bandwidth is available (e.g. space communication link), the following trade offs which can be seen in error probability plane for MFSK are possible (a) improved $P_B$ at the expense of bandwidth for a fixed $E_b/N_0$ or (b) reduction $E_b/N_0$ at the expense of bandwidth for a fixed $P_B$. For MFSK, the minimum bandwidth with assuming minimum tone spacing, is given by

$$W = \frac{M}{T_s} = MR_s$$

.........(12.32 a)

From eq(12.29) and eq(12.3a), the bandwidth efficiency of noncoherent MFSK signals can be expressed as

$$\frac{R}{W} = \frac{\log_2 M}{M} \text{bit} / \text{S} / Hz$$

.........(12.33)
Notice the important difference between the bandwidth efficiency (R/W) of MPSK expressed in eq(12.31)\[R/W=\log_2 M\] and that of MFSK expressed in eq(12.33)\[R/W=\log_2 M/M\].

With MPSK, R/W increases as the signal dimensionality M increase. With MFSK, there are two mechanism at works \[\log_2 M, M\] as M increases, the value of R/W decreases. Since \[\text{grows of } \log_2 M \text{ smaller than grows of } M\].

**Bandwidth efficient modulation**

The primary objective of spectrally efficient modulation techniques is to maximize bandwidth efficiency.

Offset QPSK\[OQPSK\] and Minimum shift Keying \[MSK\] are two examples of constant envelope modulation schemes that are attractive for satellite using nonlinear transponders. The satellite transponder requires large bandwidth efficiency with constant envelope modulation. This is the nonlinear transponder produce extraneous sidebands when passing a signal with amplitude fluctuations.

**QPSK and Offset QPSK**

-Fig (12.15) shows the partitioning of a typical pulse stream for QPSK modulation. Fig (12.15a) shows the original data stream \(d_k(t)=d_0,d_2,d_3,\ldots\) consisting of bipolar pulses. The pulses stream is divided into in-phase stream, \(d_i(t)\) and a quadrature stream \(d_Q(t)\) shown in fig(12.15b), as follows
Note that $d_i(t)$ and $d_j(t)$ each have half the bit rate of $d_k(t)$.

A convenient orthogonal realization of a QPSK waveform, $S(t)$, is achieved by amplitude modulating the in-phase and quadrature data.
streams onto the cosine and sine functions of a carrier wave as follows:

\[ S(t) = \frac{1}{\sqrt{2}} d_I(t) \cos \left( \frac{\omega_0 t + \pi}{4} \right) + \frac{1}{\sqrt{2}} d_Q(t) \sin \left( \frac{\omega_0 t + \pi}{4} \right) \] \quad (12.34)

The pulse stream \( d_I(t) \) amplitude modulates the cosine function with an amplitude of +1 as

1-This is equivalent to shifting the phase of the cosine function by 0 or \( \pi \) to produce BPSK

Similarly, the pulse stream \( d_Q(t) \) modulates the sine function, yielding a BPSK waveform orthogonal to the cosine function. The summation of these two orthogonal components of the carrier yields the QPSK waveform. Equation (12.34) can be written as

\[ S(t) = \cos \left[ \omega_0 t + \theta(t) \right] \] \quad (12.35)

The value of \( \theta(t) \) will correspond to one of the four possible combination of \( d_I(t) \) and \( d_Q(t) \) in eq(12.34). The value of \( \theta = 0^\circ, \pm 90^\circ \) or \( 180^\circ \).

-Offset QPSK[OQPSK] signaling can also represented by eq(12.34) and(12.35), the difference between two modulation schemes, QPSK
and OQPSK is the only the alignment of the two baseband waveforms as shown in fig (12.16). The duration of each original pulse is $T$ and in the partitioned streams, the duration of each pulse is $2T$. There is a time shift between $d_L$ and $d_Q$ for OQPSK as shown in fig (12.16).

OQPSK, Sometimes called staggered QPSK(SQPSK).

In standard QPSK, due to the coincident alignment of $d_I$ and $d_Q(t)$, the carrier phase change only once every $2T$. If a QPSK modulated signal pass through filter to reduce the spectral sidelopes, the resulting waveform will no longer have a constant envelope also in fact, the $180^\circ$ phase shift will cause the envelope to go to zero momentarily. Also all of the undesirable frequency sidelopes, which can interfere with adjacent channels. In OQPSK, the pulse streams $d_I(t)$ and $d_Q(t)$ are staggered and thus do not change states simultaneously. The possibility of the carrier changing phase by $180^\circ$ is eliminated. Changes are limited to

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\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig12.16}
\caption{Fig (12.16)}
\end{figure}
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every T seconds. When OQPSK signal pass through band limiting, the resulting intersymbol interference causes the envelope to drop slightly in the region of $\pm 90^\circ$ phase transition.

