Ministry of Higher Education and Scientific Research

University of Technology
Chemical Engineering Department

Optimization
Fourth Year

By
Dr. Nidhal AL - Azzaw
1. Introduction to Optimization Methods: (4 hrs)

2. Organization of Optimization problems: (4 hrs)

3. Single Variable:
   Analytical methods, numerical methods, graphical methods, numerical Search, restriction Function, unrestricted Function, Direct search, Dichotomous Search, golden Search, Fibonacci Search (12 hrs)

4. Multivariables Optimization Methods:
   - Necessary Conditions For Extreme Values in graphical Cases.
   - Solution by graphical method.
   - Simplex method.
   - Linear Programming and application in Chemical engineering (transportation mixing) (10 hrs)
OPTIMIZATION

Fourth Year B.Sc. Course
For Both Branches
Unit Operation of Chemical Industries
& Refinery Engineering of Oil and Gas

2 hrs / week
2. Units

Syllabus:
1. Introduction to Optimization Methods. 4 hrs
2. Organization of Optimization Problems. 4 hrs
3. Single Variables-
   Analytical methods, Numerical methods,
   Graphical method, Numerical search,
   Restricted Functions, Unrestricted Functions,
   Direct search, Dichotomous search,
   Golden search, Fibonacci search. 12 hrs
4. Multivariables Optimization Methods
   Necessary conditions for extreme values
   in graphical cases
   Solution by graphical method
   Simplex method
   Linear programming and applications 10 hrs
   in chemical engineering,
Introduction to Optimization

What is optimization?

* An act, process or methodology of making something (as a design, system or decision) as fully perfect, functional or effective as possible, that is to achieve optimum.

* A collective process of finding the best conditions required to achieve the best results from a given situation.

* Optimization is one of the major quantitative tools in the machinery of decision making. A wide variety of problems in the design, construction, operation and analysis of chemical plants can be resolved by optimization.

Some Other Definitions:

Optimum value - It is a technical term including quantitative measurements and mathematical analysis to determine the best setting (maximum or minimum) of a dependent variable.

Optimization Procedure: The process of determining the maximum or minimum value of some criterion function.

Optimization Problem: It is the specification of the variables that need to be optimized.
Why are engineers interested in optimization? What benefits result from optimization versus intuitive decision making?

* Engineers work to improve the initial design of process and equipment
* Engineers strive for enhancements in the operation, in order to realize:
  1. largest production,
  2. greatest profit,
  3. minimum cost,
  4. least energy usage
* In plant operation, benefits arise from improved plant performance such as:
  1. Improved yields of valuable products
  2. Reduced energy consumption.
  3. Higher processing rates.
  4. Longer times between shutdowns.
  5. Reduced maintenance costs.

In a typical industrial company there are three levels in which optimization is used:

1. Management
2. Process design and equipment specification
3. Plant operation.

Process design and equipment specification is usually performed prior to the implementation of the process, and management decisions to implement are usually made for in advance of the process design step.
Optimum Economic Design: It is based on the best or most favorable conditions that give the least cost per unit of time or maximum profit per unit of production.

Example: Determination of pipe diameter to use when pumping a given amount of fluid from one point to another.

\[
\text{Cost of pumping power} \quad \text{Fixed charge based on capital investment for installed pipe.}
\]

* Some trial results can be accomplished by using an infinite number of different pipe diameters.
* Economic balance can show that one particular pipe diameter gives the least total cost.
* Total cost = Pumping cost + Fixed cost for the installed piping system.
Pumping cost increases with decreased size of pipe diameter, why?
The fixed charges for the pipeline become lower when smaller pipe diameters are used, why?
The optimum economic diameter is found where the sum of the pumping costs and fixed costs for the pipeline becomes a minimum.

Example 2: The design of a distillation column is ordinarily based on specifications giving the degree of separation required for a feed supplied to the unit at a known composition, temperature and flow rate. The chemical engineer must determine the size of the column and reflux ratio necessary to meet the specifications.

As the reflux ratio is increased, the number of theoretical stages required for the given separation decreases.
An increase in reflux ratio may result in lower fixed charges for the column and greater costs for the reboiler heat supply and condenser coolant.

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**Fig. 2**

![Graph showing the relationship between reflux ratio and total cost with lines for variable and fixed costs](image)
Optimum Operation Design:

In every chemical engineering process there are several alternatives which can be used for any given process or operation. For example, formaldehyde can be produced by catalytic dehydrogenation of methanol, by controlled oxidation of natural gas, or by direct reaction between CO and H₂ under special conditions of catalyst, temperature and pressure. It is the responsibility of the chemical engineer to choose the best process and to incorporate into his design the equipment and methods which will give the best results. The engineers generally replace the word "best" by "Optimum". Many processes require definite conditions of temperature, pressure, contact time, catalyst type, composition and many other variables. It is often possible to make a partial separation of these optimum conditions from direct economic considerations.

So the Optimum Operation Design is a tool or step in the development of an optimum economic design.

Example: Determination of the operation conditions for the catalytic oxidation of SO₂ to SO₃.

* Suppose all variables, size, flowrate, catalyst, concentration are fixed.
* The only possible variable is the operating temperature.

Fig 3: Optimum operation temp for SO₂ to SO₃.
Example 1. Determination of Optimum Reflux Ratio

A bubble-cap distillation column is being designed to handle 700 lb feed/hr. The unit is to operate at 1 atm.

The feed contains 45 mole% benzene and 55 mole% toluene and enters at its boiling point. The overhead product must contain 92 mole% benzene and the bottoms must contain 95 mole% toluene. Determine the following:

a) The optimum reflux ratio as moles liquid returned to tower per mole of distillate product withdrawn.

b) The ratio of the Optimum Reflux Ratio/Min. Reflux Ratio.

c) The percent of the total variable cost due to steam consumption at the optimum condition.

Data

\( \Delta P \) for both benzene and toluene = 40 Btu/lb mol \( F^\circ \)

\( \Delta H \) for \( F \) = \( -13780 \) Btu/lb mol

\( \text{Uoverall} \) at \( F = 80 \text{ Btu/hr ft}^2 \text{ F}^\circ \) in the reboiler

and \( = 100 \text{ Btu/hr ft}^2 \text{ F}^\circ \) in the condenser

\( T_F = 201 \text{ F}^\circ \)

\( T_D = 179 \text{ F}^\circ \)

\( T_B = 273 \text{ F}^\circ \)

The driving force in the reboiler-condenser may be based on an average cooling-water temperature of 90 \( F \)

and the change in cooling-water temperature is 50 \( F \)

Saturated steam at 60 psia is used in the reboilers,

where the temperature of the condensing steam is 292.5 \( F \) and the heat of condensation = 915.5 Btu/lb

Column diameter based on maximum allowable vapor velocity of 2.5 \( \text{ ft/sec} \) at the top of the column.