Fig(12-17) shows QPSK and OQPSK waveforms

**Minimum shift keying**
- The main advantage of OQPSK over QPSK, that of suppressing out of band interference. Also another further improvement is possible if the OQPSK format is modified to avoid discontinuous phase transition by designing continuous phase modulation (CPM) schemes.
Minimum shift keying (MSK) is one such scheme. MSK can be viewed as either a special case of continuous phase frequency shift keying (CPFSK) or special case of OQPSK with sinusoidal symbol weighting. When viewed as (CPFSK), the MSK waveform can be defined in the form of an angle modulated signal as follows:

\[
s(t) = \cos \left[ 2\pi \left( f_0 + \frac{d_k}{4T} \right) t + x_k \right] \quad kT < t < (k + 1)T
\]  

(12.36)

Where \( f_0 \) is the carrier frequency
\( d_k = \pm 1 \) represents the bipolar data being transmitted at a rate \( R=1/T \)
\( x_k \) is the phase constant which is valid over the \( k \)th binary data interval.

if \( d_k=1 \), the frequency transmitted is \( f_0 + \frac{1}{4T} \).
If \( d_k=-1 \), the frequency transmitted is \( f_0 - \frac{1}{4T} \).

-During each \( T \)-second data interval, the value of \( x_k \) is a constant that is, \( x_k =0 \) or \( \pi \), determined by the requirement that the phase of the waveform be continuous at \( t=kT \), the value of \( x_k \) is given by:

\[
x_k = x_{k-1} + \frac{\pi k}{2} (d_{k-1} - d_k)
\]

(12.37)

Eq(12.3b) can be expressed in a quadrature representation

\[
S(t)=\cos \left[ 2\pi \left( f_0 + \frac{d_k}{4T} \right) t + x_k \right]
\]

(12.36)

\[
S(t)=[\cos
2\pi \left( f_0 + \frac{d_k}{4T} \right) t]
\]
S(t) = [\cos 2

(Since

\[ S(t) = \frac{a_k \cos \pi t}{2T} \cos 2\pi f_c t - \frac{b_k \sin \pi t}{2T} \sin 2\pi f_c t \]

where \( a_k = \cos x_k = \mp 1 \)

\( b_k = d_k \cos x_k = \mp 1 \)

The inphase component(I) is identified as

\[ \cos x_k \cos \frac{d_k}{4T} t \]

where \( \cos 2\pi f_c t \) is the carrier

\[ \cos \frac{d_k}{4T} t \]

can be regarded as a sinusoidal symbol weighting

are a data dependent terms
Similarly the quadrature component (Q) is identified as

\[ -\cos x \sin \frac{d_k}{4T} \sin \omega_c t \]

When viewed the MSK as a special case of OQPSK, eq(12.37) can be rewritten as:

Where \( d_i(t) \) and \( d_Q(t) \) have the same in-phase and quadrature data stream for OQPSK as in eq(12.34)

\[
\begin{align*}
  d_i(t) &= d_0, d_2, d_4, \ldots \ldots \text{even bits} \\
  d_Q(t) &= d_1, d_3, d_5, \ldots \ldots \text{odd bits}
\end{align*}
\]

Fig(12.18) shows MSK according to eq(12.38). Waveform of the MSK signal \( s(t) \) obtained by adding in-phase and quadrature on bit by bit basis.

The following properties of MSK modulation can be deduced from eq(12.38) and fig(12.18);

1-The waveform \( s(t) \) has constant envelope.
2-There is phase continuity in the RF carrier at the bit transitions.
3-The waveform \( s(t) \) can be regarded as an FSK waveform with signaling Frequencies.

\[
f_0 + \frac{1}{4T} \quad \text{and} \quad f_0 - \frac{1}{4T}
\]

.: The minimum tone spacing required for MSK modulation is
\[
\left( f_0 + \frac{1}{4T} \right) - \left( f_0 - \frac{1}{4T} \right) = \frac{1}{2T}
\]

![Diagram](image)

Fig(12.18)
**Error performance of OQPSK and MSK**

1-We have seen that BPSK and QPSK have the same bit error probability because QPSK is configured as two BPSK signals modulating orthogonal components of the carrier. Since staggering the bit streams does not change the orthogonality of the carriers, OQPSK has the same theoretical bit error performance as BPSK and QPSK.

2-MSK uses antipodal symbol shapes, $\cos\left(\frac{\pi t}{2T}\right)$ and $\sin\left(\frac{\pi t}{2T}\right)$ to modulate the two quadrature components of the carriers. Thus when matched filter is used to recover the data from each of the quadrature components independently, MSK as defined by eq(12.38) has the same error performance properties as BPSK, QPSK and OQPSK.

3-If MSK is coherently detected as an FSK signal over an observation interval of $T$ seconds, it would be poorer than BPSK by 3db.

4-MSK, with differentially encoded data, as defined in eq(12.36), MSK has the same error probability performance as the coherent detection of differentially encoded PSK.
\[ S(t) = \cos \left[ 2\pi \left( F_0 + \frac{d}{4T} \right) t + \chi_k \right] \]  

\[ \ldots \ldots (12.36) \]

MSK can be also nocoherently detected, this allowed inexpensive demodulation when the value of received \( \frac{E_b}{N_0} \) permits.