Overall Plate Efficiency 70%.

The unit is to operate 8500 hr per year.

Cost Data

Cost of steam - $0.05/1000 lb.

Cost of cooling water = $0.03/1000 gal.

\( c_2 = $0.036/1000 \text{ lb.} \)
The cost of piping, insulation, and instrumentation can be estimated to be 60% of the cost for the installed equipment.

Annual fixed charges amount to 15% of the total cost for installed equipment, piping, insulation, and instrumentation.

The following costs are for the installed equipment, piping, and include delivery and erection costs.

### Bubble-Cap Distillation Column
Values may be interpolated.

<table>
<thead>
<tr>
<th>Diameter, inch</th>
<th>$/plate</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>800</td>
</tr>
<tr>
<td>70</td>
<td>1000</td>
</tr>
<tr>
<td>80</td>
<td>1230</td>
</tr>
<tr>
<td>90</td>
<td>1500</td>
</tr>
<tr>
<td>100</td>
<td>1800</td>
</tr>
</tbody>
</table>

### Condenser - Shell & Tube Heat Exchanger

<table>
<thead>
<tr>
<th>Heat Transfer Area, ft²</th>
<th>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>800</td>
<td>6500</td>
</tr>
<tr>
<td>1000</td>
<td>7500</td>
</tr>
<tr>
<td>1200</td>
<td>8400</td>
</tr>
<tr>
<td>1400</td>
<td>9200</td>
</tr>
<tr>
<td>1600</td>
<td>9900</td>
</tr>
</tbody>
</table>

### Reboiler - Shell & Tube Heat Exchanger

<table>
<thead>
<tr>
<th>Heat Transfer Area, ft²</th>
<th>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>11500</td>
</tr>
<tr>
<td>1400</td>
<td>14100</td>
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<tr>
<td>1800</td>
<td>16400</td>
</tr>
<tr>
<td>2200</td>
<td>18500</td>
</tr>
<tr>
<td>2600</td>
<td>20200</td>
</tr>
</tbody>
</table>
Solution 1.

The variable costs involved are cost of column, cost of reboiler, cost of condenser, cost of steam and cost of cooling water. Each of these costs is a function of the reflux ratio, and the optimum reflux ratio occurs at the point where the sum of the annual variable costs is a minimum. The total variable cost will be determined at various reflux ratios and the optimum reflux ratio will be found by the graphical method. The following is a sample of calculation for reflux ratio = 1.5

Fig. 4: Equilibrium Data, McCabe-Thiele

The number of theoretical plates can be determined by the standard graphical method of McCabe-Thiele diagram.

- Slope of enriching operating line = \( \frac{R - 1.5}{R + 1} \) = 0.6
- Theoretical number of stages = 12.1 - 1 = 11.1
- Actual number of stages = \( \frac{11.1}{0.4} \) = 27.75

\[ \text{Eqn} = D x_0 + \text{B} x_8 \]

\[ 700 \times 0.45 = D \times 0.92 + (700 - D) \times 0.05 \]

\[ D = 322 \text{ mol/lhr} \]

\[ V_0 = D (1 + R) = 322 (1 + 1.5) = 805 \text{ mol/lhr} \]

Vapor velocity at the top of the tower = 2.5 ft/s

From perfect gas law \( \frac{V_1}{n_1} = \frac{V_2}{n_2} \)
\[ t_6 = \frac{305 \times 359 (460 + 174)}{3600 (460 + 32) \times 0.02} \]

\[ D = 7.3 \text{ ft} \quad \text{diameter of distillation column} = 82.5^\circ \]

\[ \text{Cost of column per plate} = \$1430 \]

\[ \text{Annual cost for distillation column} = 1430 \times 16 (1 + 0.15) = \$5400 \]

**Annual Cost of Condenser:**

\[ q_c = 805 \times 13700 = 11,000,000 \text{ Btu/hr} \]

\[ Q_c = \text{UAT} \]

\[ A = \frac{11,000,000}{100 (179 - 90)} = 1240 \text{ ft}^2 \]

\[ \text{Cost per ft}^2 = \frac{\$8550}{1240} \]

\[ \text{Annual cost for condenser} = \frac{\$8550}{1240} \times 1240 (1 + 0.15) \times 0.15 = \$2050 \]

**Annual Cost of Reboiler:**

The rate of heat transfer in the reboiler \( q_r \) can be determined by a total energy balance around the distillation column. Base taken at energy level of liquid at 179°F.

\[ \text{Heat input} = \text{Heat output} \]

\[ q_r + 700 \times 40 (20 + 179) = 11,000,000 + 378 \times 40 (217 + 79) \]

\[ q_r = 11,110,000 \text{ Btu/hr} = \text{UAT} \]

\[ A = \text{heat transfer area} = \frac{11,110,000}{80 (292.7 - 227)} = 2120 \text{ ft}^2 \]

\[ \text{Cost per ft}^2 = \frac{\$18100}{2120} \]

\[ \text{Annual cost for reboiler} = \frac{\$18100}{2120} \times 1240 (1 + 0.15) = \$4340 \]

**Annual Cost of Cooling Water:**

\[ q_c = 11,000,000 \text{ Btu/hr} \]

\[ c_p = 1.0 \text{ Btu/lb°F} \]

\[ \text{Annual cost of cooling water} = \frac{11,000,000 \times 0.034 \times 8500}{1.0 \times 50 \times 10,000} \]
Annual cost of steam

The rate of heat transferred in reboiler = 41,140,000 Btu/hr

\[ \text{Annual cost from steam} = \frac{41,140,000 \times 0.5 \times 8520}{915.5 \times 1000} \]

\[ = \$51700 \]

Total annual cost at reflux ratio 1.5

\[ = 5490 + 2050 + 4340 + 6740 + 51700 = \$70,320 \]

By repeating the preceding calculation for different reflux
dratios, the following table can be prepared:

<table>
<thead>
<tr>
<th>Rnin</th>
<th>Rmin</th>
<th>D</th>
<th>I</th>
<th>C</th>
<th>E</th>
<th>SX</th>
<th>Total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.14</td>
<td>0.0</td>
<td>6.7</td>
<td>6.0</td>
<td>1310</td>
<td>3960</td>
<td>5100</td>
<td>44300</td>
</tr>
<tr>
<td>1.2</td>
<td>29</td>
<td>6.8</td>
<td>8320</td>
<td>1950</td>
<td>4040</td>
<td>5940</td>
<td>45520</td>
</tr>
<tr>
<td>1.3</td>
<td>21</td>
<td>7.0</td>
<td>6620</td>
<td>1950</td>
<td>4130</td>
<td>6200</td>
<td>47520</td>
</tr>
<tr>
<td>1.4</td>
<td>18</td>
<td>7.4</td>
<td>6920</td>
<td>2000</td>
<td>4240</td>
<td>6470</td>
<td>49600</td>
</tr>
<tr>
<td>1.5</td>
<td>16</td>
<td>7.3</td>
<td>5490</td>
<td>2050</td>
<td>4340</td>
<td>6740</td>
<td>51300</td>
</tr>
<tr>
<td>1.7</td>
<td>14</td>
<td>7.7</td>
<td>5290</td>
<td>2150</td>
<td>4540</td>
<td>7190</td>
<td>55700</td>
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<tr>
<td>2.0</td>
<td>13</td>
<td>8.0</td>
<td>5210</td>
<td>2280</td>
<td>4820</td>
<td>8100</td>
<td>61800</td>
</tr>
</tbody>
</table>

\( R_{min} \quad 0.0 \quad 6.7 \quad 6.0 \quad 1310 \quad 3960 \quad 5100 \quad 44300 \quad 00 \)

\( R_{nin} \quad 1.14 \quad 29 \quad 6.8 \quad 8320 \quad 1950 \quad 4040 \quad 5940 \quad 45520 \quad 66320 \)

\( D \quad I \quad C \quad E \quad S \times \quad \text{Total cost} \)

\( a) \quad \text{The above data plotted & Ropt can be found at min. total} \)
\( \text{cost per year. From Fig. 2, Ropt = 1.25} \)

\( b) \quad \text{For Rmin, slop of } D/C = \frac{R_{nin}}{R_{min} + 1} \)

\( \frac{R_{nin}}{R_{min} + 1} = 0.532 \)

\( \frac{R_{opt} = 1.25}{R_{nin}} = 1.14 \)

\( \frac{R_{opt} = 1.25}{R_{nin}} = 1.14 \)

\( c) \quad \text{At the optimum conditions} \)

\( \text{Annual steam cost} = \$46,500 \)

\( \text{Total annual cost variable} = \$66,000 \)

\( \% \text{variable cost due to steam consumption} = \frac{46,500 \times 100}{66,000} \)

\( = 70\% \)
Examples of Applications of Optimization:
1. Determination of best sites for plant location.
2. Routing of tankers for the distribution of crude and refined products.
3. Pipeline sizing and layout.
4. Equipment and entire plant design.
5. Maintenance and equipment replacement schedule.
6. Operation of equipment, such as tubular reactors, columns, exchangers - etc.
7. Evaluation of plant data to construct a model of a process.
8. Minimization of inventory charges.
9. Allocation of resources or services among several processes.
11. Scale formation in evaporators.
12. Insulation thickness.
13. Fitting of mathematical curves to experimental data to obtain the most accurate representations.
14. Fluid dynamics, mass and heat transfer applications.
15. Blending of different types of tobacco and flavor.
16. Road construction between two certain points.
17. Quality and number of students for each college.

And Many Many Other Applications.
Optimization Problems involve specifying four quantities:

* Decisions: Items that need to be figured out to achieve max. efficiency.
* Ranking Function: A method to rank different choices of decisions.
* Rules and Restrictions: Specifying limitations on choices of decision values.
* Parameters: Information necessary in specifying ranking function and rules.

Optimization models are mathematical representation of optimized problem, it is called also mathematical model which analyze or solve the optimization problem. These models have:

- Parameters → represent the given data
- Decision variables → represent items that need to be determined.
- Constraints → represent limitations on the choice of decision variables, either internal or external imposed by the designer.
- Objective Function → give the ranking of different choices.
General Procedure of Optimization

1. Define a suitable objective for the problem.
2. Examine the restrictions imposed upon the problem.
3. Choose a system or systems to study.
4. Examine the structure of the system and the inter-relationship of the system elements.
5. Construct a mathematical model for the system.
6. Examine and define the internal restrictions planned upon the system variables.
7. Carry out the simulation by expressing the objective function.
8. Verify the proposed model.
9. Determine the optimum solution and discuss the nature of system conditions.
10. Using the information that obtained, repeat the procedure until a satisfactory results are found.
Example: We do complete formulation for a well-known problem called diet problem. Given a set of food (e.g. milk, chocolate, Raisin-Bran, Pizza) and their nutrient/calorie values, find a diet minimizing the daily cost of food.

*Problem Statement:

1. **Decision**: Quantities of different food items to be consumed daily.
2. **Objective**: Minimizing the daily cost of food.
3. **Restrictions/Constraints**:
   - a. total fat content in the diet does not exceed some limit.
   - b. total calories do not exceed some limit.
   - c. total protein intake is at least some minimum amount.
4. **Parameters**: nutrient/calorie, cost

**Decision**: For each food item, how much of it we should consume daily?

The decision variables can be defined as follows:

- \( X_m \) = Gallons of milk consumed daily.
- \( X_c \) = Bars of chocolate consumed daily.
- \( X_r \) = Packs of Raisin-Bran consumed daily.
- \( X_z \) = Pizza consumed daily.
2. Objective Function:

Suppose our objective is to minimize our daily cost.

Let \( C_i \) for \( i = m, c, r, z \) be the current cost per unit of each item \( i \).

So the Objective Function:

\[
F(x_m, x_c, x_r, x_z) = C_m x_m + C_c x_c + C_r x_r + C_z x_z
\]

Constraints of restrictions:

- Restrictions on fat, calories and protein intake.

Suppose \( f_i, w_i, p_i \) for \( i = m, c, r, z \) are the values of fat, calories and protein per unit item, respectively.