**Quadrature amplitude modulation (QAM)**

- Coherent M-ary phase shift keying (MPSK) modulation is a technique for achieving bandwidth reduction. Instead of using a binary 1 bit of information per symbol, an M symbols is used with k-bits (k=\( \log_2 M \)) per symbol.

- QPSK modulation consists of two independent streams as shown previously. One stream amplitude-modulates the cosine function of a carrier wave with levels +1 or -1 and the other stream similarly amplitude modulates the sine function. The resultant waveform is a DSB-SC.

- QAM can be considered an extension of QPSK, since QAM also consists of two independently amplitude modulated carriers in quadrature. Each block of \( K \) bits (\( K \) assumed even) can be split into two (\( k/2 \)) bit blocks which use (\( K/2 \)) bit digital to analog (D/A) converters to provide the required modulating voltage for the carriers. At the receiver, each of the two signals is independently detected using matched filters.
QAM can be viewed as a combination of amplitude shift keying (ASK) and a phase shift keying (PSK).

Fig(12.19a) shows a two dimensional signal space and a set of 16-ary QAM a signal vectors or points arranged in a rectangular constellation. Fig(12.19b) shows the QAM modulator

\[ y \]

\[ x \]

\[ (a) \]

\[ y \]

\[ x \]

\[ \cos \omega_0 t \]

\[ \sin \omega_0 t \]

\[ \Sigma \]

\[ s(t) \]

\[ \text{LPF} \]

\[ \times \]

\[ \Sigma \]

\[ r_x \]

\[ n_x \]

\[ \text{LPF} \]

\[ \times \]

\[ n_y \]

\[ r_y \]

\[ y \]

\[ x \]

\[ (b) \]

\[ (c) \]

Fig (12.19)

For a rectangular constellation, a Gaussian channel, and matched filter reception the bit error probability for M-QAM, where \( M = 2^k \) and \( K \) is even, is given by

\[ P_B \approx \frac{2(1 - L^{-1})}{\log_2 L} Q \left[ \sqrt{\left( \frac{3 \log_2 L}{L^2 - 1} \right) \frac{2E_b}{N_0}} \right] \quad (12.37) \]

Where \( Q(\chi) \) is the complementary error function
\( L = \sqrt{M} \) represents the number of an amplitude levels.

**Ex12.2**

Assume that a data stream with data rate \( R = 144 \text{ Mbit/s} \) is to be transmitted on an RF channel using a DSB modulation scheme. Assume Nyquist filtering and an allowable DSB bandwidth of 36 MHz. (a) Which modulation technique would you choose for this requirement? (b) If the available \( \frac{E_b}{N_0} \) is 20, what would be the resulting probability of bit error? (c) Explain the computation of the QAM spectral efficiency if orthogonal components of a carrier wave is used.

**Solution:**

a-The required spectral efficiency \( \frac{R}{W} \)

\[
\frac{R}{W} = \frac{144 \text{ Mbit/s}}{36 \text{ MHz}} = 4 \text{ bit/s/Hz}
\]

b-From the bandwidth efficiency plane fig(12.14), we note that 16-ary QAM, with a theoretical spectral efficiency of 4bit/s/Hz, requires a lower \( \frac{E_b}{N_0} \), than that of 16-ary PSK for the same \( P_B \). Based on these considerations we choose a 16-ary QAM modem. Using eq(12.37)

\[
P_B \approx \frac{2(1 - L^{-1})}{\log_2 L} Q\left[\sqrt{\left(\frac{3 \log_2 L}{L^2 - 1}\right) \frac{2E_b}{N_0}}\right]
\]
\[ L = \sqrt{M} = \sqrt{16} = 4 \]

\[ \therefore \rho_B \approx \frac{2(1-4^{-1})}{\log_2 4} Q \left[ \sqrt{\left(\frac{3 \log_2 4}{4^2 - 1}\right) \times 20} \right] \]

\[ \approx \frac{2 \left(\frac{3}{4}\right)}{2} Q \left[ \sqrt{\frac{3 \times 2 \times 2 \times 20}{15}} \right] \]

\[ \approx \frac{3}{4} Q[4] = 2.5 \times 10^{-5} \]

\[ \text{if } \chi > 3 \]

\[ Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \]

\[ = \frac{1}{4\sqrt{2\pi}} e^{-8} \]

\[ = 3.345 \times 10^{-5} \]

C-Bandpass channel using QAM

1-The 144Mbit/s data stream is partitioned into a 72 Mbit/s in-phase and quadrature stream.

2-One stream amplitude modulates the cosine component of a carrier over a bandwidth of 3b MHz and the other stream amplitude
modulates the sine component of the carrier over the same 36MHz bandwidth.

3-Since each 72Mbit/s stream modulates an orthogonal components of the carrier, the 36MHz suffices for both stream, or for the full 144Mbit/s.

4-Thus the spectral efficiency is

\[
\frac{144\text{Mbit}}{s} \div \frac{36\text{MHz}}{} = 4 \text{ bit/s/Hz}
\]