Then

\[
\sum f_i x_m + \sum w_i x_c + \sum p_i x_r + \sum p_i x_z \leq F
\]

\[
\sum f_i x_m + \sum w_i x_c + \sum w_i x_r + \sum w_i x_z \leq W
\]

\[
\sum p_i x_m + \sum p_i x_c + \sum p_i x_r + \sum p_i x_z \geq P
\]

So the complete model in words is:

The goal → minimum cost, such that
1. Fat requirement
2. Calorie requirement
3. Protein requirement
4. None of the intake amount is negative

\[ x_m \geq 0, \ x_c \geq 0, \ x_r \geq 0, \ x_z \geq 0 \]
Formulation of Optimization Model:

1. Type of Variables: or Decision Variables.
   These are defined to capture decisions that need to be made. These variables have different types depending on the values they can take.
   The basic variable types are:
   a. Continuous: This can take any real value such as, by how much should I change my current investment in stocks?
   b. Continuous - non-negative: This can take only non-negative values, $x \geq 0$. Such as, how much milk should I drink every day.
   c. Binary: This can take values of 0 or 1 only, $x \in \{0, 1\}$. Such as, should a new warehouse be setup in Baghdad?
   d. Integer: This can take any integer value, $x \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. If it is required to be non-negative then $x \geq 0$. Such as, how many workers should be hired to meet lunch time demand in a café.
e. Finite Sets: This can take a small set of values \( x \in S \) where \( S \) is the set of values \( x \) can take. Such as which road should I take to college today.

2. Objective Function: Can be defined as a function of decision variables whose output is a number. There are uncountable possible functions of this kind, these are classified into two groups:

a. Linear Functions:

\[ f(x_1, x_2, x_3) = x_1 + x_2 + 5x_3 \]

b. Non-Linear Functions:

- Polynomials: \( f(x, y, z) = x^2 + y^2 + z^2 \)
- Cross terms: \( f(x, y, z) = xy \)
- Exponentials: \( f(x) = e^x \)
- Maximum: \( f(x, y, z) = \max \{ x, y, z \} \)
- Absolute: \( f(x) = |x| \)

3. Constraints:

a. Linear constraints which are of three types

\[ \geq, \leq, = \]

b. Non-linear constraints

\[ x^2 + y^2 \leq 1 \quad y < x < 1 \]
Optimization Techniques:

It is the process of determining the maximum or minimum value of some criterion function (Objective function).

Let us denote the quantity to be optimized or the objective function as

\[ J(x) \]

where \( x \) may be

1. Single variable (independent).
2. A vector of such variables
3. A function of some other independent variables
4. Or even a vector of function of of several independent variables

We assume that \( J(x) \) is a scalar-valued function of \( x \). If \( J(x) \) is a continuous function of a single variable \( x \) is restricted to values in the range \( a < x < b \). Figure 5 illustrates some possible shapes for various objective functions.
For $J_1$ : There is no unique value for $x$ which gives a maximum value for the objective function.

For $J_2$ & $J_3$ : Maximum value occurs for $x$ at the boundary value $b$.

For $J_3$ & $J_4$ : The value $x = c$ gives a locally maximum value for $J$ but not the true maximum.

For $J_5$ : The value $x = c$ does give the true maximum.

* If the objective function has only one maximum value in a given range for $x$, it is called Unimodal function. From Fig 5, $J_2$ and $J_5$ are unimodal functions.

* If there is more than one maximum or minimum, a local optimum should be found for each then these optimum values should be compared to determine the true optimum

Local and Global Extremes:

Consider Figure 6, which shows that the objective function $f(x)$ has several peaks within the interval considered, where $a < x < b$. Point A gives maximum value, whereas points B and C give other maximum points for the function $f(x)$.
Point A is called Global Maximum.
Points B, C, and D are called Local Maximum.
Point E is called Global Minimum.
Points D and F are Local Minimum.

From calculus, to find min. or max. values, the first derivative = 0

\[ \frac{dJ}{dx} \bigg|_{x^*} = 0 \]

where \( x^* \) is the optimum value of \( x \).

\( J''(x^*) < 0 \) (−ve) local max.

\( J''(x^*) > 0 \) (+ve) local min.

\( J''(x^*) = 0 \) Point of inflection or saddle point.
There are two approaches to the optimization problems to be solved; either analytical method or numerical method.

1. **Analytical Method**
   - Used for single variable problems (unrestricted).
   - Rules about extremes (stationary points) are:
     - a) Extremes of \( J(x) \) can occur only where \( \frac{dJ}{dx} = 0 \)
     - b) If at points determined from (a) certain derivative vanish (\( = 0 \)), then the next derivative which does not vanish is examined for sign:
       - IF \( \frac{dJ}{dx} = \frac{d^2J}{dx^2} = \ldots = \frac{d^nJ}{dx^n} = 0 \)
       - then \( \frac{d^{n+1}J}{dx^{n+1}} \) is either (+ve) or (-ve).
     - IF \( n \) is even → point of inflection
     - IF \( n \) is odd → \( n+1 \) examined → (-ve) max. → (+ve) min.
     - c) IF \( \frac{dJ}{dx} \) changes from +ve to zero to negative (-ve) it gives max.
     - IF \( \frac{dJ}{dx} \) changes from -ve to zero to +ve it gives min.
     - IF sign does not change, there is no extremes.
Example 1:

$$y = a^2 - x^2$$  Find the extremes

The necessary conditions for extremes (max or min.) is $$y' = 0$$

Then $$y' = -2x = 0$$ at $$x = 0$$

So the extreme at $$x = 0$$ & $$y = a^2$$ is max

$$y'' = -2 \quad (-ve)$$ so it is indeed a maximum.

Example 2: $$y = x^3 - 2x^2 + 4x$$  Find the extremes.

$$y' = 3x^2 - 4x + 4 = 0$$

$$(3x - 1)(x - 2) = 0$$

$$x = 1, \quad x = 2$$  Two extreme points

$$y'' = 6x - 12$$

$$y''(x = 1) = 6 \quad \rightarrow \text{ max}$$

$$y''(x = 2) = 6 \quad \rightarrow \text{ min}$$

Example 3: Assume $$x$$ is the insulation thickness of a pipe transferring steam between two points. The two costs which has to be considered are

Fixed Cost = $$ax + b$$

Cost of heat losses = $$\frac{c}{x} + d$$

Find the optimum insulation thickness to minimize the total cost.

$$x \geq 0, \quad a, b, c, d \text{ constants.}$$
Solution:

Total Cost = \( C_T = ax + b + \frac{c}{x} + d \)

where \( a, b, c \) and \( d \) are constants.

For minimum cost \( \frac{dc_T}{dx} = 0 \)

\[
\frac{dc_T}{dx} = a - \frac{c}{x^2} = 0
\]

\[
x = \left(\frac{c}{a}\right)^{1/2}
\]

This is either the optimum point or a point of inflection.

\[
\frac{d^2c_T}{dx^2} = \frac{2c}{x^3}
\]

so if \( c \) positive and \( x \) must be positive.

Then

\[
\frac{d^2c_T}{dx^2} > 0 \quad (+ve) \rightarrow \min.
\]

Then

\[x = \left(\frac{c}{a}\right)^{1/2}\]

is the value of \( x \) which gives min. cost.

Example:

\[y = 7 + 0.5x - 0.5x^3\]

\[y = 0.5 - x = 0 \quad \therefore x = 0.5\]

\[y = 0.5 \quad (+ve \rightarrow \min)\]
H.W.

It is necessary to find optimum L/D ratio that minimize the cost. Find this minimum using P.3 from example 3 page 25.

Restricted Functions:

If there are restrictions on the independent variables, the optimum may not then be at stationary point but lies at the restriction boundary

Example 1: Find the maximum of

\[ y = \frac{x^5}{5} - \frac{5x^4}{2} + \frac{25}{3} x^3 - 25x^2 + 24x - 4 \]

For

(a) subjected to the restriction \( g = x^2 - (4.1)^2 = 0 \)

(b) \( g = x^2 - (4.1)^2 < 0 \)

Solution:

a) \( x^2 - (4.1)^2 = 0 \) \( \Rightarrow x = \pm 4.1 \)

\( y(4.1) = 3.5 \)

\( y(-4.1) = -2264.88 \)

b) when \(-4.1 < x < 4.1\)

Consider the function as unrestricted and choose a stationary point \( y=0 \), the optimum values at stationary must be compared with objective function at the boundary

\[ y = x^4 - 10x^3 + 35x^2 - 50x + 24 = 0 \]

\( (x-1)(x-2)(x-3)(x+1) = 0 \)

\( x = 1, 2, 3, -1 \)

\( y'' = 4x^3 - 30x^2 + 70x - 50 \)
\[ y^2 = (x-1)(x-2)(2x-7) + (x-3)(x-4)(2x-3) \]

The results are

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( y' )</th>
<th>( y'' )</th>
<th>type of pt.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.37</td>
<td>0</td>
<td>-6 (-ve)</td>
<td>max</td>
</tr>
<tr>
<td>2</td>
<td>3.73</td>
<td>0</td>
<td>2 (+ve)</td>
<td>min</td>
</tr>
<tr>
<td>3</td>
<td>4.10</td>
<td>0</td>
<td>-2 (-ve)</td>
<td>max</td>
</tr>
<tr>
<td>4</td>
<td>3.47</td>
<td>0</td>
<td>6 (+ve)</td>
<td>min</td>
</tr>
</tbody>
</table>

* Two maxs. at \( x = 1 \) & \( x = 3 \), but at \( x = 1 \) gives larger value for \( y \) where \( y = 4.37 \)

* \( y \) must be tested at the boundary \( x = 4.1 \)

\[ y = 3.5 \]

* So at \( x = 1 \) the global max. occurs which is the required value (at stationary not at the boundary).

Fig 7
Example: A plant produces refrigerators at a rate of \( P \) units per day. The variable costs per refrigerator have been found to be \( \$ (47.73 + 0.1P^{1.2}) \). The total daily fixed charges are \( \$ 1750 \), and all other expenses are constant at \( \$ 7325 \) per day. If the selling price per refrigerator is \( \$ 173 \), determine:

a) The daily profit at a production schedule giving the minimum cost per refrigerator.
b) The daily profit at a production schedule giving the maximum daily profit.
c) The production schedule at the break-even point.

Solution:

a) Total cost per refrigerator is

\[
C_T = 47.73 + 0.1P^{1.2} + \frac{1750}{P} + 7325
\]

So the decision variable is \( P \) (no. of refrigerators/day) and the function of \( C_T \) is the objective function.

For minimum cost per refrigerator at production schedule

\[
\frac{dC_T}{dP} = 0 = 0.12P^{0.2} - \frac{9075}{P^2}
\]

\( P_0 = 165 \) units per day for minimum cost per unit.

The daily profit at the production schedule for minimum cost per refrigerator

\[
\text{Profit} = \left[ 173 - (47.73 + 0.1P^{1.2} + \frac{9075}{P}) \right] P_0
\]

\[= \left[ 173 - (47.73 + 0.1(165)^{1.2} + \frac{9075}{165}) \right] 165 \]
b) The Daily Profit at a production schedule (max.)

\[ R = (173 - 47.73 - 0.1 P^{1.2} - \frac{1750 + 7325}{P}) P \]

For maximum daily profit

\[ \frac{dR}{dP} = 0 = 125.24 - 0.22 P^{1.2} \]

\[ P_0 = 198 \text{ units per day for maximum daily profit} \]

\[ R_{\text{max}} = (173 - 47.73 - 0.1 P^{1.2}_{198}) - \frac{9075}{198} \]

\[ = \$4400 \]

C) Total daily profit at break-even point

\[ R = 0 = (173 - 47.73 - 0.1 P^{1.2} - \frac{1750 + 7325}{P}) P \]

\[ P = 88 \text{ units per day at the break-even point} \]
Gradient Discontinuity:

The function suddenly jumps from one value to another without taking any of the intermediate values, for example

\[ y = \frac{1}{x} \]

The function is discontinuous at \( x = 0 \)
- \( y > 0 \) when \( x > 0 \)
- \( y < 0 \) when \( x < 0 \)

Another example

\[ y = |x^3| \]

\[ y = \begin{cases} 
  x^3 & \text{for } x > 0 \\
  -x^3 & \text{for } x < 0 
\end{cases} \]

\[ y' = 3x^2 \quad \text{at } x = 0 \quad y' = 0 \]
\[ y'' = 6x \quad \text{at } x = 0 \quad y'' = 0 \]
\[ y''' = 6 \quad \text{discontinuous at } x = 0 \]

The value of \( y'''(x) \) at \( x = 0 \) equal to either 6 or -6 depending upon whether it is approached from a higher or lower value of \( x \) so that optimum lying at the discontinuity.
Numerical Methods:

The analytical method requires the continuity of the function and its derivatives together with a solution for the points at which the first derivative vanishes. Because of these restrictions, alternative methods have been suggested, that is some type of search procedure called Numerical Search Method.

Direct Search:

Assume that for a specified value \( x_i \), we can find the value \( f(x_i) \) with known accuracy, and that \( f(x_i) \) is unimodal for the range \( a < x < b \). Also, assume that we wish to make a fixed number of measurements to determine the optimum of \( f(x) \).

For example, assume \( n \) values of \( x \)

\[ x_1, x_2, x_3, ..., x_n \]

Then find \( f(x_1), f(x_2), ..., f(x_n) \)

From results \( f(x^*) \) will be estimated.

The placing of the trial points \( x_i \) is called Search Plan.

For the case of two trials \( x_1 \) & \( x_2 \) where \( x_1 < x_2 \)

and the function is unimodal at the range \( a < x < b \),

there are three possible results (assume search for minimum)

a) \( f(x_1) > f(x_2) \) \( \Rightarrow \) \( 0 \leq x^* \leq x_2 \)

b) \( f(x_1) < f(x_2) \) \( \Rightarrow \) \( x_1 \leq x^* \leq 1 \)

c) \( f(x_1) = f(x_2) \) \( \Rightarrow \) \( x_1 \leq x^* \leq x_2 \)

as shown in the following figures.
Numerical Search - Unrestricted Function: (open ended)

1. Fixed Step Size: Unimodal Function

This method is based on the assumptions:

- Starting from a base point
  \( x_0 \) (or end point in previous search or arbitrary defined) \( f(x) \)

with fixed step size \( S \)

- \( x^*_1 = x_0 + S \)
- \( x^*_n = x_{n-1} + S \)

For maximum

- If \( f(x^*_1) > f(x^*_0) \)
  then \( x^* \geq x^*_1 \)
- If \( f(x^*_1) \leq f(x^*_0) \)
  then \( x^* \leq x^*_1 \)

* Search continued using a new base point until the final point \( x_4 \) which shows an increase (1st stage search terminated at \( x^*_4 \))
* Search continued with a new base pt. \( x^*_1 \) for higher accuracy
Example: Determine the minimum of the function
\[ y = (x - 100)^2 \]

Solution:
1. Choose a base point \( x_0 = 0 \)
2. Evaluate \( y(x_0) = (0 - 100)^2 = 10,000 \)
3. Choose a step size \( s = -3 \)
4. Evaluate \( x_1 = x_0 + s = -3 \)
5. Evaluate \( y(x_1) = (-3 - 100)^2 = 10,609 \)

Since \( y(x_1) > y(x_0) \)
minimum can not be found in the region of \( x = -3 \)
6. Take \( s = -s = 3 \)
\[ x_2 = 3 \quad \& \quad y(x_2) = 9406 \]
7. First Stage Search
\[ x_1 \rightarrow 0 \rightarrow -3 \rightarrow 3 \rightarrow 6 \rightarrow 9 \rightarrow 93 \rightarrow 96 \rightarrow 99 \rightarrow 102 \]
\[ y(x_1) \rightarrow 10,000 \rightarrow 10,609 \rightarrow 9,406 \rightarrow 8,836 \rightarrow 8,281 \rightarrow 49 \rightarrow 16 \rightarrow 4 \]
8. Second Stage Search: For higher accuracy \( s = 0.3 \) and \( x^2 = 99 \)
\[ x^2 \rightarrow 99 \rightarrow 99.3 \rightarrow 99.6 \rightarrow 99.9 \rightarrow 100.2 \rightarrow 100.5 \]
\[ y(x^2) \rightarrow 1 \rightarrow 0.99 \rightarrow 0.16 \rightarrow 0.1 \rightarrow 0.04 \rightarrow 0.25 \]
9. Further reduction can be used by making 3rd stage search between 100.2 and 100.3 taking \( s = 0.01 \) to get highly accurate result.
2. Direct Search with Acceleration (DSC).

The previous example repeated with acceleration (doubled step size) as follows:

<table>
<thead>
<tr>
<th>Step No.</th>
<th>S</th>
<th>X_i</th>
<th>Y(X_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10000</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>9409</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>9</td>
<td>8281</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>21</td>
<td>6241</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>45</td>
<td>3025</td>
</tr>
<tr>
<td>5</td>
<td>48</td>
<td>93</td>
<td>49</td>
</tr>
<tr>
<td>6</td>
<td>96</td>
<td>189</td>
<td>7920</td>
</tr>
</tbody>
</table>

The number of measurements required is reduced compared with previous method.

The optimum value \( X^* \) will be:

\[
X^* = \frac{1}{2} \left( X_{i+2} + X_{i-1} \right) - \frac{\beta_1}{\beta_{11}}
\]

\[
\beta_1 = \frac{y_{i+1} - y_{i-2}}{x_{i+1} - x_{i-2}}, \quad \beta_2 = \frac{y_{i+1} - y_{i-2}}{x_i - x_{i-2}}
\]

\[
\beta_{11} = \frac{\beta_2 - \beta_1}{x_{i+1} - x_{i-1}}
\]

Apply these for the data obtained gives:

\[
\beta_1 = \frac{49 - 3025}{93 - 45} = -62
\]

\[
\beta_2 = \frac{7921 - 3025}{189 - 45} = 34
\]
\[ \beta_{11} = \frac{34 - (-62)}{189 - 93} = 2 \]

\[ x^* = \frac{1}{2} \left( 45 + 93 - \frac{-62}{2 \times 1} \right) = 100 \]

Example: Find the maximum of \( y = 5 + 3x - x^3 \) starting at \( x_0 = 0 \) using ODE for \( \delta = 0.1 \).

Solution:

<table>
<thead>
<tr>
<th>Step No.</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( y(x_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.1</td>
<td>5.299</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.3</td>
<td>5.873</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>0.7</td>
<td>6.757</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
<td>1.5</td>
<td>6.125</td>
</tr>
<tr>
<td>5</td>
<td>1.6</td>
<td>3.1</td>
<td>-15.491</td>
</tr>
<tr>
<td>6</td>
<td>-0.4</td>
<td>1.1</td>
<td>6.969</td>
</tr>
</tbody>
</table>

If \( x_i = 1.1 \), \( y_i = 6.969 \)  
\( \beta_i = -0.79 \)  
\( \beta_{ii} = 3.3 \)

If \( x_i = 3.1 \), \( y_i = -15.491 \)  
\( \beta_i = -0.79 \)  
\( \beta_{ii} = -5.3 \)

\( x^* = 0.98 \)  
\( x^* = 1.02547 \)

So we can choose any of these two values.
Restricted Functions:

a- In the direct search techniques, the variable $x$ is subjected to certain restrictions, i.e.
\[ a \leq x \leq b \]
The search region is divided equally into a certain No. of intervals.

b- Allow a set of experiments or measurements to be performed and evaluate the objective function.

Uniform Search Method (Preplanned Experiments):
( Simultaneous search plan)

Let $L_0 = b - a$
and $L_0$ is divided into $N+1$ equal intervals,
where $N$ is the number of experiments without evaluating $y(a)$ or $y(b)$, then

\[ x_i = a + \frac{i(b-a)}{N+1} \]
where $i = 1, 2, \ldots, N$

The optimum will be located within $L_N$

\[ L_N = \frac{2L_0}{N+1} \]

The accuracy $\alpha$ will be

\[ \alpha = \frac{L_N}{L_0} = \frac{2L_0}{L_0(N+1)} \]

\[ \alpha = \frac{2}{N+1} \]

\[ N' \geq \frac{2}{\alpha} - 1 \]
For example: let \(0 \leq x \leq 10\), take \(N = 4\), find the optimum max.

1st case: if \(y_4 > y_2\) and \(y_4 > y_6\),
then the optimum lies in the range \(2 < x < 6\).

\[ L_N = 4 = \frac{2 \times 10}{4+1} \]

2nd case: if \(y_6 > y_4\) and \(y_6 > y_8\),
then the optimum lies in the range \(4 < x < 8\).

\[ L_N = 4 \quad \text{also} \]

hence the accuracy \( \epsilon X = \frac{L_N}{L_0} = \frac{4}{10} = 0.4 \)

2. Sequential Uniform Methods

Two Experiments:

placing two experiments in each cycle, \(f(x)\) must be evaluated at \(\frac{1}{2}\) and \(\frac{2}{3}\) of the search interval. So for two experiments \(X_1\) & \(X_2\) placed within \((L_0 = b - a)\), then

\[ X_1 = a + \frac{1}{2} L_0 \]

These are from the equation

\[ X_1 = a + \frac{1}{N+1} (b-a) \]

\[ X_2 = a + \frac{2}{3} L_0 \]

\(N = \text{number of experiments}\)
For $N =$ number of experiments and $m =$ number of cycles

$$L_N = L_{2m} = \left(\frac{2}{3}\right)^m L_0$$

i.e.

For the 1st cycle and 2 experiments

$$L_2 = \frac{2}{3} L_0$$

and for the 2nd cycle i.e. 4 experiments

$$L_4 = \left(\frac{2}{3}\right)^2 L_0$$

Since $\alpha = \frac{L_N}{L_0}$ then

$$\alpha = \left(\frac{2}{3}\right)^m \frac{L_0}{L_0} = \left(\frac{2}{3}\right)^{N/2}$$

If $N$ is even $N = 2m \geq 2$ $\frac{\log \alpha}{\log \frac{2}{3}}$

If $N$ is odd $N = 1 + 2m \geq 1 + 1$ $\frac{\log \alpha}{\log \frac{2}{3}}$

b) Three experiments

$P(x)$ is evaluated at $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ of $L_0$

The search interval

For $0 < x < 1$ will be

$$0 - \frac{1}{2} \quad \frac{1}{4} - \frac{3}{4} \quad \frac{3}{4} - 1$$

For $N = 3$

1st cycle $L_3 = \frac{1}{2} L_0$

2nd cycle

$$L_5 = \left(\frac{1}{2}\right)^2 L_0$$

where $L_5 = L_{1+2m}$

3rd cycle

$$L_N = L_{1+2m} = \left(\frac{1}{2}\right)^m L_0 = \left(\frac{1}{2}\right)^{N/2} L_0$$
Example: Find the optimum minimum point of
\[ y = x^2 - 6x + 2 \] in the interval 0 \leq x \leq 10
using sequential search method with two
experiments. The accuracy \( \alpha = 0.06 \).

Solution:
\[ N = 2m > \frac{2 \log x}{\log 2} > \frac{2 \log 0.06}{\log 0.6667} \]
\[ N \geq 13.8 \approx 14 \text{ experiments} \]
No. of cycles \( m = \frac{N}{2} = \frac{14}{2} = 7 \text{ cycles} \)

1st Cycle
\[ x_1 = \alpha + \frac{1}{2} b = 3.333 \text{ since } b = 10 \]
\[ x_2 = \alpha + \frac{2}{3} b = 6.667 \]
From the Function
\[ y_1 = -6.89 \]
\[ y_2 = 6.395 \]

2nd Cycle \( 0 \leq x \leq 6.667 \)
\[ x_1 = 2.223 \quad y_1 = -6.89 \]
\[ x_2 = 4.445 \quad y_2 = -4.89 \]

3rd Cycle \( 0 \leq x \leq 4.445 \)
\[ x_1 = 1.4819 \quad y_1 = -4.6941 \]
\[ x_2 = 2.9629 \quad y_2 = -6.9986 \]

4th Cycle \( 1.481 \leq x \leq 4.445 \)
\[ x_1 = 2.4691 \quad y_1 = -6.718 \]
\[ x_2 = 3.156 \quad y_2 = -6.791 \]
5th Cycle  \[ 2.469 < x < 4.445 \]
\[ x_1 = 3.128 \] \[ y_1 = -6.983 \]
\[ x_2 = 3.786 \] \[ y_2 = -6.382 \]

6th Cycle  \[ 2.469 < x < 3.786 \]
\[ x_1 = 2.908 \] \[ y_1 = -6.991 \]
\[ x_2 = 3.347 \] \[ y_2 = -6.879 \]

7th Cycle  \[ 2.469 < x < 3.347 \]
\[ x_1 = 2.761 \] \[ y_1 = -6.943 \]
\[ x_2 = 3.054 \] \[ y_2 = -6.997 \]

\[ \hat{x} = 3.054 \] which is the optimum to give minimum \[ y = -6.997 \]

\[ L_N = 3.347 - 2.761 = 0.5852 \]
\[ \hat{x} = \frac{0.5852}{10} = 0.05852 \]

H.W.

1. Find the maximum optimum of \( y = x \sin x \) with search interval \( L_0 = 0 \rightarrow 10 \) when \( x \) in radians using three equally spaced experiments with \( x = 0.05 \)...
   ans. \( \hat{x} = 7.9687 \)

2. Solve the previous example (page 28) using a) uniform search method (Preplanned experiments), b) then with \( 4 \) experiments.
   ans. \( \hat{x} = 2.9411 \)
Example:

Find the optimum minimum point of

\[ y = x^2 - 6x + 2 \quad 0 \leq x \leq 10 \]

using sequential method with three experiments

use \( \alpha = 0.06 \).

Solution:

\[ 0 \leq x \leq 10 \]

Take \( x \) at \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \) of \( 10 \)

1st Cycle:

\( x_1 = 2.5 \)
\( y_1 = -6.75 \)
\( x_2 = 5 \)
\( y_2 = -3 \)
\( x_3 = 7.5 \)
\( y_3 = 13.25 \)

2nd Cycle \( 0 \leq x \leq 5 \)

\( x_1 = 1.25 \)
\( y_1 = -3.9375 \)
\( x_2 = 2.5 \)
\( y_2 = -6.75 \)
\( x_3 = 3.75 \)
\( y_3 = -6.4375 \)

3rd Cycle \( 1.25 \leq x \leq 3.75 \)

\( x_1 = 1.875 \)
\( y_1 = -5.9343 \)
\( x_2 = 2.5 \)
\( y_2 = -6.75 \)
\( x_3 = 3.125 \)
\( y_3 = -6.9843 \)

4th Cycle \( 2.5 \leq x \leq 3.75 \)

\( x_1 = 2.8125 \)
\( y_1 = -6.9648 \)
\( x_2 = 3.125 \)
\( y_2 = -6.9848 \)
\( x_3 = 3.4375 \)
\( y_3 = -6.8085 \)
For the 4th Cycle: $L_N = 3.125 - 2.5 = 0.625$

\[ \alpha = \frac{e^{0.625}}{10} = \frac{L_N}{L_0} = 0.0625 > 0.06 \]

so we need another cycle to get an accuracy not more than 0.06

5th Cycle: $2.8125 \leq x \leq 3.4375$

$x_1 = 2.9687 \quad \delta_1 = -6.999$

$x_2 = 3.125 \quad \delta_2 = -6.989$

$x_3 = 3.281 \quad \delta_3 = -6.9208$

So

$L_N = 3.125 - 2.8125 = 0.3125$

\[ \alpha = \frac{L_N}{L_0} = \frac{0.3125}{10} = 0.03 < 0.06 \]

\[ x^* = 2.9687 \quad \delta_{\text{min}}^* = -6.999 \]
C) Dichotomous Search Plan

In interval of known limits, the assumption of unimodality is exploited to reduce the interval of uncertainty about the optimum as quickly as possible. For two points placed in a known interval, the interval of uncertainty is minimized for two points placed symmetrically about the midpoint. This is the basis of a dichotomous search. The distance between the two points is called the "Resolution Distance" and is denoted by \( S \) (Delta). The optimal placement of points is given by:

\[
\begin{align*}
X_1 &= b - \frac{b-a+S}{2} \\
X_2 &= a + \frac{b-a+S}{2}
\end{align*}
\]

The search proceeds with the placement of symmetrical points within the remaining interval of uncertainty. For example, if the search is for maximum and \( y(x_1) < y(x_2) \), the interval of uncertainty is now \( x_2 \leq x \leq b \) and the two new measurements are placed at:

\[
\begin{align*}
x_3 &= x_2 + \frac{b-x_2+S}{2} \\
x_4 &= b - \frac{b-x_2+S}{2}
\end{align*}
\]

Since \( S \) limits the resolution of measurements, the interval of uncertainty must be greater than \( S \).
and \[ x_1 = b - \frac{b-a+s}{2} \]
\[ x_2 = a + \frac{b-a+s}{2} \]

The number of experiments is given by
\[ (2^k + 1) \leq (b-a) \]

Where \( k \) is the number of pairs of search points.

Example:

Determine the optimum maximum of
\[ y = x e^{-x} \quad 0 \leq x \leq 2 \]

by dichotomous search, assume \( s = 0.05 \)

Solution:

The No. of experiments is
\[ (2^k + 1) \times 0.05 \leq 2 \]

\( k = 5 \) which is the greatest value of \( k \) satisfying this equation

1. First pair
\[ x_1 = 2 - \frac{2-0+0.05}{2} = 0.975 \]
\[ x_2 = 0 + \frac{2-0+0.05}{2} = 1.025 \]
\[ y_1 = 0.367763 \]
\[ y_2 = 0.367766 \]

\[ y_2 > y_1 \] so the new interval \( 0.775 \leq x \leq 2 \)

\[ \text{2nd Pair} \]
\[ x_3 = 2 - \frac{2 \cdot 0.975 + 0.05}{2} = 1.4625 \] \[ y_3 = 0.338797 \]
\[ x_4 = 0.975 + \frac{2 \cdot 0.975 + 0.05}{2} = 1.5125 \] \[ y_4 = 0.333292 \]

\[ y_3 > y_4 \]
The new interval \( 0.975 \leq x \leq 1.5125 \), Continue to obtain the following table:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( y_i )</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.975</td>
<td>0.367763</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1.025</td>
<td>0.367766</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1.4625</td>
<td>0.338797</td>
<td>0.975</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>1.5125</td>
<td>0.333292</td>
<td>0.975</td>
<td>1.5125</td>
</tr>
<tr>
<td>5</td>
<td>1.21875</td>
<td>0.360267</td>
<td>0.975</td>
<td>1.5125</td>
</tr>
<tr>
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<td>0.356575</td>
<td>0.975</td>
<td>1.26875</td>
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<tr>
<td>7</td>
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<td>0.366261</td>
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</tr>
<tr>
<td>8</td>
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<td>0.364279</td>
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<tr>
<td>9</td>
<td>1.035938</td>
<td>0.364747</td>
<td>0.975</td>
<td>1.146875</td>
</tr>
<tr>
<td>10</td>
<td>1.085938</td>
<td>0.366376</td>
<td>0.975</td>
<td>1.146875</td>
</tr>
</tbody>
</table>
The interval of uncertainty \((b-a)\) must be greater than 8 and from the previous table \(b-a=0.171875\) which is exceeded 25, so the best estimate of optimum \(x\) can be determined as follow:

\[ x_1 = \frac{1}{2} (a+b) = \frac{1}{2} (0.975 + 1.085938) \]
\[ x_1 = 1.030469 \]

and \( y_1 = 0.367712 \)

but from the table the value of \( y \) of \( x=1.025\) gives higher value

\[ x_2 = 1.025 \quad \quad y_2 = 0.367766 \]

\[ x^\ast = 1.025 \]

If we use analytical method:

\[ y = xe^{-x} \]
\[ \frac{dy}{dx} = e^{-x} - xe^{-x} = e^{-x}(1-x) \]

at \( y = 0 \), \( x^\ast = 1 \) and \( y^\ast = 0.367879 \)

\[ \frac{d^2y}{dx^2} = -e^{-x} - [e^{-x} - xe^{-x}] \]
\[ = -e^{-x}(x-2) \]

For \( x = 1 \)

\[ y^\ast = 0.368(1-2) = -0.736 \quad \text{min.} \]